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Minimum Energy Broadcast on Rectangular Grid Wireless Networks

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Abstract

The minimum energy broadcast problem is to assign a transmission range to each node in an ad hoc wireless network to construct a spanning tree rooted at a given source node such that any non-root node resides within the transmission range of its parent. The objective is to minimize the total energy consumption, i.e., the sum of the δth powers of a transmission range ($\delta \geq 1$). In this paper, we consider the case that $\delta = 2$, and that nodes are located on a 2-dimensional rectangular grid. We prove that the minimum energy consumption for an n-node $k \times l$-grid with $n = kl$ and $k \leq l$ is at most $\frac{n}{\pi} + O(\frac{n}{k^{0.68}})$ and at least $\frac{n}{\pi} + \Omega(\frac{n}{k}) - O(k)$. Our bounds close the previously known gap of upper and lower bounds for square grids. Moreover, our lower bound is $\frac{n}{3} - O(1)$ for $3 \leq k \leq 18$, which matches a naive upper bound within a constant term for $k \equiv 0 \pmod 3$.

Keywords: energy minimization, broadcast, grid, ad hoc wireless network

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1. Introduction

In ad hoc wireless networks, communication is established via a sequence of wireless connections between neighboring nodes. It is well known that a transmission power at least $\gamma \cdot \mathrm{dist}(u, v)^\delta$ is necessary for a node u to directly transmit a data message to a node v, where $\mathrm{dist}(u, v)$ is the distance between u and v, and $\gamma \geq 1$ and $\delta \geq 1$ are the transmission-quality parameter and the distance-power gradient, respectively, which depend on environment [1]. In what follows, we fix $\gamma = 1$ and assume that nodes are located on the Euclidean plane.

It is important to save energy consumption in ad hoc wireless networks because wireless nodes are often driven by batteries. The *minimum energy broadcast problem*, i.e., the problem of transferring a data message to all nodes in an ad hoc network with the minimum total energy consumption has extensively been studied. Formally, this problem is to assign a transmission range $r_u \geq 0$ to each node u so that there exists a spanning tree rooted at a given source node and satisfying $\mathrm{dist}(u, v) \leq r_u$ for any node u and its child v, and that the cost $\sum_u r_u^\delta$ is minimized.

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It is known that the minimum energy broadcast problem is NP-hard for any $\delta > 1$ [2]. Approximation ratios for this problem have been proved in [3, 4, 5, 6]. The best known algorithm achieving the approximation ratio of 4.2 for any $\delta \geq 2$ on the Euclidean plane was presented in [7]. Calamoneri, Clementi, Ianni, Lauria, Monti, and Silvestri [8] considered the case that $\delta = 2$, and that n nodes are located on a square grid with side length $\sqrt{n}-1$. They proved that the minimum cost is between $\frac{n}{\pi} - O(\sqrt{n})$ and $1.01013\frac{n}{\pi} + O(\sqrt{n})$. They also conjectured that a broadcast on the square grid based on a circle packing called the Apollonian gasket would achieve a cost matching the lower bound asymptotically.

In this paper, we demonstrate that a simple application of early results on the Apollonian gasket answers the conjecture. Specifically, we prove that a broadcast on an n-node square grid based on Apollonian gaskets achieves a cost of $\frac{n}{\pi} + O(n^{\frac{S}{2}+\epsilon})$, where S is the Hausdorff dimension of an Apollonian gasket. Because it is well known that $S < 1.314534$ [9], our upper bound matches the lower bound of [8] within an $o(n)$ term. We also generalize these results to rectangular grids. The upper bound on square grids is extended to $\frac{n}{\pi} + O(k^{S-2+\epsilon}n)$ for any $k \times l$-grid with $n = kl$ and $k \leq l$. Moreover, we present a lower bound of $\frac{n}{\pi} + \Omega(\frac{n}{k}) - O(k)$. Thus, we can obtain upper and lower bounds matching within an $o(n)$ term as long as $k = \omega(1)$. Although we do not know a tight factor of n for all $k = O(1)$, our lower bound is $\frac{n}{3} - O(1)$ for $3 \leq k \leq 18$, which matches a naive upper bound of $\frac{n}{3} + O(k)$ for $k \equiv 0 \pmod 3$.

Our upper bounds can be obtained by polynomial time algorithms, whose main ideas are from [8]. Moreover, we prove our lower bounds using a refined technique of the proof of [8], which is introduced in order to obtain better bounds for smaller k and is the technically interesting contribution for rectangular grids.

The paper is organized as follows: In Section 2, we describe the definition of Apollonian gaskets and some facts that we use in the following sections. In Section 3, we prove our upper bound on square grids. Finally, we generalize upper and lower bounds to rectangular grids in Section 4.

2. Apollonian Gasket

Let $T(a, b, c)$ be the range bounded by the curvilinear triangle of three mutually tangent disks of curvatures (i.e., reciprocals of a radius) a, b, and c, where $a, b, c \geq 0$, and at most one of a, b, and c equals 0. The *Apollonian gasket* of $T(a, b, c)$ is a set of infinite disks $\{D_1, D_2, \ldots\}$ such that D_i has the maximal radius of all the disks contained in $T(a, b, c) \setminus \bigcup_{j=1}^{i-1} D_j$. D_1 is said to be of level 1. D_i with $i \geq 2$ is said to be of level j if it is tangent to a disk of level $j-1$ but not to a disk of a higher level than $j-1$. For any disk D of level $j \geq 2$, we call the unique disk of level $j-1$ tangent to D the parent of D. The *exponent* of $\{D_i\}$ is defined as $S := \inf\{t \mid \sum_{i=1}^{\infty} r_i^t < \infty\} = \sup\{t \mid \sum_{i=1}^{\infty} r_i^t = \infty\}$, where r_i is the radius of D_i. It is well known that S does not depend on a, b, or c, and that S is equal to the Hausdorff dimension of an Apollonian gasket [10]. Currently best provable bounds on S were presented by Boyd:

Theorem A ([9]). $1.300197 < S < 1.314534$.

We denote $\sigma(a, b, c, t) := \sum_{i=1}^{\infty} r_i^t$, which is finite for any $t > S$.

3. Broadcast on Square Grids

In this section, we assume that $n = m^2$ nodes are located on points with coordinates (x, y) of integers $0 \leq x, y < m$.

Our algorithm to construct a broadcast on an $m \times m$-grid is based on an idea mentioned in [8] of naturally generalizing an Apollonian gasket to a circle packing of the square Q with side length $m - 1$ bounding the grid. Specifically, the algorithm, called AGBS, is defined as follows:

1. Locate a maximal disk D_1 of level 1 contained in Q.
2. For $j \geq 2$, we have $4 \cdot 3^{j-2}$ ranges in $Q \setminus$ (union of disks of lower level than j). Locate a maximal disk of level j in each range if such a disk has radius at least 1. Repeat this step until we have no range to locate a maximal disk of radius at least 1.
3. Let $\mathcal{D} := \{D_i\}_{i \geq 1}$ be the set of located disks. For each $i \geq 1$, let r_i be the radius of D_i. For each $i \geq 2$, let t_i be the tangency point of D_i and the parent of D_i.
4. For each $i \geq 1$, move and enlarge D_i so that it is centered at a nearest node c_i to the original center and has radius $r'_i := r_i + 1 + \frac{3\sqrt{2}}{2}$.
5. Locate disks of radius 1 centered at grid points on the line segments from (x, y) to (x', y) and from (x', y) to (x', y'), where (x, y) and (x', y') are the coordinates of a source node s and c_1, respectively.
6. For each $i \geq 2$, locate disks of radius 1 centered at grid points on the line segments from (x, y) to (x', y) and from (x', y) to (x', y'), where (x, y) and (x', y') are the coordinates of a nearest node t'_i to t_i and c_i, respectively.
7. Assign each node v the maximum radius of a disk centered at v if such a disk exists, 0 otherwise.

Figure 1 illustrates a broadcast constructed by AGBS.

Lemma 1. *AGBS constructs a broadcast.*

Proof. After Step 2, any range T bounded by a curvilinear triangle in $Q \setminus \bigcup_i D_i$ cannot contain a disk of radius 1. If $T = T(a, b, c)$ with $a \geq 0$ and $b, c > 0$, then any point p in T can be covered by a disk of radius less than 1 that is contained in T and tangent to a disk D of curvature b or c. Thus, p is covered by D by increasing the radius of D by 2 (Fig. 2(a)). If T is a curvilinear triangle with one curve of a disk D and two line segments of Q, then T is covered by D by increasing the radius of D by $1 + \sqrt{2}$ (Fig. 2(b)). Therefore, Q is covered by $\bigcup_i D_i$ by increasing all the radii by $1 + \sqrt{2}$. Moreover, t'_i is covered by the parent of D_i after the increase of its radius because $\mathrm{dist}(t_i, t'_i) \leq \frac{\sqrt{2}}{2}$. Because the distance of c_i and the original center of D_i is at most $\frac{\sqrt{2}}{2}$, after Step 4, Q is covered by D_is centered at grid points. Steps 5 and 6 guarantee that a data message from s is transferred to all the nodes covered by $\bigcup_i D_i$. Thus, AGBS constructs a broadcast. □

Let C be the set of disks located in Steps 5 and 6. Then, the cost of AGBS is $\mathrm{cost} = \sum_i r'^2_i + |C|$.

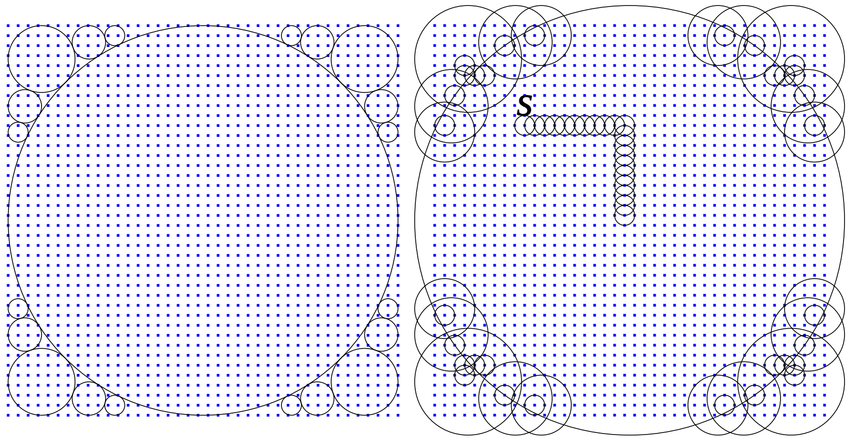
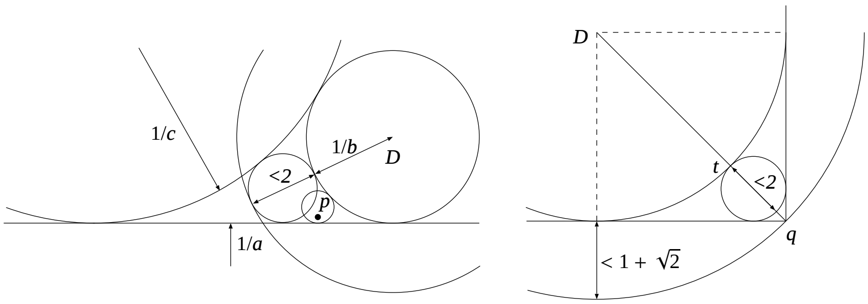


Figure 1: A broadcast on a 40×40-grid constructed by AGBS: disks located after Step 2 (left) and the completed broadcast (right).



(a) $T = T(a, b, c)$ with $a \geq 0$ and $b, c > 0$.

(b) T is bounded by D and two line segments.

Figure 2: Increasing radii so that every point of Q is covered by a disk.

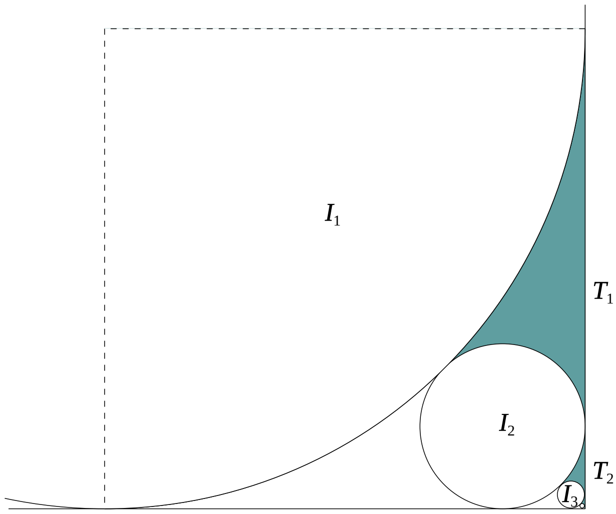


Figure 3: I_j and T_j.

Lemma 2. $|C| \leq \sum_i(\sqrt{2}\, r_i + 2) + m.$

Proof. Suppose $i \geq 2$. Because $\mathrm{dist}(t_i, t'_i) \leq \frac{\sqrt{2}}{2}$ and the distance of c_i and the original center of D_i is at most $\frac{\sqrt{2}}{2}$, it follows that $\mathrm{dist}(t'_i, c_i) \leq r_i + \sqrt{2}$. This means that the number of disks located in Step 6 from t'_i to c_i is at most $\sqrt{2}\, r_i + 2$. The number of disks located in Step 5 is obviously at most $2\lceil (m-1)/2 \rceil \leq m$. □

Lemma 3. *For any $\epsilon > 0$, it follows that $\sum_i r_i = O(m^{S+\epsilon})$.*

Proof. Consider disks after Step 2. Let $I_1 := D_1$ and I_2 be one of the four disks of level 2. Then, for $j \geq 3$, let I_j be the disk tangent to I_{j-1} and to two line segments of Q. For $j \geq 1$, let T_j be a range bounded by the curvilinear triangle of I_j, I_{j+1}, and Q (Fig. 3), and let $\mathcal{T}_j$ be the set of disks contained in T_j. It follows that $\sum_i r_i \leq 4\sum_j(\text{radius of } I_j) + 8\sum_j \sum_{D_i \in \mathcal{T}_j} r_i$. We can observe that for $j \geq 2$, T_j is similar to T_{j-1} with the shrink factor of $3 - 2\sqrt{2}$. Moreover, $\sum_j(\text{radius of } I_j) \leq \frac{\sqrt{2}}{2} m$ because the sum is at most half of a diagonal of Q. Thus, we have

$$\sum_i r_i \leq 2\sqrt{2}\, m + \frac{8}{1-(3-2\sqrt{2})} \sum_{D_i \in \mathcal{T}_1} r_i = 2\sqrt{2}\, m + 4(\sqrt{2}+1) \sum_{D_i \in \mathcal{T}_1} r_i. \tag{1}$$

Because $T_1 = T(0, \frac{2}{m-1}, \frac{2(3+\sqrt{2})}{m-1})$ is similar to $T(0, 2, 2(3+\sqrt{2}))$ with the scale factor of $m-1$, and because every disk in $\mathcal{T}_1$ has radius at least 1, it follows that

$$\begin{aligned} \sum_{D_i \in \mathcal{T}_1} r_i \leq \sum_{D_i \in \mathcal{T}_1} r_i^{S+\epsilon} &\leq \sigma\left(0, \frac{2}{m-1}, \frac{2(3+\sqrt{2})}{m-1}, S+\epsilon\right) \\ &< \sigma(0, 2, 2(3+\sqrt{2}), S+\epsilon) m^{S+\epsilon}. \end{aligned} \tag{2}$$

Because $\sigma(0, 2, 2(3+\sqrt{2}), S+\epsilon)$ is a finite value[1] independent of m, by (1) and (2), we have the lemma. $\square$

Theorem 1. *For any $\epsilon > 0$, AGBS has a cost of $\frac{n}{\pi} + O(n^{\frac{S}{2}+\epsilon})$.*

Proof. Because $\pi \sum_i r_i^2 \leq (m-1)^2 < m^2$, it follows from Lemmas 2 and 3 that $\mathsf{cost} = \sum_i {r'_i}^2 + |C| = \sum_i r_i^2 + O(\sum_i r_i) + m < \frac{m^2}{\pi} + O(m^{S+\epsilon}) = \frac{n}{\pi} + O(n^{\frac{S}{2}+\epsilon})$. $\square$

By Lemmas 2 and 3, the running time of AGBS is $n+O(|\mathcal{D}|+|C|) = n+O(\sum_i r_i+m) = n + O(n^{\frac{S}{2}+\epsilon}) = O(n)$.

4. Broadcast on Rectangular Grids

In this section, we assume that $n = kl$ nodes ($k \leq l$) are located on points with coordinates (x, y) of integers $0 \leq x < l$ and $0 \leq y < k$.

4.1. Upper Bounds

Our broadcast algorithm on rectangular grids is based on a simple application of AGBS to maximal square grids contained in a given rectangular grid. Specifically, the algorithm, called AGBR, is defined as follows:

1. Let $k_1 := k$ and $l_1 := l$. For each $i \geq 1$ with $k_i > 0$, recursively define $k_{i+1} := l_i \bmod k_i$, $l_{i+1} := k_i$, and $l'_i := l_i - k_{i+1}$.
2. Let G_1 be a $k_1 \times l_1$-grid and s be a source node. For each $i \geq 1$ with $k_i > 0$, repeat (a) and (b).
   (a) Divide G_i into a $k_i \times l'_i$-grid G'_i and a $k_{i+1} \times k_{i+1}$-grid G_{i+1}.
   (b) Divide G'_i into l'_i/k_i square grids, and apply AGBS on each square grid with setting a nearest node to s as the source node.
3. For each square grid Q appeared in Step 2(b) and not containing s, the nearest node to s is adjacent to a node v of another square grid Q' closer to s. Locate a disk of radius 1 centered at v, so that a broadcast message from s is transferred to Q via Q'.
4. Assign each node v the maximum radius of a disk located centered at v if such a disk exists, 0 otherwise.

Figure 4 illustrates a broadcast constructed by AGBR.

Theorem 2. *For any $\epsilon > 0$, AGBR has a cost of $\frac{n}{\pi} + O(k^{S-2+\epsilon}n)$.*

Proof. Let C be the set of disks located in Step 3. Then, by Theorem 1, the cost of AGBR is

$$\mathsf{cost} \leq \sum_{i\geq 1} \left(\frac{k_i^2}{\pi} + O(k_i^t) \right) \frac{l'_i}{k_i} + |C| = \frac{n}{\pi} + O\left(\sum_{i\geq 1} k_i^{t-1} l'_i \right) + |C|, \tag{3}$$

[1]In fact, we can guarantee $\sigma(0, 2, 2(3+\sqrt{2}), S+\epsilon)$ to be reasonably small if we are allowed to have a certain ϵ. For example, we can estimate $\sigma(0, 2, 2(3+\sqrt{2}), 1.4) \leq 0.97$ using the recurrence presented in [9].

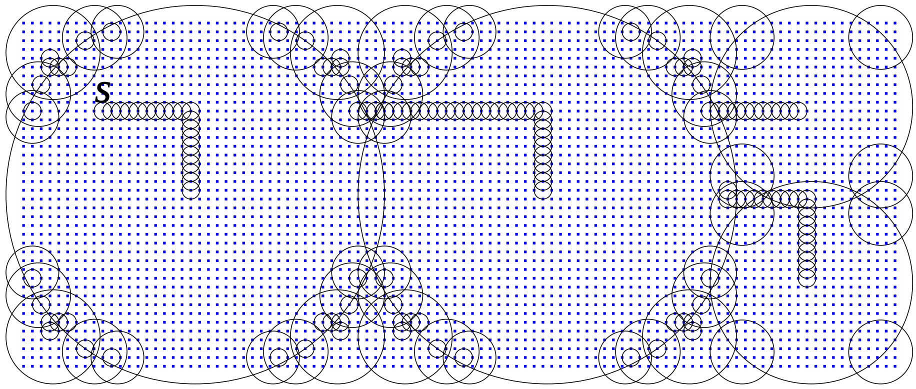


Figure 4: A broadcast on a 40×100-grid constructed by AGBR.

where $t := S + \epsilon$. We can observe that $k_i \leq l'_i$ for any $i \geq 1$, and that $l'_i = l_i - k_{i+1} = k_{i-1} - k_{i+1} \leq k_{i-1}$ for any $i \geq 2$. Therefore, it follows that for $i \geq 3$,

$$k_i^{t-1} l'_i \leq k_i^{t-1} k_{i-1} \leq k_i^{t-1}(k_{i-2} - k_i) \leq \frac{(t-1)^{t-1}}{t^t} k_{i-2}^t \leq \frac{(t-1)^{t-1}}{t^t} k_{i-2}^{t-1} l'_{i-2}. \tag{4}$$

Here, we have used the fact that $x^\alpha(\beta - x)$ with $\alpha, \beta > 0$ is maximized at $x = \frac{\alpha\beta}{1+\alpha}$. It follows from (4) that

$$\begin{aligned} \sum_{i \geq 1} k_i^{t-1} l'_i &= \sum_{i \geq 1} (k_{2i-1}^{t-1} l'_{2i-1} + k_{2i}^{t-1} l'_{2i}) \\ &= O(k_1^{t-1} l'_1 + k_2^{t-1} l'_2) = O(k^{t-1} l) = O(k^{t-2} n). \end{aligned} \tag{5}$$

Moreover,

$$|C| \leq \sum_{i \geq 1} \frac{l'_i}{k_i} \leq \sum_{i \geq 1} l'_i \leq l + k \leq 2l = \frac{2n}{k}. \tag{6}$$

By (3), (5), and (6), we have the theorem. □

Because the running time of AGBS is $O(n)$, the running time of AGBR is $n + \sum_{i \geq 1} O(k_i^2) + |C| = O(n)$.

Theorem 2 is not useful to bound a factor of n for the case $k = O(1)$. The following theorem is simple but provides an explicit factor of n for any $k \geq 3$.

Theorem 3. *For a $k \times l$-grid with $n = kl$ and $k \geq 3$, the minimum cost is at most $\frac{n}{3} + \frac{2}{3}k - 1$ if $k \bmod 3 = 0$, $(1 + \frac{1}{k})\frac{n}{3} + \frac{2}{3}k - \frac{1}{3}$ otherwise.*

Proof. It can easily be verified that the following algorithm constructs a desired broadcast:

1. Locate a disk of radius 1 centered at every (x, y) with $0 \leq x \leq l-2$, $0 \leq y \leq k-2$, and $y \bmod 3 = 1$.

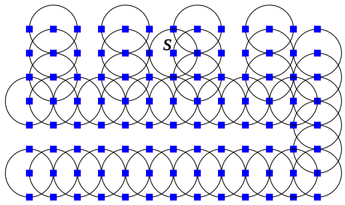


Figure 5: A broadcast on a 8×13-grid based on Theorem 3.

2. Locate a disk of radius 1 centered at every $(l-1, y)$ with $1 \le y \le k-2$.
3. If $k \bmod 3 \ge 1$, then locate a disk of radius 1 centered at every $(x, k-1-i)$ with $1 \le x \le l-2$, $x \bmod 3 = 1$, and $0 \le i \le k \bmod 3$.
4. Locate a disk of radius 1 centered at a source node.
5. Assign each node v the transmission range of 1 if there exists a disk centered at v, 0 otherwise.

$\square$

Figure 5 illustrates a broadcast of Theorem 3. The running time of the algorithm of Theorem 3 is obviously $O(n)$.

4.2. Lower Bounds

Proof Sketch. Let $R := \{1, \sqrt{2}, 2, \sqrt{5}, 2\sqrt{2}, 3, \sqrt{10}, \ldots\}$ be the set of radii of disks centered at a node and having at least one node on the boundary. Suppose that $\mathcal{D} := \{D_1, D_2, \ldots\}$ is a broadcast on a $k \times l$-grid with the minimum cost denoted by cost, and that D_1 is centered at a source node s. It should be noted that any D_i has a radius $r_i \in R$. The proof of the lower bound for square grids in [8] is as follows: For any $D_i \in \mathcal{D}$ not covering s, there exists a sequence $\mathcal{H}_i$ of disks *activating* D_i, i.e., transferring a data message from the outside of D_i to the center c_i of D_i. We can observe that $n \le \sum_i N(r_i) - \sum_{D_i \not\ni s} M(r_i)$, where $N(r_i)$ and $M(r_i)$ are the numbers of nodes in D_i and $D_i \cap \bigcup_{A \in \mathcal{H}_i} A$, respectively. Moreover, the following inequalities are proved in [8]:

$$N(r) < \pi r^2 + 2\sqrt{2}\,r - 5 \text{ for any } r \in R \text{ with } r > \sqrt{10}, \tag{7}$$

$$\sum_{D_i \ni s} r_i = O(r_{\max}), \text{ and} \tag{8}$$

$$M(r) \ge 2\sqrt{2}\,r - 5 \text{ for any } r \in R.$$

Here, $r_{\max} := \max_i\{r_i\}$, which is $O(\sqrt{n})$ on a square grid. Thus, we have $n \le \sum_{D_i \ni s} N(r_i) + \sum_{D_i \not\ni s}(N(r_i) - M(r_i)) = \pi \sum_i r_i^2 + O(\sqrt{n}) = \pi \cdot \mathsf{cost} + O(\sqrt{n})$. To obtain a lower bound of $\frac{n}{\alpha}$ with $\alpha < \pi$ by this proof, we need to improve bounds of $N(r)$ and/or $M(r)$ so that $N(r) - M(r) \le \alpha r^2$. However, there is no effective room for such improvement. Our key idea is to estimate the overlap of D_i and $\mathcal{H}_i$ by the cost instead of by $M(r_i)$. If $\mathrm{dist}(v, c_i) \ge a_i$ for every node v covered by the first disk $\tilde{D}_i$ in $\mathcal{H}_i$, then the total cost of disks in $\mathcal{H}_i \setminus \{\tilde{D}_i\}$ is at least a_i. Therefore, if we can choose $\mathcal{Z} \subset \mathcal{D}$ such that $D_i \in \mathcal{Z}$

implies $\mathcal{H}_i \cap \mathcal{Z} = \{\tilde{D}_i\}$, and that any $A \notin \mathcal{Z}$ activates a unique disk of $\mathcal{Z}$, then we have $n \le \sum_{D_i \ni s} N(r_i) + \sum_{s \notin D_i \in \mathcal{Z}} (N(r_i) - L(a_i, r_i))$ and $\mathsf{cost} \ge \sum_{D_i \ni s} r_i^2 + \sum_{s \notin D_i \in \mathcal{Z}} (r_i^2 + a_i)$, where $L(a_i, r_i)$ is the number of nodes covered by $D_i \cap \tilde{D}_i$. From these inequalities, we can obtain sufficient conditions $N(r_i) \le \alpha r_i^2 + \beta r_i$ and $N(r_i) - L(a_i, r_i) \le \alpha(r_i^2 + a_i)$ for the lower bound of $\frac{n}{\alpha} - O(\frac{\beta}{\alpha} r_{\max})$. Because $L(a_i, r_i) + \alpha a_i$ is minimized at $a_i \simeq r_i$, by (7) and $r_{\max} = O(k)$, we can prove that the sufficient conditions are satisfied with $\alpha = \pi - \Omega(k^{-1})$ and $\beta = O(1)$.

Now we describe our formal proof. Suppose that $\mathcal{G}$ is the directed graph with the node set $\mathcal{D}$ and edge set $\{(D, D') \mid D \text{ covers the center of } D'\}$. Because $\mathcal{D}$ is a broadcast, there exists a path from D_1 to every $D \in \mathcal{D} \setminus \{D_1\}$ in $\mathcal{G}$. Therefore, there exists a spanning tree $\mathcal{T} := (\mathcal{D}, \mathcal{E})$ of $\mathcal{G}$ such that D_1 is the root of $\mathcal{T}$, and that D is the parent of D' for each $(D, D') \in \mathcal{E}$. For each $D \in \mathcal{D}$, let $\tilde{D}^{\mathcal{T}}$ be the nearest ancestor to D that covers a node not covered by D if such an ancestor exists, D_1 otherwise. It should be noted that if $\mathcal{A}_i^{\mathcal{T}}$ is the set of disks between $\tilde{D}_i^{\mathcal{T}}$ and D_i on $\mathcal{T}$ (excluding both $\tilde{D}_i^{\mathcal{T}}$ and D_i), then every node covered by $A \in \mathcal{A}_i^{\mathcal{T}}$ is covered also by D_i. Therefore, $\tilde{D}_i^{\mathcal{T}}$ covers also a node in D_i. Let $\mathcal{Z}^{\mathcal{T}}$ be the set of disks $D \in \mathcal{D}$ such that there exists a sequence of disks $Z_1, \ldots, Z_h \in \mathcal{D}$ ($h \ge 1$), where $Z_1 = D$, $Z_j = \tilde{Z}_{j+1}^{\mathcal{T}}$ for $1 \le j < h$, and Z_h is a leaf of $\mathcal{T}$. It should be noted that D_1 and all the leaves of $\mathcal{T}$ are contained in $\mathcal{Z}^{\mathcal{T}}$.

Lemma 4. $\bigcup_i D_i = D_1 \cup \bigcup_{D \in \mathcal{Z}^{\mathcal{T}} \setminus \{D_1\}} (D \setminus \tilde{D}^{\mathcal{T}})$.

Proof. For any disk $D' \notin \mathcal{Z}^{\mathcal{T}}$, there exists $D \in \mathcal{Z}^{\mathcal{T}}$ covering every node covered by D'. Therefore, it follows that $\bigcup_i D_i = \bigcup_{D \in \mathcal{Z}^{\mathcal{T}}} D$. Moreover, by the definition of $\mathcal{Z}^{\mathcal{T}}$, $D \in \mathcal{Z}^{\mathcal{T}} \setminus \{D_1\}$ implies that $\tilde{D}^{\mathcal{T}} \in \mathcal{Z}^{\mathcal{T}}$, and that there is no sequence $Z_1, \ldots, Z_h = D$ such that $Z_j = \tilde{Z}_{j+1}^{\mathcal{T}}$ for $1 \le j < h$ and $Z_h = \tilde{Z}_1^{\mathcal{T}}$. Therefore, for each node $v \in D \cap \tilde{D}^{\mathcal{T}}$, there exists an ancestor $A \in \mathcal{Z}^{\mathcal{T}}$ of D such that $v \in A \setminus \tilde{A}^{\mathcal{T}}$, or $v \in D_1$. Thus, the lemma holds. □

Lemma 5. *For any spanning tree $\mathcal{T}$ associated with $\mathcal{D}$ and rooted at D_1, and for any leaves Y_p and Z_q of $\mathcal{T}$, let $\mathcal{Z}^{\mathcal{T}}(Y_p) := \{Y_1, \ldots, Y_p\}$ and $\mathcal{Z}^{\mathcal{T}}(Z_q) := \{Z_1, \ldots, Z_q\}$, where $Y_1 = Z_1 = D_1$, $Y_j = \tilde{Y}_{j+1}^{\mathcal{T}}$ for $1 \le j < p$, and $Z_j = \tilde{Z}_{j+1}^{\mathcal{T}}$ for $1 \le j < q$. Then, there exists $\mathcal{T}$ satisfying the following conditions for any pair of leaves Y_p and Z_q of $\mathcal{T}$:*

1. *The nearest common ancestor A to Y_p and Z_q in $\mathcal{T}$ is contained in $\mathcal{Z}^{\mathcal{T}}(Y_p) \cap \mathcal{Z}^{\mathcal{T}}(Z_q)$.*
2. *There exists $1 \le a \le \min\{p, q\}$ such that $Y_j = Z_j$ for $1 \le j < a$, and that $Y_a = Z_a = A$.*

Proof. It should be noted that Condition 2 is implied by Condition 1 because $Y_1, \ldots, Y_{a-1}$ ($Z_1, \ldots, Z_{a-1}$, resp.) are uniquely determined by Y_a (Z_a, resp.) and $\mathcal{T}$. Therefore, we prove that we can obtain $\mathcal{T}$ satisfying Condition 1 for any pair of leaves Y_p and Z_q of $\mathcal{T}$.

Fix Y_p and Z_q, and assume $A \notin \mathcal{Z}^{\mathcal{T}}(Y_p)$ and $A \in \mathcal{A}_i^{\mathcal{T}}$ for some $D_i \in \mathcal{Z}^{\mathcal{T}}(Y_p)$. Let $(A, D) \in \mathcal{E}$ such that D is on the path between A and Z_q in $\mathcal{T}$. Because every node covered by A is covered also by D_i, we can obtain another spanning tree $\mathcal{T}' = (\mathcal{D}, \mathcal{E}')$

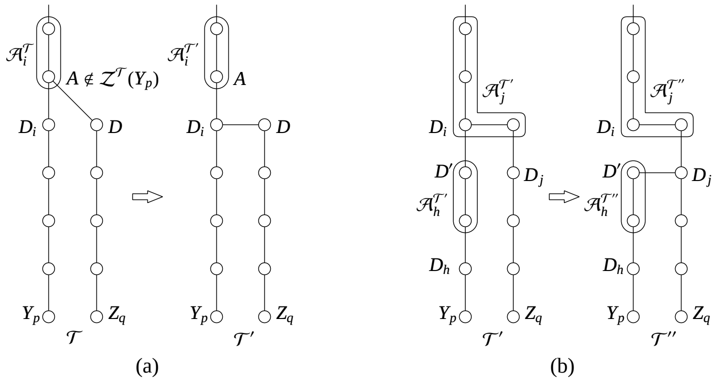


Figure 6: Modifications of $\mathcal{T}$. If $D_i \in \mathcal{Z}^{\mathcal{T}'}(Z_q)$ in (a), then the modification for Y_p and Z_q is finished. Otherwise, we modify $\mathcal{T}'$ as shown in (b).

from $\mathcal{T}$ by replacing (A, D) with (D_i, D), so that D_i becomes the nearest common ancestor to Y_p and Z_q in $\mathcal{T}'$ (Fig. 6 (a)). If Y_p is not a leaf of $\mathcal{T}'$, or if $D_i \in \mathcal{Z}^{\mathcal{T}'}(Z_q)$, then $\mathcal{T}'$ satisfies Condition 1 with respect to Y_p and Z_q fixed here.

Otherwise, assume $D_i \notin \mathcal{Z}^{\mathcal{T}'}(Z_q)$ and $D_i \in \mathcal{A}_j^{\mathcal{T}'}$ for some $D_j \in \mathcal{Z}^{\mathcal{T}'}(Z_q)$. Let $(D_i, D') \in \mathcal{E}'$ such that D' is on the path between D_i and Y_p in $\mathcal{T}'$. Because every node covered by D_i is covered also by D_j, we can obtain another spanning tree $\mathcal{T}''$ from $\mathcal{T}'$ by replacing (D_i, D') by (D_j, D'), so that D_j becomes the nearest common ancestor to Y_p and Z_q in $\mathcal{T}''$ (Fig. 6 (b)). Let $D_h \in \mathcal{Z}^{\mathcal{T}'}(Y_p)$ with $\tilde{D}_h^{\mathcal{T}'} = D_i$. It should be noted that $D_h \in \mathcal{Z}^{\mathcal{T}''}(Y_p)$, and that if Z_q is a leaf of $\mathcal{T}''$, then $D_j \in \mathcal{Z}^{\mathcal{T}''}(Z_q)$. Moreover, $\tilde{D}_h^{\mathcal{T}''} = D_j$ holds. This is because any disk from D' to D_h on $\mathcal{T}'$ is contained in $\mathcal{A}_h^{\mathcal{T}'}$, and hence, in $\mathcal{A}_h^{\mathcal{T}''}$, and because every node covered by D_i is covered by D_j, which means that D_j covers a node not covered by D_h. Thus, $D_j \in \mathcal{Z}^{\mathcal{T}''}(Y_p) \cup \mathcal{Z}^{\mathcal{T}''}(Z_q)$ if Z_q is a leaf of $\mathcal{T}''$.

By repeating the above argument until every pair of leaves satisfies Condition 1, we will obtain a desired spanning tree. This process will be finished in finite steps because replacing edges in the process increases $\sum_{D\in\mathcal{D}}$(distance between D_1 and D in $\mathcal{T}$), which is at most $|\mathcal{D}|(|\mathcal{D}| - 1)/2 \leq n(n-1)/2$. □

In what follows, we fix a spanning tree $\mathcal{T} = (\mathcal{D}, \mathcal{E})$ satisfying the conditions of Lemma 5 and omit the superscript $\mathcal{T}$ from each symbol.

Definition 1. *For any $r \geq 1$ and an integer $a \geq 0$, let $L(a, r)$ be the minimum number of grid points of an infinitely large grid that is covered by two disks D of radius r and $\tilde{D}$ satisfying the following conditions:*

1. *D and $\tilde{D}$ are centered at grid points.*
2. *$\tilde{D}$ covers a grid point not covered by D, and a grid point of coordinates (x, y) such that D covers (x, y), $(x, y \pm 1)$, and $(x \pm 1, y)$.*
3. *The shortest Manhattan distance between a node in $D \cap \tilde{D}$ and the center of D is a.*

Let $N(r)$ be the number of grid points of an infinitely large grid that is covered by a disk of radius r centered at a grid point. We define $X(a,r) := \frac{N(r)-L(a,r)}{r^2+a}$, which can be used to estimate a lower bound of cost as follows:

Lemma 6. *If $N(r) \leq \alpha r^2 + \beta r$ and $X(a,r) \leq \alpha$ for any $r \in R$ with $r \leq r_{\max}$ and any $a \geq 0$, then $\mathsf{cost} \geq \frac{n}{\alpha} - O(\frac{\beta}{\alpha} r_{\max})$.*

Proof. Let $D_i \in \mathcal{Z} \setminus \{D_1\}$. We first claim that $\mathcal{A}_i \cap (\mathcal{A}_j \cup \{D_j, \tilde{D}_j\}) = \emptyset$ for any $D_j \in \mathcal{Z} \setminus \{D_i\}$. Let Y_i and Y_j be leaves of $\mathcal{T}$ such that $D_i \in \mathcal{Z}(Y_i)$ and $D_j \in \mathcal{Z}(Y_j)$, respectively. If D_i is an ancestor or a descendant of D_j in $\mathcal{T}$, then $D_i \in \mathcal{Z}(Y_j)$ and $D_j \in \mathcal{Z}(Y_i)$ by Condition 2 of Lemma 5 if $Y_i \neq Y_j$, simply by the definition of $\mathcal{Z}$ otherwise. This means that $\mathcal{A}_i \cap (\mathcal{A}_j \cup \{D_j, \tilde{D}_j\}) = \emptyset$. If D_i is neither an ancestor nor a descendant of D_j in $\mathcal{T}$, then the nearest common ancestor A to Y_i and Y_j is also the nearest common ancestor to D_i and D_j. Because $A \in \mathcal{Z}(Y_i) \cap \mathcal{Z}(Y_j)$ by Condition 1 of Lemma 5, it follows that $\mathcal{A}_i \cap (\mathcal{A}_j \cup \{D_j, \tilde{D}_j\}) = \emptyset$.

Let a_i be the shortest Manhattan distance between a node in $D_i \cap \tilde{D}_i$ and the center of D_i. Because $\mathcal{A}_i \cap (\mathcal{A}_j \cup \{D_j, \tilde{D}_j\}) = \emptyset$ for any $D_j \in \mathcal{Z} \setminus \{D_i\}$, $\mathcal{A}_i$ plays only a role of transferring a data message from $\tilde{D}_i$ to D_i, which requires a cost at least the cost of a_i disks of radius 1. Thus, we have $\mathsf{cost} \geq \sum_{D_i \ni s} r_i^2 + \sum_{s \notin D_i \in \mathcal{Z}} (r_i^2 + a_i)$. Moreover, if D is a disk with $(\tilde{D}_i, D) \in \mathcal{E}$ and $D \in \mathcal{A}_i \cup \{D_i\}$, then the center (x,y) of D is covered by $\tilde{D}_i$, and (x,y), $(x, y \pm 1)$, and $(x \pm 1, y)$ are covered by D_i. Thus, (# nodes in D_i) − (# nodes in $D_i \cap \tilde{D}_i$) is at most $X(a_i, r_i) \cdot (r_i^2 + a_i)$ if $s \notin D_i$. It should be noted that this holds even if D_i covers fewer than $N(r_i)$ nodes due to its location close to the boundary of the underlying $k \times l$-grid. Thus, it follows from (8) and Lemma 4 that

$$
\begin{aligned}
n &\leq N(r_1) + \sum_{D_i \in \mathcal{Z} \setminus \{D_1\}} \Big((\#\text{ nodes in } D_i) - (\#\text{ nodes in } D_i \cap \tilde{D}_i) \Big) \\
&\leq \sum_{D_i \ni s} N(r_i) + \sum_{s \notin D_i \in \mathcal{Z}} \Big((\#\text{ nodes in } D_i) - (\#\text{ nodes in } D_i \cap \tilde{D}_i) \Big) \\
&\leq \sum_{D_i \ni s} (\alpha r_i^2 + \beta r_i) + \sum_{s \notin D_i \in \mathcal{Z}} X(a_i, r_i) \cdot (r_i^2 + a_i) \leq \alpha \cdot \mathsf{cost} + O(\beta r_{\max}),
\end{aligned}
$$

by which we obtain the lemma. □

We bound $X(a,r)$ and $r_{\max}$ from above by the following lemmas. We can easily verify the following lemma by (7) and simple calculation.

Lemma 7. *For any $r \in R \setminus \{1\}$, it follows that $N(r) < \pi(r^2 + r - c)$, where $c := \sqrt{2} - \frac{2}{\pi} \approx 0.778$.*

For any $r > 0$, let $N'(r)$ be the minimum number of nodes of an infinitely large grid that is covered by a disk of radius r centered at any point (i.e., not necessarily a grid point) on the Euclidean plane.

Lemma 8. *For any $r \geq \frac{\sqrt{2}}{2}$, it follows that $N'(r) \geq \pi(r - \frac{\sqrt{2}}{2})^2$.*

Proof. Let D be a disk of radius r and centered at a point v. If we locate a square of side length 1 centered at each grid point covered by D, then $N'(r)$ equals the area of the range U covered by these squares. Let p be a point not contained in U. Then, there exists a grid point q not contained in U such that p is covered by a square of side length 1 centered at q. Thus, we have $\mathrm{dist}(p, v) \geq \mathrm{dist}(q, v) - \mathrm{dist}(p, q) > r - \frac{\sqrt{2}}{2}$. This means that U contains the disk of radius $r - \frac{\sqrt{2}}{2}$ centered at v. Thus, the lemma holds. □

Lemma 9. *For any $r \geq 1$, $X(a, r)$ is maximized in the case that $a \leq \lfloor r \rfloor - 1$.*

Proof. Suppose that disks D and $\tilde{D}$ satisfies the conditions of Definition 1. Then, $\tilde{D}$ covers a grid point of coordinates (x, y) such that D covers (x, y), $(x, y\pm1)$, and $(x\pm1, y)$. We may assume without loss of generality that $y \leq x$ and that D is centered at (w, z) with $w \leq x$ and $z \leq y$. Because $(x + 1, y)$ is covered by D and $(x - w) + (y - z) \geq a$, if $a \geq \lfloor r \rfloor$, then $y > z$. Hence, eight points $(x + p, y + q)$ with $(p, q) \in \{-1, 0, 1\} \times \{-1, 0, 1\} \setminus \{(1, 1)\}$ are covered by D. Thus, at any grid point $\tilde{D}$ is centered, at least three of the eight points are covered by $D \cap \tilde{D}$. This yields $X(a, r) \leq \frac{N(r)-3}{r^2+\lfloor r \rfloor}$ for any $a \geq \lfloor r \rfloor$.

On the other hand, if $\tilde{D}$ has radius 1 and is centered at $(w + \lfloor r \rfloor, z)$, then at most four points $(w + \lfloor r \rfloor - 1, z)$, $(w + \lfloor r \rfloor, z)$, and $(w + \lfloor r \rfloor, z \pm 1)$ are covered by $D \cap \tilde{D}$. This means that $X(\lfloor r \rfloor - 1, r) \geq \frac{N(r)-4}{r^2+\lfloor r \rfloor-1} \geq \frac{N(r)-3}{r^2+\lfloor r \rfloor}$. The last inequality holds because we can easily observe that $N(r) \geq r^2 + \lfloor r \rfloor + 3$ for any $r \geq 1$. □

Lemma 10. *For any $r \in R \setminus \{1\}$ and any $a \geq 0$, it follows that $X(a, r) < \frac{N(r)}{r^2+r-c}$.*

Proof. By Lemma 9, we may assume $a \leq \lfloor r \rfloor - 1$. By Lemma 7, we can observe that for any $r \in R \setminus \{1\}$, $\frac{N(r)-L(a,r)}{r^2+a} < \frac{N(r)}{r^2+r-c}$ holds if $L(a, r) \geq \pi(b - c)$, where $b := r - a$.

For any pair of disks D and $\tilde{D}$ satisfying the conditions of Definition 1, a disk of radius $b/2$ is contained in $D \cap \tilde{D}$. Therefore, it follows from Lemma 8 that for $b \geq \sqrt{2}$, $L(a, r) \geq \frac{\pi}{4}(b - \sqrt{2})^2$, which is larger than $\pi(b - c)$ for any $b \geq 6$.

Assume $1 \leq b < 6$. Let $\lambda(b) := L(0, b')$, where $b' \in R$ is the largest value with $b' \leq b$. By the definition of $L(a, r)$, we can observe that $L(a, r) \geq L(a - i, r - i)$ for any integer i with $0 \leq i \leq a$, and that $L(0, r) \geq L(0, r')$ for any $1 \leq r' \leq r$. Therefore, $\lambda(b)$ is a lower bound of $L(a, r)$ and a non-decreasing function, and hence, we have the lemma if $\lambda(b) \geq \pi(b - c)$ for $1 \leq b < 6$. This can be verified by evaluating $L(0, b)$ for each $b \in R$ with $b \leq 4$ and observing $L(0, 4) = 17 > \pi(6 - c) \approx 16.4$ as shown in Fig. 7. □

Lemma 11. *For any $r \in R$ with $r \leq \sqrt{202}$ and any $a \geq 0$, it follows that $X(a, r) \leq 3$.*

Proof. We can verify by numerical computation that $\frac{N(r)}{r^2+r-c} < 3$ for any $r \in R \setminus \{1, \sqrt{2}, \sqrt{5}\}$ with $r \leq \sqrt{202}$. Thus, by Lemma 10, we have the lemma for such r. For $r \in \{1, \sqrt{2}, \sqrt{5}\}$, we can verify $\frac{N(r)-L(a,r)}{r^2+a} \leq 3$ by evaluating $L(a, r)$ for every possible combination of a and r, i.e., $N(1) = 5$, $N(\sqrt{2}) = 9$, $N(\sqrt{5}) = 21$, $L(0, 1) = 2$, $L(0, \sqrt{2}) = 4$, $L(0, \sqrt{5}) = 7$, and $L(1, \sqrt{5}) = 4$. □

Lemma 12. *For any $k \geq 3$, it follows that $r_{\max} \leq \frac{2}{3}k + \frac{13}{6}$.*

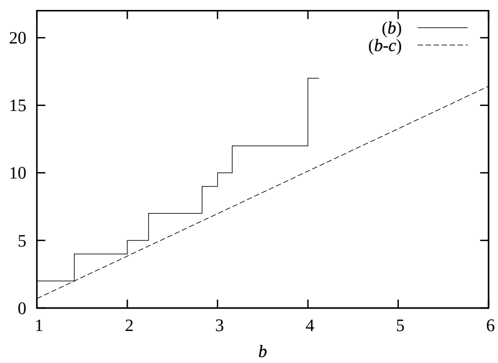


Figure 7: Plots of $\lambda(b)$ and $\pi(b-c)$.

Proof. On a $k \times l$-grid, a disk D of radius $r_{\max}$ centered at a node v covers at most $(2r_{\max}+1)k$ nodes of a $k \times (2r_{\max}+1)$-grid. By Theorem 3, there exists a broadcast on the $k \times (2r_{\max}+1)$-grid with a cost at most $\frac{(2r_{\max}+1)k}{3} + \frac{2r_{\max}+1}{3} + \frac{2}{3}k - \frac{1}{3} = \frac{2}{3}(k+1)r_{\max} + k$. This cost is at least $r_{\max}^2$, for otherwise, we can obtain a broadcast on the $k \times l$-grid with a cost less than cost by replacing D with the broadcast of Theorem 3. Thus, we have

$$r_{\max} \le \frac{k+1}{3} + \sqrt{\left(\frac{k+1}{3}\right)^2 + k} < \frac{2}{3}k + \frac{13}{6}.$$

□

Theorem 4. $\mathsf{cost} \ge \frac{n}{\pi} + \Omega(\frac{n}{k}) - O(k)$. *In particular,* $\mathsf{cost} \ge \frac{n}{3} - O(1)$ *if* $3 \le k \le 18$.

Proof. By Lemmas 6 and 10–12, it suffices to prove the following claims:

1. There exist α with $\alpha^{-1} = \pi^{-1} + \Omega(k^{-1})$ and $\beta = O(1)$ such that for any $r \in R$ with $\sqrt{10} < r \le r_{\max}$, $N(r) \le \alpha r^2 + \beta r$ and $X(a, r) \le \alpha$.
2. There exists $\beta = O(1)$ such that for any $r \in R$ with $r \le \sqrt{202}$, $N(r) \le 3r^2 + \beta r$.
3. $r_{\max} \le \sqrt{202}$ if $k \le 18$.

The second claim is immediate because $r = O(1)$. Moreover, the third claim can be verified simply by applying Lemma 12. As for the first claim, it follows from (7) and Lemma 10 that for $r \in R$ with $\sqrt{10} < r \le r_{\max}$,

$$\begin{aligned} X(a, r) &\le \frac{\pi r^2 + 2\sqrt{2}\,r - 5}{r^2 + r - c} = \pi - \frac{(\pi - 2\sqrt{2})r + (5 - \pi c)}{r^2 + r - c} \\ &< \pi - \frac{\pi - 2\sqrt{2}}{r} \le \pi - \frac{\pi - 2\sqrt{2}}{r_{\max}}. \end{aligned} \tag{9}$$

If we set $\alpha := \pi - \frac{\pi - 2\sqrt{2}}{r_{\max}}$ and $\beta := \pi$, then it follows that

$$\begin{aligned} N(r) - (\alpha r^2 + \beta r) &\le \pi r^2 + 2\sqrt{2}\,r - 5 - (\alpha r^2 + \beta r) \\ &< \frac{\pi - 2\sqrt{2}}{r_{\max}} r^2 - (\pi - 2\sqrt{2})r \le 0. \end{aligned} \tag{10}$$

Moreover, it follows from Lemma 12 that

$$\alpha^{-1} = \pi^{-1}\left(1 + \frac{\pi - 2\sqrt{2}}{\pi r_{\max} - \pi + 2\sqrt{2}}\right) = \pi^{-1} + \Omega(k^{-1}). \tag{11}$$

By (9)–(11), we have the first claim. □

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