

NONISOTROPIC DILATIONS AND THE METHOD OF ROTATIONS WITH WEIGHT

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ABSTRACT. We consider maximal functions $Mf(x, \theta)$, singular integrals $Hf(x, \theta)$, and maximal singular integrals $H_*f(x, \theta)$ on $\mathbb{R}^n \times S^{n-1}$ associated with homogeneous curves, for functions f on \mathbb{R}^n . We prove certain weighted mixed norm estimates for them. These results are applied to the theory of singular integrals with variable kernels via the method of rotations of Calderón-Zygmund.

1. INTRODUCTION

Let $\{A_t\}_{t>0}$ be a dilation group on \mathbb{R}^n defined by $A_t = t^P = \exp((\log t)P)$, where P is an $n \times n$ real matrix whose eigenvalues have positive real parts. We assume $n \geq 2$. We can define a norm function r on \mathbb{R}^n from $\{A_t\}_{t>0}$ such that:

- (1) $r(x) \geq 0$, $r(x) = r(-x)$ for all $x \in \mathbb{R}^n$, and $r(x) = 0$ if and only if $x = 0$;
- (2) r is continuous on \mathbb{R}^n and infinitely differentiable in $\mathbb{R}^n \setminus \{0\}$;
- (3) $r(A_t x) = tr(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$;
- (4) $r(x + y) \leq C(r(x) + r(y))$ for some $C > 0$;
- (5) $\Sigma = \{\theta \in \mathbb{R}^n : \langle B\theta, \theta \rangle = 1\}$ for a positive symmetric matrix B , where $\Sigma = \{x \in \mathbb{R}^n : r(x) = 1\}$ and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in \mathbb{R}^n ;
- (6) $dx = t^{\gamma-1} d\mu dt$, that is,

$$\int_{\mathbb{R}^n} f(x) dx = \int_0^\infty \int_\Sigma f(A_t \theta) t^{\gamma-1} d\mu(\theta) dt, \quad d\mu = \omega d\mu_0,$$

where ω is a strictly positive C^∞ function on Σ , $d\mu_0$ is the Lebesgue surface measure on Σ and $\gamma = \text{trace } P$;

- (7) there are positive constants $c_1, c_2, c_3, c_4, \alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\begin{aligned} c_1|x|^{\alpha_1} < r(x) < c_2|x|^{\alpha_2} & \quad \text{if } r(x) \geq 1, \\ c_3|x|^{\beta_1} < r(x) < c_4|x|^{\beta_2} & \quad \text{if } 0 < r(x) \leq 1. \end{aligned}$$

See [2, 18, 22] for more details.

Let A_t^* denote the adjoint of a matrix A . Then $A_t^* = \exp((\log t)P^*)$. We define a norm function s from $\{A_t^*\}$ in the same way as we define r from $\{A_t\}$. The function s has properties similar to those of r .

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For $t < 0$, define A_t by $A_t = (\operatorname{sgn} t)A_{|t|} = -A_{|t|}$. Let S^{n-1} denote the unit sphere of \mathbb{R}^n . For $(x, \theta) \in \mathbb{R}^n \times S^{n-1}$, we define

$$(1.1) \quad Mf(x, \theta) = \sup_{h>0} h^{-1} \left| \int_0^h f(x - A_t \theta) dt \right|,$$

$$(1.2) \quad Hf(x, \theta) = \text{p.v.} \int_{-\infty}^{\infty} f(x - A_t \theta) dt/t,$$

$$(1.3) \quad H_* f(x, \theta) = \sup_{0 < \epsilon < R} \left| \int_{\epsilon < |t| < R} f(x - A_t \theta) dt/t \right|.$$

Let w be a weight function. We recall that

$$\|F\|_{L_w^p(L^q)} = \left(\int_{\mathbb{R}^n} \left(\int_{S^{n-1}} |F(x, \theta)|^q d\sigma(\theta) \right)^{p/q} w(x) dx \right)^{1/p}$$

for functions $F \in L_w^p(L^q(S^{n-1}))$, with usual modifications when $q = \infty$ or $p = \infty$, where $d\sigma$ denotes the Lebesgue surface measure on S^{n-1} . Also, we write $\|f\|_{L_w^p} = \|fw^{1/p}\|_{L^p} = \|fw^{1/p}\|_p$ for $f \in L_w^p(\mathbb{R}^n)$.

For $1 \leq p, q \leq \infty$, let $\Delta_{(p,q)} \subset [0, 1] \times [0, 1]$ be the interior of the convex hull of the points $(0, 0), (1, 1), (0, 1), (1/p, 1/q)$. Put $q_n(p) = p(n-1)/(n-p)$, $p_n = \max(2, (n+1)/2)$, where we define $q_n(p) = \infty$ when $p \geq n$. Then the following result was proved in [8] (see [14] for earlier work).

Theorem A. *Suppose $P = E$ (the identity matrix). Let $(1/p, 1/q) \in \Delta_{(p_n, q_n(p_n))}$. Then, M, H and H_* are bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^q)$.*

This result for M, H was extended to the case of nonisotropic dilations by [1] as follows.

Theorem B. *Let M and H be as in (1.1) and (1.2). Then:*

- (1) *If $(1/p, 1/q) \in \Delta_{(p_n, q_n(p_n))}$, M is bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^q)$;*
- (2) *the operator H is bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^q)$ whenever $(1/p, 1/q) \in \Delta_{(2, q_n(2))}$.*

See [5, 6, 20] for previous results. To prove Theorem B, Bez [1] showed certain $L^2(S^{n-1})$ estimates for trigonometric integrals (see Lemma 2 below) by using the decay estimates for the Fourier transform of $d\sigma$. Further, to prove part (1) for $p > 2$, [1] found that a result of [15] (see also [16]) can be applied; at the corresponding point in the isotropic case of [8], a result for X-ray transforms of [7] was used.

In this note we shall prove an analogue of Theorem B (2) for the maximal singular integral operator H_* . We assume that $\Sigma = S^{n-1}$, $d\mu = \omega d\sigma$ and ω is even, where $\Sigma, d\mu, \omega$ are as in statements (5), (6) above (see [2, 11, 18] for relevant results).

Theorem 1. *Let H_* be as in (1.3). Suppose that $(1/p, 1/q) \in \Delta_{(2, q_n(2))}$. Then H_* is bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^q)$.*

We recall the following result for H_* shown by Chen [5].

Theorem C. *Suppose $n = 2$ and $P = \operatorname{diag}(\alpha_1, \alpha_2)$ with $1 < \alpha_2/\alpha_1 < 4/3$. Then H_* is bounded from $L^p(\mathbb{R}^2)$ to $L^p(L^q(S^1))$ whenever $(1/p, 1/q) \in \Delta_{(2,4)}$.*

Theorem 1 improves on Theorem C when $n = 2$. It is known that if H is bounded from L^p to $L^p(L^q)$, $p \in (1, \infty)$, then $q \leq q_n(p)$ (see Theorem 6 (1) of [1]). This implies the same result for H_* . Thus, in particular, we can see that Theorem 1 is a sharp result when $n = 2$ (we note that $\Delta_{(2, q_2(2))} = \Delta_{(2, \infty)}$).

If B is a subset of \mathbb{R}^n such that $B = \{x \in \mathbb{R}^n : r(x-a) < t\}$ for some $a \in \mathbb{R}^n$ and $t > 0$, then we call B an r -ball. Let w be a weight function on \mathbb{R}^n . For $1 \leq p < \infty$, we recall the Muckenhoupt class \mathcal{A}_p . We say $w \in \mathcal{A}_p$, $1 < p < \infty$, if

$$\sup_B \left(|B|^{-1} \int_B w(x) dx \right) \left(|B|^{-1} \int_B w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all r -balls B . The class \mathcal{A}_1 is defined to be the set of weight functions w satisfying $M_{HL}w \leq Cw$ a.e., where M_{HL} is the Hardy-Littlewood maximal operator defined by

$$M_{HL}g(x) = \sup_{t>0} t^{-\gamma} \int_{r(x-y)<t} |g(y)| dy.$$

We note that $\mathcal{A}_p \subset \mathcal{A}_u$ if $p \leq u$. We shall prove the following weighted estimates.

Theorem 2. *Let $2 \leq q < q_n(2)$. Then, M, H, H_* are bounded from $L_w^2(\mathbb{R}^n)$ to $L_w^2(L^q)$ for $w \in \mathcal{A}_1^\tau$, $\tau = 2(n-1)/q - n + 2$, where \mathcal{A}_1^τ is a subclass of \mathcal{A}_1 defined by $\mathcal{A}_1^\tau = \{v^\tau : v \in \mathcal{A}_1\}$.*

The weight class \mathcal{A}_1^τ is required by our methods using interpolation with a change of measure. By [22] we know that the operator H_* is bounded from $L^p(\mathbb{R}^n)$ to $L^p(L^p)$, $1 < p < \infty$, so Theorem 1 follows by interpolation from the part of Theorem 2 concerning H_* with $w = 1$. By Theorem 2 with $q = 2$ and the extrapolation theorem of Rubio de Francia, the operators M, H, H_* are bounded from $L_w^p(\mathbb{R}^n)$ to $L_w^p(L^2)$ for $w \in \mathcal{A}_{p/2}$, $p \geq 2$. Using this and known estimates, we can obtain some other results not explicitly stated in Theorem 2. For example, we can prove some weighted estimates for M by interpolation with the $L^{p_n}-L^{p_n}(L^q)$ ($q < q_n(p_n)$) estimates in Theorem B (1). At present we do not know whether we can get better results by applying in the weighted norm cases the arguments due to [1] of Theorem B (1) for $(1/p, 1/q)$ near $(1/p_n, 1/q_n(p_n))$.

In Section 2, we shall prove $L_w^2-L_w^2(L^q)$ estimates for M of Theorem 2. As in [8], we apply the Plancherel theorem for functions with values in the Sobolev space $L_\alpha^2(S^{n-1})$ and the Littlewood-Paley theory. We prove certain orthogonality estimates (see Lemma 1 below), from which we deduce results for $L^q(S^{n-1})$ valued functions via the Sobolev embedding theorem (see (2.8)). Also, we prove certain weighted estimates (see (2.7)). Interpolation with a change of measure between these estimates will readily imply the result of Theorem 2 for M .

In Sections 3 and 4, $L_w^2-L_w^2(L^q)$ estimates of Theorem 2 will be proved for H and H_* , respectively. We again need the Littlewood-Paley theory and orthogonality estimates for vector valued functions along with interpolation arguments. To handle H_* , we apply a method of [10] designed to treat certain maximal operators, where the results for M, H obtained in Sections 2 and 3 will be used. We note that the maximal operator M cannot be used to control H, H_* as in Lemma 4.1 of [8]; a reason for this is that weighted inequalities which will be required in the arguments are not yet available in the case of nonisotropic dilations.

Finally, in Section 5, we consider singular integrals and maximal singular integrals with variable kernels and we shall prove certain L^p and L_w^p boundedness of

them (Theorems 3 and 4) by applying Theorems 1 and 2 via the Calderón-Zygmund method of rotations (see [3]). Theorem 3 is an extension to the case of nonisotropic dilations of a result due to [9] for isotropic dilation.

2. $L_w^2(L^q)$ ESTIMATES FOR MAXIMAL FUNCTIONS

In this section we prove

$$(2.1) \quad \|Mf\|_{L_w^2(L^q)} \leq C\|f\|_{L_w^2}, \quad 2 \leq q < q_n(2), \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class, and q and w are related as in Theorem 2.

Define $f_t(x) = t^{-\gamma} f(A_t^{-1}x)$, $t > 0$. We denote by \hat{f} the Fourier transform of f :

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x, \xi \rangle} dx.$$

Let $\{D_k\}_{-\infty}^{\infty}$ be a sequence of nonnegative functions in $C^\infty((0, \infty))$ such that

$$\begin{aligned} \text{supp}(D_k) &\subset [2^{-k-1}, 2^{-k+1}], \quad \sum_k D_k(t)^2 = 1, \\ |(d/dt)^m D_k(t)| &\leq c_m/t^m \quad (m = 1, 2, \dots). \end{aligned}$$

To apply the Littlewood-Paley theory, we define S_k by

$$(S_k(f))^\wedge(\xi) = D_k(s(\xi))\hat{f}(\xi), \quad k \in \mathbb{Z},$$

where \mathbb{Z} denotes the set of integers. For $k \in \mathbb{Z}$, let

$$N_k f(x, \theta) = \int_{-\infty}^{\infty} f(x - A_t\theta)\varphi_k(t) dt - \int_{\mathbb{R}^n} f(x - y)\Phi_{2^k}(y) dy,$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geq 0$, $\text{supp}(\varphi) \subset (1/2, 2)$, $\int \varphi dt = 1$, $\varphi_k(t) = 2^{-k}\varphi(2^{-k}t)$, and $\Phi \in C_0^\infty(\mathbb{R}^n)$, $\int \Phi dx = 1$. Put $\tilde{S}_k = S_k^2$. Then, $\sum_k \tilde{S}_k f = f$. To prove (2.1), we may assume $f \geq 0$. We note that

$$\begin{aligned} (2.2) \quad Mf(x, \theta) &\leq C \sup_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x - A_t\theta)\varphi_k(t) dt \\ &\leq C \sup_k |N_k f(x, \theta)| + CM_{HL}f(x) \\ &\leq C \sum_{j=-\infty}^{\infty} \left(\sum_k |N_k \tilde{S}_{j+k} f(x, \theta)|^q \right)^{1/q} + CM_{HL}f(x). \end{aligned}$$

Let $2 \leq q < q_n(2)$. Since $q \geq 2$, (2.2) and the Hardy-Littlewood maximal theorem imply

$$\begin{aligned} (2.3) \quad \|Mf\|_{L_w^2(L^q)} &\leq C \sum_j \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^q(S^{n-1})}^q \right)^{1/q} \right\|_{L_w^2} + C\|f\|_{L_w^2} \\ &\leq C \sum_j \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^q(S^{n-1})}^2 \right)^{1/2} \right\|_{L_w^2} + C\|f\|_{L_w^2}, \end{aligned}$$

for $w \in \mathcal{A}_2$. We prove the following result.

Lemma 1. *If $0 \leq \alpha < 1/2$, then*

$$\left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^2_\alpha(S^{n-1})}^2 \right)^{1/2} \right\|_2 \leq C 2^{-\epsilon|j|} \|f\|_2 \quad \text{for some } \epsilon > 0.$$

Proof. Let $0 < a < 1/2 - \alpha$. It suffices to prove

$$(2.4) \quad \int_{\mathbb{R}^n} \|N_k \tilde{S}_{j+k} f(x, \cdot)\|_{L^2_\alpha}^2 dx \leq C \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 |D_{k+j}(s(\xi))|^4 \min(|A_{2^k}^* \xi|, |A_{2^k}^* \xi|^{-1})^{2a} d\xi.$$

Note that $(N_k f(\cdot, \theta))^\wedge(\xi) = \hat{f}(\xi) \Psi(A_{2^k}^* \xi, \theta)$, where

$$\Psi(\xi, \theta) = \Psi_0(\xi, \theta) - \hat{\Phi}(\xi), \quad \Psi_0(\xi, \theta) = \int_{-\infty}^\infty \exp(-2\pi i \langle A_t \theta, \xi \rangle) \varphi(t) dt.$$

Therefore, to obtain (2.4), it suffices to show the pointwise inequality

$$(2.5) \quad \|\Psi(\xi, \cdot)\|_{L^2_\alpha}^2 \leq C \min(|\xi|, |\xi|^{-1})^{2a}.$$

If $|\xi| \leq 1$, this is easily obtained, since $\Psi(\xi, \theta)$ is C^∞ and vanishes when $\xi = 0$. The estimate (2.5) for $|\xi| > 1$ follows from the following result of [1] and integration by parts, as in [8].

Lemma 2. *Let $0 < c_1 < c_2$ and $\xi \in \mathbb{R}^n$, $|\xi| > 1$. Then*

$$\int_{S^{n-1}} \left| \int_{c_1}^{c_2} \exp(i \langle A_t \theta, \xi \rangle) dt \right|^2 d\sigma(\theta) \leq C_\delta |\xi|^{-1+\delta}$$

for all $\delta > 0$.

This follows from Lemma 8 of [1] and an argument including a change of variables. □

We can easily see that

$$(2.6) \quad \|N_k f\|_{L^2_w(L^2)} \leq C \|f\|_{L^2_{M_{HL}(w)}}.$$

Therefore, if $w \in \mathcal{A}_1$, by the Littlewood-Paley inequality we have

$$(2.7) \quad \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^2(S^{n-1})}^2 \right)^{1/2} \right\|_{L^2_w}^2 \leq C \sum_k \|\tilde{S}_{j+k} f\|_{L^2_w}^2 \leq C \|f\|_{L^2_w}^2.$$

If $2 \leq q < q_n(2)$, then by the Sobolev embedding theorem we have $L^2_\alpha(S^{n-1}) \subset L^q(S^{n-1})$ for some $\alpha = \alpha(q) \in [0, 1/2)$. Thus, Lemma 1 implies

$$(2.8) \quad \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^q(S^{n-1})}^2 \right)^{1/2} \right\|_2 \leq C 2^{-\epsilon|j|} \|f\|_2$$

for some $\epsilon > 0$. Interpolating between (2.7) and (2.8) with change of measure, we get

$$(2.9) \quad \left\| \left(\sum_k \|N_k \tilde{S}_{j+k} f\|_{L^q(S^{n-1})}^2 \right)^{1/2} \right\|_{L^2(w^\tau)} \leq C 2^{-\epsilon|j|} \|f\|_{L^2(w^\tau)}$$

for some $\epsilon > 0$, where q and τ are related as in Theorem 2. The estimate (2.1) follows from (2.9) and (2.3).

Remark 1. Let $0 < c_1 < c_2$ and $\eta, \zeta \in \mathbb{R}^n \setminus \{0\}$. Then, we have

$$\left| \int_{c_1}^{c_2} \exp(i\langle A_t \eta, \zeta \rangle) dt \right| \leq C |\langle P\eta, \zeta \rangle|^{-1/d}$$

for some positive constant C independent of η and ζ , where d is the degree of the minimal polynomial of P (see [21, Corollary in Section 4]). We note that this result implies Lemma 2 when $d = 1, 2$.

3. $L_w^2(L^q)$ ESTIMATES FOR SINGULAR INTEGRALS

Let H be as in (1.2). Let $2 \leq q < q_n(2)$. In this section we prove

$$(3.1) \quad \|Hf\|_{L_w^2(L^q)} \leq C \|f\|_{L_w^2}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

where q and w are related as in Theorem 2.

Decompose

$$Hf(x, \theta) = \sum_{k=-\infty}^{\infty} H_k f(x, \theta), \quad H_k f(x, \theta) = \int_{-\infty}^{\infty} f(x - A_t \theta) \psi_k(t) dt,$$

where $\psi_k(t) = 2^{-k} \psi(2^{-k}t)$, $\psi \in C_0^\infty(\mathbb{R})$, $\text{supp}(\psi) \subset \{1/2 \leq |t| \leq 2\}$, $\int \psi(t) dt = 0$. We write

$$Hf = \sum_k H_k f = \sum_j U_j f, \quad U_j f = \sum_k H_k S_{j+k}^2 f,$$

where S_j is as in Section 2. Let $0 \leq \alpha < 1/2$. We prove

$$(3.2) \quad \|U_j f\|_{L^2(L_\alpha^2)} \leq C 2^{-\epsilon|j|} \|f\|_2,$$

for some $\epsilon > 0$. Then, arguing as in Section 2, from (3.2) we can get

$$(3.3) \quad \|U_j f\|_{L^2(L^q)} \leq C 2^{-\epsilon|j|} \|f\|_2, \quad 2 \leq q < q_n(2).$$

Let

$$\tilde{\Psi}(\xi, \theta) = \int_{-\infty}^{\infty} \exp(-2\pi i \langle A_t \theta, \xi \rangle) \psi(t) dt.$$

We have an estimate similar to (2.5) for $\tilde{\Psi}(\xi, \theta)$, and $(H_k f(\cdot, \theta))^\wedge(\xi) = \hat{f}(\xi) \tilde{\Psi}(A_{2^k}^* \xi, \theta)$, where the operator H_k is as in the previous paragraph. Therefore, by the Littlewood-Paley theory for vector valued functions,

$$\begin{aligned} \|U_j f\|_{L^2(L_\alpha^2)}^2 &\leq C \sum_k \|H_k S_{j+k}^2 f\|_{L^2(L_\alpha^2)}^2 \\ &\leq C \sum_k \int_{\mathbb{R}^n} |D_{j+k}(s(\xi)) \hat{f}(\xi)|^2 \min(|A_{2^k}^* \xi|, |A_{2^k}^* \xi|^{-1})^{2a} d\xi, \end{aligned}$$

where $0 < a < 1/2 - \alpha$. This implies

$$\|U_j f\|_{L^2(L_\alpha^2)}^2 \leq C 2^{-\epsilon|j|} \sum_k \int_{\mathbb{R}^n} |D_{j+k}(s(\xi)) \hat{f}(\xi)|^2 d\xi \leq C 2^{-\epsilon|j|} \|f\|_2^2$$

for some $\epsilon > 0$, which proves (3.2).

If $w \in \mathcal{A}_1$, from an estimate similar to (2.6) and the Littlewood-Paley inequality, we have

$$(3.4) \quad \begin{aligned} \|U_j f\|_{L_w^2(L^2)} &\leq C \left(\sum_k \|H_k S_{j+k} f\|_{L_w^2(L^2)}^2 \right)^{1/2} \\ &\leq C \left(\sum_k \|S_{j+k} f\|_{L_w^2}^2 \right)^{1/2} \leq C \|f\|_{L_w^2}. \end{aligned}$$

Interpolation between (3.3) and (3.4) implies

$$(3.5) \quad \|U_j f\|_{L_{w^\tau}^2(L^q)} \leq C 2^{-\epsilon|j|} \|f\|_{L_{w^\tau}^2}$$

for some $\epsilon > 0$, where q and τ are related as in Theorem 2. Using (3.5) and the triangle inequality, we can obtain (3.1) as follows:

$$\|Hf\|_{L_w^2(L^q)} \leq \sum_j \|U_j f\|_{L_w^2(L^q)} \leq C \sum_j 2^{-\epsilon|j|} \|f\|_{L_w^2} \leq C \|f\|_{L_w^2},$$

where q and w are as in (3.1).

4. $L_w^2(L^q)$ ESTIMATES FOR MAXIMAL SINGULAR INTEGRALS

Let H_* be as in (1.3). Let q, w be as in Theorem 2. In this section we prove

$$(4.1) \quad \|H_* f\|_{L_w^2(L^q)} \leq C \|f\|_{L_w^2}, \quad f \in \mathcal{S}(\mathbb{R}^n).$$

To prove (4.1) we use the following.

Lemma 3. *Let*

$$H_{**} f(x, \theta) = \sup_{N \in \mathbb{Z}} \left| \sum_{k=N}^{\infty} H_k f(x, \theta) \right|.$$

Then

$$\|H_{**} f\|_{L_w^2(L^q)} \leq C \|f\|_{L_w^2},$$

where q, w are as in (4.1).

We need the following result, for $p \leq q$, to show Lemma 3.

Lemma 4. *Let $1 < p < \infty$, $1 < q \leq \infty$, $w \in \mathcal{A}_p$. For functions $F(x, \theta)$ on $\mathbb{R}^n \times S^{n-1}$, define $(M_{HL} F)(x, \theta) = (M_{HL} F(\cdot, \theta))(x)$. Then*

$$\|M_{HL} F\|_{L_w^p(L^q)} \leq C \|F\|_{L_w^p(L^q)}.$$

Proof. If $p = q$, this is obvious since M_{HL} is bounded on $L_w^p(\mathbb{R}^n)$. Also, when $q = \infty$, the inequality holds since M_{HL} is a positive operator. The conclusion follows from interpolation between these results when $p \leq q$. The case $p > q$ can be treated by the duality arguments of [13]. We omit the details (see also [19]). \square

Proof of Lemma 3. Let $\hat{Q} \in C_0^\infty(\mathbb{R}^n)$, $\text{supp}(\hat{Q}) \subset \{s(\xi) < 2\}$, $\hat{Q}(\xi) = 1$ if $s(\xi) < 1$. Decompose

$$(4.2) \quad \sum_{k=N}^{\infty} H_k f = Q_{2^N} * Hf - Q_{2^N} * \sum_{k=-\infty}^{N-1} H_k f + (\delta - Q_{2^N}) * \sum_{k=N}^{\infty} H_k f,$$

where δ denotes the delta function and the convolution is taken with respect to the x variable.

The first term on the right hand side of (4.2) can be handled by (3.1) and Lemma 4 as follows:

$$\left\| \sup_N |Q_{2^N} * Hf| \right\|_{L_w^2(L^q)} \leq C \|M_{HL}Hf\|_{L_w^2(L^q)} \leq C \|Hf\|_{L_w^2(L^q)} \leq C \|f\|_{L_w^2}.$$

Also, by inspection we see that

$$\sup_N \left| Q_{2^N} * \sum_{k=-\infty}^{N-1} H_k f(x, \theta) \right| \leq C M_{HL}f(x)$$

with the constant C independent of θ (see the proof of Theorem E in [10]). Therefore, the second term on the right hand side of (4.2) can be handled by the weighted norm inequality for the Hardy-Littlewood maximal operator.

It remains to estimate

$$I(f) = \sup_N \left| (\delta - Q_{2^N}) * \sum_{k=N}^{\infty} H_k f \right|.$$

We note that

$$(4.3) \quad I(f) \leq \sum_{j=0}^{\infty} I_j(f), \quad I_j(f) = \sup_{N \in \mathbb{Z}} |(\delta - Q_{2^N}) * H_{N+j}f|.$$

Let $0 \leq \alpha < 1/2$ and $0 < a < 1/2 - \alpha$. Then, as in the proof of (3.2), we have

$$\|(\delta - Q_{2^N}) * H_{N+j}f\|_{L^2(L_\alpha^2)}^2 \leq C \int_{\mathbb{R}^n} |(1 - \hat{Q}(A_{2^N}^* \xi)) \hat{f}(\xi)|^2 |A_{2^{N+j}}^* \xi|^{-2a} d\xi.$$

Therefore,

$$\sum_N \|(\delta - Q_{2^N}) * H_{N+j}f\|_{L^2(L_\alpha^2)}^2 \leq C 2^{-j\epsilon} \|f\|_2^2,$$

and hence, if $2 \leq q < q_n(2)$, the Sobolev embedding theorem implies

$$(4.4) \quad \sum_N \|(\delta - Q_{2^N}) * H_{N+j}f\|_{L^2(L^q)}^2 \leq C 2^{-j\epsilon} \|f\|_2^2$$

for some $\epsilon > 0$.

We write $\delta - Q_{2^N} = \sum_{m \leq N} \Delta_m$, $\hat{\Delta}_m(\xi) = \Gamma(A_{2^m}^* \xi)$, where $\Gamma \in C_0^\infty$, $\text{supp}(\Gamma) \subset \{c_1 < s(\xi) < c_2\}$ for some $c_1, c_2 > 0$. Then, by Plancherel's theorem we have

$$(4.5) \quad \|\Delta_m * H_{N+j}f\|_{L^2(L^2)}^2 \leq C 2^{-\epsilon(N-m+j)} \|f\|_2^2.$$

On the other hand, if $w \in \mathcal{A}_1$, as in (3.4), we see that

$$(4.6) \quad \|\Delta_m * H_{N+j}f\|_{L_w^2(L^2)}^2 \leq C \|f\|_{L_w^2}^2.$$

For $w \in \mathcal{A}_1$, choose $b > 1$ such that $w^b \in \mathcal{A}_1$. Then, interpolating between (4.5) and (4.6) with w^b in place of w , we get

$$(4.7) \quad \|\Delta_m * H_{N+j}f\|_{L_w^2(L^2)}^2 \leq C 2^{-\epsilon(N-m+j)} \|f\|_{L_w^2}^2$$

for some $\epsilon > 0$. Choose $G_m \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{G}_m(\xi) = F(A_{2^m}^* \xi)$, $F \in C_0^\infty$, $\text{supp}(F) \subset \{d_1 < s(\xi) < d_2\}$ for some $d_1, d_2 > 0$, and $\Delta_m * G_m * f = \Delta_m * f$. Then

the Littlewood-Paley inequality and (4.7) imply

$$\begin{aligned}
 (4.8) \quad \sum_{N \in \mathbb{Z}} \|(\delta - Q_{2^N}) * H_{N+j} f\|_{L_w^2(L^2)}^2 &\leq C \sum_{N \in \mathbb{Z}} \sum_{m \leq N} \|\Delta_m * H_{N+j} f\|_{L_w^2(L^2)}^2 \\
 &\leq C \sum_{N \in \mathbb{Z}} \sum_{m \leq N} 2^{-\epsilon(N-m+j)} \|G_m * f\|_{L_w^2}^2 \\
 &\leq C \sum_{m \in \mathbb{Z}} 2^{-j\epsilon} \|G_m * f\|_{L_w^2}^2 \\
 &\leq C 2^{-j\epsilon} \|f\|_{L_w^2}^2.
 \end{aligned}$$

Interpolation between (4.4) and (4.8) implies

$$(4.9) \quad \sum_N \|(\delta - Q_{2^N}) * H_{N+j} f\|_{L_w^2(L^q)}^2 \leq C 2^{-j\epsilon} \|f\|_{L_w^2}^2$$

for some $\epsilon > 0$, where q, w are as in Theorem 2. Since

$$I_j(f) \leq \left(\sum_{N \in \mathbb{Z}} |(\delta - Q_{2^N}) * H_{N+j} f|^q \right)^{1/q}$$

and $q \geq 2$, by (4.3) and (4.9) we have

$$\begin{aligned}
 \|I(f)\|_{L_w^2(L^q)} &\leq \sum_{j=0}^{\infty} \left\| \left(\sum_{N \in \mathbb{Z}} \|(\delta - Q_{2^N}) * H_{N+j} f\|_{L^q(S^{n-1})}^q \right)^{2/q} \right\|_{L_w^1}^{1/2} \\
 &\leq \sum_{j=0}^{\infty} \left(\sum_{N \in \mathbb{Z}} \|(\delta - Q_{2^N}) * H_{N+j} f\|_{L_w^2(L^q)}^2 \right)^{1/2} \\
 &\leq C \sum_{j=0}^{\infty} 2^{-j\epsilon/2} \|f\|_{L_w^2} \leq C \|f\|_{L_w^2},
 \end{aligned}$$

where q, w are as in Theorem 2. This completes the proof of Lemma 3. \square

Proof of (4.1). We can easily prove the pointwise inequality

$$H_* f(x, \theta) \leq C H_{**} f(x, \theta) + C M f(x, \theta) + C M f(x, -\theta).$$

Therefore, (4.1) follows from (2.1) and Lemma 3. \square

5. APPLICATIONS

Let $K(x, y)$ be a kernel on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$K(x, A_t y) = t^{-\gamma} K(x, y) \quad \text{for all } t > 0 \text{ and } (x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}).$$

We assume that $K(x, y)$ is locally integrable with respect to y in $\mathbb{R}^n \setminus \{0\}$ and

$$\int_{a \leq r(y) \leq b} K(x, y) dy = 0 \quad \text{for all } a, b, 0 < a < b,$$

for every $x \in \mathbb{R}^n$. We consider the singular integral

$$Tf(x) = \text{p.v.} \int K(x, y) f(x - y) dy = \lim_{\epsilon \rightarrow 0} \int_{r(y) \geq \epsilon} K(x, y) f(x - y) dy$$

and the maximal singular integral

$$T_*f(x) = \sup_{\epsilon, R > 0} \left| \int_{\epsilon \leq r(y) \leq R} K(x, y) f(x - y) dy \right|.$$

We can apply Theorems 1 and 2 to study mapping properties of T and T_* .

Theorem 3. *Let $(1/p, 1/q') \in \Delta_{(2, q_n(2))}$, $q' = q/(q-1)$. Suppose that $K(x, y)$ is odd in y , that is, $K(x, -y) = -K(x, y)$ for all $(x, y) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, and suppose that*

$$\sup_{x \in \mathbb{R}^n} \left(\int_{S^{n-1}} |K(x, \theta)|^q d\sigma(\theta) \right)^{1/q} = \|K\|_{L^\infty(L^q)} < \infty.$$

Then T_ is bounded on $L^p(\mathbb{R}^n)$.*

This is an analogue for T_* of Theorem 12 of [1] concerning T . See [4, 8, 9] for the case of isotropic dilation.

Theorem 4. *Let $2(n-1)/n < q \leq 2$, $w \in \mathcal{A}_1$. Suppose that $\|K\|_{L^\infty(L^q)} < \infty$. Then T and T_* are bounded on L_w^2 , $w \in \mathcal{A}_1^\tau$, $\tau = n - 2(n-1)/q$.*

Since $w^b \in \mathcal{A}_1$ for some $b > 1$ when $w \in \mathcal{A}_1$, from Theorem 4 we readily obtain the following result.

Corollary 1. *Suppose that $\|K\|_{L^\infty(L^q)} < \infty$ for all $q < 2$. Then T and T_* are bounded on L_w^2 for all $w \in \mathcal{A}_1$.*

Using this result and the extrapolation theorem of Rubio de Francia, we can obtain the L_w^p boundedness of T and T_* for $w \in \mathcal{A}_{p/2}$, $p \geq 2$. As in Theorem 3, the kernels K are assumed to be odd in y in Theorem 4 and Corollary 1.

Proof of Theorem 3. The method of rotations of Calderón-Zygmund and Hölder's inequality imply

$$\begin{aligned} T_*f(x) &\leq C \int_{S^{n-1}} |K(x, \theta)| H_*f(x, \theta) d\sigma(\theta) \\ &\leq C \|K\|_{L^\infty(L^q(S^{n-1}))} \|H_*f(x, \cdot)\|_{L^{q'}}. \end{aligned}$$

Thus, the conclusion follows from Theorem 1. □

Similarly, Theorem 4 follows from Theorem 2.

Remark 2. Introducing nonisotropic Riesz transforms as in [12], we expect that Theorems 3 and 4 extend to the case where kernels $K(x, y)$ are even in y (see [3, 8, 17]). We do not pursue this here.

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