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## LETTER

# The Complexity of Embedding of Acyclic Graphs into Grids with Minimum Congestion

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**SUMMARY** It is known that the problem of determining, given a planar graph  $G$  and integers  $m$  and  $n$ , whether there exists a congestion-1 embedding of  $G$  into a two dimensional  $m \times n$ -grid is NP-complete. In this paper, we show that the problem is still NP-complete if  $G$  is restricted to an acyclic graph.

**key words:** graph embedding, graph layout, VLSI layout, grid

## 1. Introduction

The problem of efficiently implementing parallel algorithms on parallel machines and the problem of efficiently laying out VLSI systems onto VLSI chips have been studied as the graph embedding problem, which is to embed a guest graph within a host graph with certain constraints and optimization criteria, such as dilation and/or congestion.

We consider minimal (edge) congestion embeddings of graphs into grids, or *layouts*. The grids are well known not only as a model of VLSI chips but also as one of the most popular processor interconnection graphs for parallel machines. It is well known that the minimal congestion embedding is very important for a parallel machine that uses cut-through switching techniques, which are well used in recent architectures for node-to-node communication. Also in VLSI layout, the minimal congestion embedding is crucial since the congestion is a lower bound for the number of layers.

It is known that the following problem is NP-complete\*\* [1][3]:

GRAPH LAYOUT

**Instance** A graph  $G$  and integers  $m$  and  $n$ .

**Question** Does there exist a layout of  $G$  into an  $m \times n$ -grid?

In particular, GRAPH LAYOUT is NP-complete if  $G$  is restricted to a connected planar graph [1]. It is not only interesting but also important to investigate the complexity of variants of GRAPH LAYOUT in which  $G$  is restricted to a smaller class than planar graphs. One

of the most important graphs of such classes are trees. However, the complexity of the problem of laying out trees is still open.

In this paper, we show by proving the following theorem that GRAPH LAYOUT is still NP-complete even if  $G$  is restricted to an acyclic graph:

**Theorem 1:** The problem of determining, given an acyclic graph  $G$  and integers  $m$  and  $n$ , whether there exists a layout of  $G$  into an  $m \times n$ -grid is NP-complete.

## 2. Preliminaries

Let  $G$  be a graph and let  $V(G)$  and  $E(G)$  denote the vertex set and edge set of  $G$ , respectively.  $\Gamma_G(v)$  is the set of edges incident to  $v \in V(G)$ .  $|\Gamma_G(v)|$  is called the *degree* of  $v$  and denoted by  $\deg_G(v)$ . We define that  $V_k(G) = \{v \in V(G) \mid \deg_G(v) = k\}$ .  $G[U]$  is the subgraph of  $G$  induced by  $U \subseteq V(G)$ . Similarly, we denote by  $G[S]$  the subgraph of  $G$  induced by  $S \subseteq E(G)$ . For graphs  $G$  and  $H$ ,  $G \cup H$  is the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H)$ .

The *one dimensional  $n$ -grid* denoted by  $M(n)$  is the graph with vertex set  $\{0, 1, \dots, n-1\}$  and edge set  $\{(i, i+1) \mid 0 \leq i \leq n-2\}$ . A Cartesian product  $M(m) \times M(n)$  is called a *two dimensional  $m \times n$ -grid* and denoted by  $M(m, n)$ .

An *embedding*  $\langle \phi, \rho \rangle$  of a graph  $G$  into a graph  $H$  is defined by a one-to-one mapping  $\phi : V(G) \rightarrow V(H)$ , together with a mapping  $\rho$  that maps each edge  $(u, v) \in E(G)$  onto a set of edges of  $H$  which induces a path connecting  $\phi(u)$  and  $\phi(v)$ . The *(edge) congestion* of  $\langle \phi, \rho \rangle$  is  $\max_{e_H \in E(H)} |\{e_G \in E(G) \mid e_H \in \rho(e_G)\}|$ . For  $U \subseteq V(G)$ , let  $\phi(U) = \{\phi(v) \mid v \in U\}$ . Also, let  $\rho(S) = \bigcup_{e \in S} \rho(e)$  for  $S \subseteq E(G)$ . Moreover, for a subgraph  $G'$  of  $G$ , we define that  $\langle \phi, \rho \rangle(G') = H[\rho(E(G'))]$ .

An embedding of a graph  $G$  into a two dimensional grid  $H$  is called a *layout* of  $G$  into  $H$  if it has unit congestion.

Throughout this paper, we contract redundant parentheses for simplicity. For example, we denote  $\deg_{M(m,n)}((i, j))$  simply by  $\deg_{M(m,n)}(i, j)$  for  $(i, j) \in V(M(m, n))$ .

## 3. Proof of Theorem 1

In this section, we prove the following problem is NP-

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\*\*Although it has not been proved formally so far that GRAPH LAYOUT is in NP, the proof is given in this paper.

complete:

### ACYCLIC GRAPH LAYOUT

**Instance** An acyclic graph  $G$  and integers  $m$  and  $n$ .

**Question** Does there exist a layout of  $G$  into  $M(m, n)$ ?

### 3.1 NP-Hardness of ACYCLIC GRAPH LAYOUT

We show the NP-hardness of ACYCLIC GRAPH LAYOUT by constructing a pseudo-polynomial reduction from 3-PARTITION, which is well known to be NP-complete in the strong sense [2].

#### 3-PARTITION

**Instance** A set of  $3r$  integers  $A = \{a_0, a_1, \dots, a_{3r-1}\}$  and a positive integer  $b$  such that  $b/4 < a_l < b/2$  ( $0 \leq l \leq 3r-1$ ) and  $\sum_{l=0}^{3r-1} a_l = rb$ .

**Question** Can  $A$  be partitioned into  $r$  disjoint sets  $A_0, \dots, A_{r-1}$  such that  $\sum_{a \in A_k} a = b$  for  $0 \leq k \leq r-1$ ?

#### 3.1.1 Translation of Instance

For integers  $a_0, \dots, a_{3r-1}$ , and  $b$  given as an instance of 3-PARTITION, we construct an instance of ACYCLIC GRAPH LAYOUT by setting  $m = 2(b+7)r+3$  and  $n = 3$  and by constructing  $G = F \cup \bigcup_{0 \leq l \leq 3r-1} P_l$ , where  $P_l = M(a_l+1)$  for  $0 \leq l \leq 3r-1$  and  $F$  is the graph obtained from components  $C_0, \dots, C_{2(b+7)r+2}$  defined as follows:

- For  $1 \leq i \leq (b+7)r$ , let  $C_{2i} = M(2)$  and  $s_{2i} \in V(C_{2i})$ .
- For  $0 \leq k \leq r$ , let  $C_{2(b+7)k+1}$  be a 7-vertex complete binary tree, and let  $s_{2(b+7)k+1}$  be the root of  $C_{2(b+7)k+1}$ . In what follows,  $C_{2(b+7)k+1}$  is also denoted by  $Y_k$  for  $1 \leq k \leq r-1$ .
- For  $0 \leq k \leq r-1$  and  $1 \leq i \leq b+6$ , let  $C_{2(b+7)k+2i+1} = M(3)$ , and let  $s_{2(b+7)k+2i+1}$  be the unique vertex in  $V_2(C_{2(b+7)k+2i+1})$ .
- For  $i \in \{0, 2(b+7)r+2\}$ , let  $C_i$  be a trivial graph consisting of the single vertex  $s_i$ .

$F$  is defined as the graph obtained from  $C_0, \dots, C_{2(b+7)r+2}$  by joining  $s_i$  and  $s_{i+1}$  by an additional edge for  $0 \leq i \leq 2(b+7)r+1$ .

It should be noted that  $V_0(G) = \emptyset$  and  $V_4(G) = \{s_{2i+1} \mid 0 \leq i \leq (b+7)r\}$ . We denote  $s_{2i+1}$  by  $q_i$  for  $0 \leq i \leq (b+7)r$ . Moreover, we define the subgraphs  $W_0, \dots, W_{r-1}$  of  $F$  as follows:

$$\begin{aligned} W_0 &= F[\bigcup_{i=0}^{2(b+7)} V(C_i)], \\ W_k &= F[\bigcup_{i=2}^{2(b+7)} V(C_{2(b+7)k+i})] \quad (1 \leq k \leq r-2), \\ W_{r-1} &= F[\bigcup_{i=2}^{2(b+7)+2} V(C_{2(b+7)(r-1)+i})]. \end{aligned}$$

Figure 1 shows  $F$ .

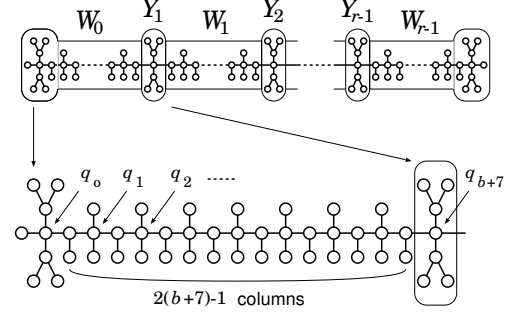


Fig. 1 Construction of  $F$

#### 3.1.2 Correspondence of Answers

Now we show that  $A$  can be partitioned into disjoint sets  $A_0, \dots, A_{r-1}$  such that  $\sum_{a \in A_k} a = b$  for  $0 \leq k \leq r-1$  if and only if there exists a layout of  $G$  into  $H = M(m, n)$ .

We first show the necessity. We define the vertex sets  $U_0, \dots, U_{r-1}$  of  $H$  as follows:

$$\begin{aligned} U_0 &= \{(i, j) \mid 0 \leq i \leq 2(b+7), 0 \leq j \leq 2\}, \\ U_k &= \{(2(b+7)k+i, j) \mid 2 \leq i \leq 2(b+7), \\ &\quad 0 \leq j \leq 2\} \quad (1 \leq k \leq r-2), \\ U_{r-1} &= \{(2(b+7)(r-1)+i+2, j) \mid \\ &\quad 0 \leq i \leq 2(b+7), 0 \leq j \leq 2\}. \end{aligned}$$

**Lemma 2:** If  $A$  can be partitioned into disjoint sets  $A_0, \dots, A_{r-1}$  such that  $\sum_{a \in A_k} a = b$  for  $0 \leq k \leq r-1$ , then there exists a layout of  $G$  into  $H$ .

*Proof* By definition, we can layout  $F$  into  $H$  so that  $U_k$  ( $0 \leq k \leq r-1$ ) has just  $b+3$  vertices to which no vertex of  $F$  is mapped and that if  $a_l \in A_k$  ( $0 \leq l \leq 3r-1$ ), then  $P_l$  can be laid out into  $H[U_k]$ . This clearly gives a desired layout (Fig. 2).  $\square$

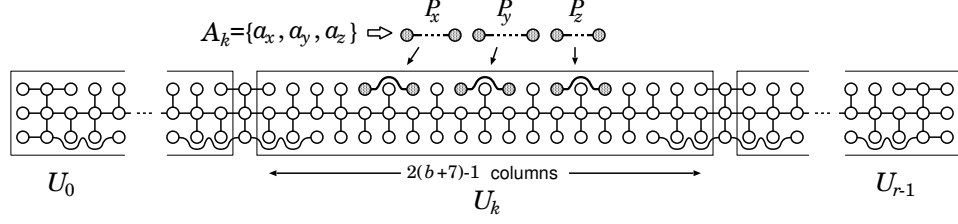
It remains to show the sufficiency. Assume that there exists a layout  $\varepsilon = \langle \phi, \rho \rangle$  of  $G$  into  $H$ . We show by a series of lemmas that  $A$  can be partitioned into disjoint sets as desired.

For  $v \in V(H)$ , let  $\phi^{-1}(v)$  be  $u \in V(G)$  such that  $\phi(u) = v$ . It should be noted that since  $|V(G)| = |H(G)|$ ,  $\phi^{-1}(v)$  exists for every vertex  $v \in V(H)$ . The following two lemmas are immediate from the assumption that  $\varepsilon$  has unit congestion:

**Lemma 3:** For  $v \in V(H)$ ,  $\deg_{\varepsilon(G)}(v) = \deg_G(\phi^{-1}(v)) + 2|\{e \in E(G) - \Gamma_G(\phi^{-1}(v)) \mid v \in V(H[\rho(e)])\}|$ .  $\square$

**Lemma 4:** For  $(u, v) \in E(H)$ , if  $\deg_G(\phi^{-1}(u)) = \deg_H(u)$  and  $\deg_G(\phi^{-1}(v)) \geq \deg_H(v) - 1$ , then  $(\phi^{-1}(u), \phi^{-1}(v)) \in E(G)$  and  $\rho(\phi^{-1}(u), \phi^{-1}(v)) = \{(u, v)\}$ .  $\square$

The following lemma can easily be shown by Lemma 3 and the fact that  $V_0(G) = \emptyset$ :

Fig. 2 Layout of  $G$  into  $H$ 

**Lemma 5:** For  $e_1, e_2 \in E(G)$ , if both  $H[\rho(e_1)]$  and  $H[\rho(e_2)]$  contain  $v \in V(H)$ , then  $\phi^{-1}(v)$  is incident to  $e_1$  or  $e_2$ .  $\square$

**Lemma 6:**  $\phi(V_4(G)) = \{(2i+1, 1) \mid 0 \leq i \leq (b+7)r\}$ .

*Proof* Since  $|V_4(G)| = (b+7)r + 1$  from the definition of  $G$ , it suffices to show that  $\phi(V_4(G)) \supseteq \{(2i+1, 1) \mid 0 \leq i \leq (b+7)r\}$ . We show this by contradiction. Assume that there exists  $0 \leq i' \leq (b+7)r$  such that  $(2i'+1, 1) \notin \phi(V_4(G))$ .

It clearly follows that  $\phi(V_4(G)) \subseteq V_4(H) = \{(i, 1) \mid 1 \leq i \leq 2(b+7)r + 1\}$ . Moreover, since any two vertices  $u, v \in V_4(G)$  are not adjacent, it follows from Lemma 4 that  $\phi(u)$  and  $\phi(v)$  are not adjacent. Thus, we have that  $|V_4(G)| = |\phi(V_4(G))| \leq \lceil \{(i, 1) \mid 1 \leq i \leq 2i'\} / 2 \rceil + \lceil \{(i, 1) \mid 2i'+2 \leq i \leq 2(b+7)r + 1\} / 2 \rceil = i' + (b+7)r - i' = (b+7)r < |V_4(G)|$ , a contradiction.  $\square$

**Lemma 7:** Let  $v = (2i, 1) \in V(H)$  ( $1 \leq i \leq (b+7)r$ ) and  $v_k = (2i+k, 1) \in V(H)$  for  $k \in \{-1, +1\}$ .

- (a) For each  $k \in \{-1, +1\}$ , there exists  $e_k \in \Gamma_G(\phi^{-1}(v_k))$  such that  $(v, v_k) \in \rho(e_k)$ ;
- (b)  $\deg_G(\phi^{-1}(v)) \in \{1, 3\}$ ;
- (c) If  $\deg_G(\phi^{-1}(v)) = 1$ , then
  - there exist  $k \in \{-1, +1\}$  and  $l \in \{0, 2\}$  such that  $\rho(e_k) = \{(v, v_k)\}$  and  $\rho(e_{-k}) \supseteq \{(v, v_{-k}), (v, (2i, l))\}$ , and
  - $\Gamma_{\varepsilon(G)}(v) \subseteq \rho(\{e_{-1}, e_{+1}\})$ .

*Proof* Since  $\deg_G(\phi^{-1}(v_k)) = 4$  ( $k \in \{-1, +1\}$ ) by Lemma 6, (a) is clearly satisfied. Thus, it follows from Lemma 5 that  $\phi^{-1}(v)$  is incident to  $e_{-1}$  or  $e_{+1}$ . Since every vertex in  $V_4(G)$  is adjacent to a vertex in  $V_1(G) \cup V_3(G)$  from the definition of  $G$ , (b) is satisfied.

It remains to show that (c) is satisfied. Assume that  $\deg_G(\phi^{-1}(v)) = 1$ . By this assumption,  $\phi^{-1}(v)$  is incident to either  $e_{-1}$  or  $e_{+1}$ . Suppose that  $\phi^{-1}(v)$  is incident to  $e_k$  but not to  $e_{-k}$  ( $k \in \{-1, +1\}$ ). Then, we have that  $\rho(e_k) = \{(v, v_k)\}$ , for otherwise,  $\rho(e_k)$  contains another edge than  $(v, v_k)$ , which means that  $H[\rho(e_k)]$  is not a path connecting  $v$  and  $v_k$ . Moreover, there clearly exists  $l \in \{0, 2\}$  such that  $\rho(e_{-k})$  contains  $(2i, l)$  but not  $(2i, l-2)$ . Thus, we have that  $\rho(e_{-k}) \supseteq \{(v, v_{-k}), (v, (2i, l))\}$  and  $\Gamma_{\varepsilon(G)}(v) \subseteq \rho(\{e_{-1}, e_{+1}\})$ .  $\square$

**Lemma 8:** For  $1 \leq i \leq (b+7)r$ ,  $\deg_{\varepsilon(G)}(2i, 0) \leq 2$  or  $\deg_{\varepsilon(G)}(2i, 2) \leq 2$ .

*Proof* By Lemma 3, we have that  $\deg_{\varepsilon(G)}(v) \bmod 2 = \deg_G(\phi^{-1}(v)) \bmod 2$  for  $v \in V(H)$ . From this fact and (b) in Lemma 7, we have that  $\deg_{\varepsilon(G)}(2i, 1) < \deg_H(2i, 1) = 4$ . Thus, it follows from (a) in Lemma 7 that there exists  $l \in \{0, 2\}$  such that  $((2i, 1), (2i, l)) \notin E(\varepsilon(G))$ . Therefore, we have that  $\deg_{\varepsilon(G)}(2i, l) \leq \deg_H(2i, l) - 1 = 2$ .  $\square$

**Lemma 9:** Let  $P$  be a path of  $F$  connecting two vertices of degree 3 or more. If  $\varepsilon(P)$  contains  $v = (2i, j) \in V(H)$  ( $1 \leq i \leq (b+7)r$ ,  $j \in \{0, 2\}$ ), then

- (a)  $\deg_{\varepsilon(G)}(v) = 3$ , and
- (b) if there exists  $e \in E(G)$  incident to no vertex of  $P$  such that  $H[\rho(e)]$  contains  $v$ , then  $\Gamma_G(\phi^{-1}(v)) = \{e\}$ .

*Proof* Let  $e' \in E(P)$  such that  $H[\rho(e')]$  contains  $v$ .

We first show that (a) is satisfied. Since every vertex of  $P$  has degree 3 or more by definition, if  $\phi^{-1}(v) \in V(P)$ , then  $3 \leq \deg_G(\phi^{-1}(v)) \leq \deg_{\varepsilon(G)}(v) \leq \deg_H(v) = 3$ . If  $\phi^{-1}(v) \notin V(P)$ , then by Lemma 3 and the fact that  $V_0(G) = \emptyset$ , we have that  $1 \leq \deg_G(\phi^{-1}(v)) \leq \deg_{\varepsilon(G)}(v) - 2|\{e'\}| \leq 3 - 2 = 1$ , i.e., there exists  $e'' \in E(G)$  such that  $\Gamma_G(\phi^{-1}(v)) = \{e''\}$  and  $\deg_{\varepsilon(G)}(v) = |\{e''\}| + 2|\{e'\}| = 3$ .

We next show that (b) is satisfied. If there exists  $e \in E(G)$  incident to no vertex of  $P$  such that  $H[\rho(e)]$  contains  $v$ , then it follows from Lemma 3 and the definitions of  $P$  and  $e$  that  $\phi^{-1}(v) \notin V(P)$ . Thus, it follows from the proof of (a) that  $\Gamma_G(\phi^{-1}(v)) = \{e\}$ .  $\square$

**Lemma 10:**  $\{\phi(q_0), \phi(q_{(b+7)r})\} = \{(1, 1), (m-2, 1)\}$ .

*Proof* From the regularity of  $G$ , it suffices to show that  $\phi(q_0) \in \{(1, 1), (m-2, 1)\}$ . We prove this by contradiction. Assume that  $\phi(q_0) \notin \{(1, 1), (m-2, 1)\}$ . Since  $q_0 \in V_4(G)$ , it follows by this assumption and Lemma 6 that there exists  $1 \leq i \leq (b+7)r - 1$  such that  $\phi(q_0) = (2i+1, 1)$ . By Lemma 6 again, there exists  $q_x$  and  $q_y$  ( $1 \leq y < x \leq (b+7)r$ ) such that  $\{\phi(q_x), \phi(q_y)\} = \{(2i-1, 1), (2i+3, 1)\}$ . We may assume without loss of generality that  $\phi(q_x) = (2i-1, 1)$ . Let  $P$  be a path of  $F$  connecting  $q_x$  and  $q_y$ . By definition, there exists  $e_P \in E(P)$  such that  $\rho(e_P)$  contains an edge in  $\{((2i, j), (2i+1, j)) \mid 0 \leq j \leq 2\}$ . Since  $q_0 \notin V(P)$  and  $\deg_G(q_0) = \deg_H(\phi(q_0))$ , it follows that  $E(\varepsilon(P)) \cap \Gamma_H(2i+1, 1) = \emptyset$ . Thus, we may assume without loss of generality that  $((2i, 0), (2i+1, 0)) \in \rho(e_P)$ .

Let  $u, v$ , and  $w$  be vertices of  $G$  such that  $\phi(u) = (2i + 1, 0)$ ,  $\phi(v) = (2i, 1)$ , and  $\phi(w) = (2i, 0)$ . Moreover, let  $e_u$  and  $e_v$  be edges in  $\Gamma_G(q_0)$  such that  $(\phi(q_0), \phi(u)) \in \rho(e_u)$  and  $(\phi(q_0), \phi(v)) \in \rho(e_v)$ .

It follows from (b) in Lemma 7 that  $\deg_G(v) \in \{1, 3\}$ . If  $\deg_G(v) = 3$ , then it follows from Lemma 4 that  $v$  is adjacent to both  $q_0$  and  $q_x$ , contradicting the assumption that  $x > 1$ . Thus, we have that  $\deg_G(v) = 1$ . Moreover, since  $e_u$  is not incident to a vertex of  $P$ , it follows from (b) in Lemma 9 that  $e_u = (q_0, u)$  and  $\deg_G(u) = 1$ . Thus,  $v$  and  $q_0$  are not adjacent since  $q_0$  is adjacent to exactly one vertex of degree 1, which is  $u$ . Since  $H[\rho(e_P)]$  contains  $\phi(w)$ , it follows from (a) in Lemma 9 that  $\deg_{\varepsilon(G)}(\phi(w)) = 3$ , and hence, there exists  $e \in E(G)$  such that  $(\phi(v), \phi(w)) \in \rho(e)$ . Thus, it follows from (c) in Lemma 7 that  $(\phi(v), \phi(q_0)) \in \rho(e)$ . Since  $(\phi(v), \phi(q_0)) \in \rho(e_v)$  by definition, we have that  $e_v = e$  and  $H[\rho(e_v)]$  contains  $\phi(w)$ . Thus, since  $e_v$  is not incident to a vertex of  $P$ , it follows from (b) in Lemma 9 that  $e_v = (q_0, w)$  and  $\deg_G(w) = 1$ . This means that  $q_0$  is adjacent to two vertices  $u$  and  $w$  of degree 1, contradicting the definition of  $F$ .  $\square$

From the regularity of  $G$ , we may assume without loss of generality that  $\phi(q_0) = (1, 1)$  and  $\phi(q_{(b+7)r}) = (m - 2, 1)$ .

**Lemma 11:** For  $0 \leq i \leq (b + 7)r$ ,  $\phi(q_i) = (2i + 1, 1)$ .

*Proof* It follows from Lemma 6 that, for  $0 \leq i \leq (b + 7)r$ , there exists  $0 \leq h_i \leq (b + 7)r$  such that  $\phi(q_i) = (2h_i + 1, 1)$ . It suffices to show that  $h_x < h_y$  for  $0 \leq x < y \leq (b + 7)r$ . We prove this by contradiction. Assume that there exist  $q_x$  and  $q_y$  such that  $h_x > h_y$  ( $x < y$ ). Since  $\phi(q_0) = (1, 1)$  and  $\phi(q_{(b+7)r}) = (m - 2, 1)$ , it follows by assumption that  $1 \leq x < y \leq (b + 7)r - 1$  and  $1 \leq h_y < h_x \leq (b + 7)r - 1$ .

Let  $P_0, P_1$ , and  $P_2$  be paths of  $F$  which connect  $q_0$  and  $q_x, q_x$  and  $q_y$ , and  $q_y$  and  $q_{(b+7)r}$ , respectively. By the definitions of  $q_x, q_y, P_0, P_1$ , and  $P_2$ , for each edge  $e \in \{((2h_x, j), (2h_x + 1, j)) \mid 0 \leq j \leq 2\}$ , there exists  $P \in \{P_0, P_1, P_2\}$  such that  $e \in E(\varepsilon(P))$ . Thus, it follows from (a) in Lemma 9 that  $\deg_{\varepsilon(G)}(2h_x, 0) = \deg_{\varepsilon(G)}(2h_x, 2) = 3$ , contradicting Lemma 8.  $\square$

**Lemma 12:** For  $v \in V_3(Y_k)$  ( $1 \leq k \leq r - 1$ ),  $\phi(v) \in \{(2(b + 7)k + 1, j) \mid j \in \{0, 2\}\}$ .

*Proof* We prove the lemma by contradiction. Assume that there exists  $v \in V_3(Y_k)$  ( $1 \leq k \leq r - 1$ ) such that  $\phi(v) \notin \{(2(b + 7)k + 1, j) \mid j \in \{0, 2\}\}$ . Since  $v$  is not adjacent to a vertex in  $V_4(G) - \{q_{(b+7)k}\}$ , it follows from assumption and Lemmas 4 and 11 that  $\phi(v)$  is not adjacent in  $H$  to a vertex in  $\{(2i + 1, 1) \mid 0 \leq i \leq (b + 7)r\}$ . Since  $\phi(v) \notin V_2(H)$  clearly, there exist  $1 \leq i \leq (b + 7)r$  and  $j \in \{0, 2\}$  such that  $\phi(v) = (2i, j)$ .

It follows from (b) in Lemma 7 that  $\deg_G(\phi^{-1}(2i, 1)) \in \{1, 3\}$ . If  $\deg_G(\phi^{-1}(2i, 1)) = 3$ , then  $(v, \phi^{-1}(2i, 1)) \in$

$E(G)$  from Lemma 4. However, this is a contradiction because  $v$  is not adjacent to a vertex of degree 3. Thus, we have that  $\deg_G(\phi^{-1}(2i, 1)) = 1$ . Let  $e \in E(G)$  be the unique edge incident to  $\phi^{-1}(2i, 1)$ . Since  $\deg_G(v) = \deg_H(\phi(v)) = 3$ , there exists  $e' \in \Gamma_G(v)$  such that  $(\phi(v), (2i, 1)) \in \rho(e')$ . Thus, it follows from (c) in Lemma 7 that  $\Gamma_{\varepsilon(G)}(2i, 1) \subseteq \rho(\{e, e'\})$ .

Let  $P$  be the path of  $G$  which connects  $q_{i-1}$  and  $q_i$ . By the definition of  $P$  and Lemma 11,  $\varepsilon(P)$  contains a vertex in  $\{(2i, j') \mid 0 \leq j' \leq 2\}$ . Since  $\deg_G(v) = \deg_H(\phi(v))$  and  $v \notin V(P)$ , it follows that  $(2i, j) (= \phi(v)) \notin V(\varepsilon(P))$ . Also, since  $\{e, e'\} \cap E(P) = \emptyset$ , it follows that  $(2i, 1) \notin V(\varepsilon(P))$ . Thus, we have that  $(2i, 2 - j) \in V(\varepsilon(P))$ . Therefore, it follows from (a) in Lemma 9 that  $\deg_{\varepsilon(G)}(2i, 2 - j) = 3$ , contradicting Lemma 8 and the fact that  $\deg_{\varepsilon(G)}(2i, j) = \deg_G(v) = 3$ .  $\square$

It follows from Lemmas 11 and 12 that  $\phi(V(Y_k) \cap (V_3(G) \cup V_4(G))) = \{(2(b + 7)k + 1, j) \mid 0 \leq j \leq 2\}$  for  $1 \leq k \leq r - 1$ . Let  $y_{k,j} = \phi^{-1}(2(b + 7)k + 1, j)$  ( $1 \leq k \leq r - 1, 0 \leq j \leq 2$ ). Since  $\deg_H(\phi(y_{k,j})) = \deg_G(y_{k,j})$ , there exist edges  $e_{k,j}^-, e_{k,j}^+ \in \Gamma_G(y_{k,j})$  such that  $((2(b + 7)k + 1, j), (2(b + 7)k, j)) \in \rho(e_{k,j}^-)$  and  $((2(b + 7)k + 1, j), (2(b + 7)k + 2, j)) \in \rho(e_{k,j}^+)$ . Let  $E_k^- = \{e_{k,j}^- \mid 0 \leq j \leq 2\}$  and  $E_k^+ = \{e_{k,j}^+ \mid 0 \leq j \leq 2\}$  for  $1 \leq k \leq r - 1$ . The following lemma is immediate from the definitions of  $E_k^-$  and  $E_k^+$  ( $1 \leq k \leq r - 1$ ).

**Lemma 13:** For a connected subgraph  $G'$  of  $G$  and  $1 \leq k \leq r - 1$ ,

- (a) if  $E(G') \cap E_k^- = \emptyset$ , then  $V(\varepsilon(G')) \subseteq \{(i, j) \mid 0 \leq i \leq 2(b + 7)k, 0 \leq j \leq 2\}$  or  $V(\varepsilon(G')) \subseteq \{(i, j) \mid 2(b + 7)k + 1 \leq i \leq 2(b + 7)r + 2, 0 \leq j \leq 2\}$ ,
- (b) if  $E(G') \cap E_k^+ = \emptyset$ , then  $V(\varepsilon(G')) \subseteq \{(i, j) \mid 0 \leq i \leq 2(b + 7)k + 1, 0 \leq j \leq 2\}$  or  $V(\varepsilon(G')) \subseteq \{(i, j) \mid 2(b + 7)k + 2 \leq i \leq 2(b + 7)r + 2, 0 \leq j \leq 2\}$ ,  $\square$

**Lemma 14:** For  $0 \leq k \leq r - 1$ ,  $\phi(V(W_k)) \subset U_k$ .

*Proof* By definition,  $W_k$  is a connected graph which does not contain an edge in  $\bigcup_{1 \leq i \leq r-1} E_i^- \cup E_i^+$ . Moreover, it follows from Lemma 11 that  $\phi(V_4(W_k)) \subset U_k$ . Thus, this lemma holds from Lemma 13.  $\square$

**Lemma 15:** For  $1 \leq k \leq r - 1$ ,  $|\phi(V_1(Y_k)) \cap U_{k-1}| = |\phi(V_1(Y_k)) \cap U_k| = 2$ .

*Proof* Since  $\deg_G(y_{k,j}) = \deg_H(\phi(y_{k,j}))$  for  $1 \leq k \leq r - 1$  and  $0 \leq j \leq 2$ , it follows from Lemma 4 that  $(y_{k,1}, y_{k,j}) \in E(G)$  and  $\rho(y_{k,1}, y_{k,j}) = \{(\phi(y_{k,1}), \phi(y_{k,j}))\}$  ( $1 \leq k \leq r - 1, j \in \{0, 2\}$ ). Thus, it follows that  $(y_{k,1}, y_{k,j}) \notin E_k^- \cup E_k^+$ . This means that, for  $1 \leq k \leq r - 1$ ,  $e_{k,j}^-$  and  $e_{k,j}^+$  ( $j \in \{0, 2\}$ ) are incident to vertices  $v_{k,j}^-, v_{k,j}^+ \in V_1(Y_k)$ , respectively.

Clearly,  $G[\{e_{k,j}^-\}]$  is a connected graph which does

not contain an edge in  $\bigcup_{1 \leq i \leq r-1} E_i^- \cup E_i^+ - E_k^-$ . Moreover,  $H[\rho(e_{k,j}^-)]$  contains the vertex  $(2(b+7)k, j)$ . Thus, it follows from Lemma 13 that  $V(H[\rho(e_{k,j}^-)]) \subset U_{k-1} \cup \{\phi(y_{k,j})\}$ . This means that  $\phi(v_{k,j}^-) \in U_{k-1}$ . Also, we can show  $\phi(v_{k,j}^+) \in U_k$  by a similar argument. Therefore, we have that  $|\phi(V_1(Y_k)) \cap U_{k-1}| = |\{v_{k,0}^-, v_{k,2}^-\}| = 2$  and  $|\phi(V_1(Y_k)) \cap U_k| = |\{v_{k,0}^+, v_{k,2}^+\}| = 2$ .  $\square$

**Lemma 16:**  $A$  can be partitioned into disjoint sets  $A_0, \dots, A_{r-1}$  such that  $\sum_{a \in A_k} a = b$  for  $0 \leq k \leq r-1$ .

*Proof* It follows from Lemmas 11, 12, 14, and 15 that  $U_k$  ( $0 \leq k \leq r-1$ ) has just  $b+3$  vertices to which no vertex of  $F$  is mapped. Since  $P_l$  ( $0 \leq l \leq 3r-1$ ) has no edge in  $\bigcup_{1 \leq i \leq r-1} E_i^- \cup E_i^+$ , by Lemma 13, there exists  $0 \leq k \leq r-1$  such that  $\phi(V(P_l)) \subset U_k$ . Since  $|V(P_l)| = a_l + 1$ , it follows that  $A$  can be partitioned into disjoint sets  $A_0, \dots, A_{r-1}$  such that  $\sum_{a \in A_k} a = b$  for  $0 \leq k \leq r-1$ .  $\square$

By Lemmas 2 and 16 and the fact that  $V(G) = O(br)$ , we have obtained a pseudo-polynomial reduction. Since 3-PARTITION is NP-complete in the strong sense, ACYCLIC GRAPH LAYOUT is NP-hard.

### 3.2 ACYCLIC GRAPH LAYOUT is in NP

Obviously, ACYCLIC GRAPH LAYOUT is in NP if GRAPH LAYOUT is in NP. However, this has not been proved formally so far. This is not straightforward in the sense that every layout of  $G$  into  $M(m, n)$  itself may not be a witness of polynomial size if  $m$  or  $n$  is much greater than  $|V(G)|$ . However, it is guaranteed as shown in the following lemma that there exists a witness of polynomial size for any instance for which the answer is “yes”.

**Lemma 17:** Let  $G$ ,  $m$ , and  $n$  be given as an instance of GRAPH LAYOUT. If  $G$  can be laid out into  $M(m, n)$ , then there exists a layout of  $G$  into a subgraph  $H$  of  $M(m, n)$  which has polynomial size of  $s = O(|V(G)| + \log m + \log n)$ , i.e., the size of the instance.

*Proof* The lemma is immediate if both  $m$  and  $n$  has polynomial size of  $s$ . If neither  $m$  nor  $n$  has polynomial size of  $s$ , then the lemma holds from the result of [5] that an  $N$ -vertex graph with maximum vertex degree at most 4 can be laid out into  $M(3N, 3N)$ . Moreover, if either  $m$  or  $n$ , say  $m$ , has polynomial size of  $s$ , then the lemma also holds from the result of [4] that an  $N$ -vertex graph which can be laid out into  $M(m, n)$  can be laid out into  $M(m, 2mN)$ .  $\square$

Thus, GRAPH LAYOUT and hence ACYCLIC GRAPH LAYOUT is in NP.

Sections 3.1 and 3.2 complete the proof of Theorem 1.

## 4. Concluding Remarks

To show the NP-completeness of ACYCLIC GRAPH LAYOUT, we construct a reduction by setting  $n = 3$ . We do not know the complexity of the problem with  $n$  to be a much greater integer, such as  $m$ . Also, we do not know the complexity of the problem with  $n$  to be a fixed integer 2, 4 or more. In this connection, it is known that the problem of determining, given a graph  $G$  and integer  $m$ , whether there exists a layout of  $G$  into  $M(m, k)$  is NP-complete for any fixed integer  $k \geq 3$  [4].

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