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Unified Closed-form Expression of Logit and Weibit and its Extension to a Transportation Network Equilibrium Assignment

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Abstract

This study proposes a generalized multinomial logit model that allows heteroscedastic variance and flexible utility function shape. The novelty of our approach is that the model is theoretically derived by applying a generalized extreme-value distribution to the random component of utility, while retaining its closed-form expression. In addition, the weibit model, in which the random utility is assumed to follow the Weibull distribution, is a special case of the proposed model. This is achieved by utilizing the *q*-generalization method developed in Tsallis statistics. Then, our generalized logit model is incorporated into a transportation network equilibrium model. The network equilibrium model with a generalized logit route choice is formulated as an optimization problem for uncongested networks. The objective function includes Tsallis entropy, a type of generalized entropy. The generalization of the Gumbel and Weibull distributions, logit and weibit models, and network equilibrium model are formulated within a unified framework with *q*-generalization or Tsallis statistics.

Keywords: Closed-form expression; Logit; Weibit; Tsallis entropy; Transportation network equilibrium assignment

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1. Introduction

The multinomial logit model has been indispensable in transportation studies for several decades. Stochastic user equilibrium models with a logit-based route choice are among the most widely used network equilibrium models. The multinomial logit model can be expressed in closed form and is easily applied. Such a simple closed formulation is desirable, especially considering the embedding of route choice models in network equilibrium analysis. Calculation of route choice probabilities is iterative and requires significant computational cost in network equilibrium algorithms. The closed-form logit formulation is derived from the assumption that the utilities are distributed independently and identically.

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Castillo *et al.* (2008) proposed the weibit model, a closed-form discrete choice model with Weibull-distributed utility. Fosgerau & Bierlaire (2009) considered a multiplicative error term and derived a closed-form model similar to the weibit model of Castillo *et al.* (2008). Li (2011) extended the logit and weibit models for other distributions and offered other alternative error distributions for discrete-choice models. Kitthamkesorn & Chen (2013, 2014) proposed a stochastic user equilibrium model with a weibit route choice. The weibit model considers heterogeneous perceived variances with respect to different travel costs, while the (multinomial) logit model has homogeneity in the variance of its error terms. Bhat (1995, 1997), DeShazo & Fermo (2002), Caussade *et al.* (2005), and Koppelman & Sethi (2005) considered an additive error term or scale parameter to relax the homogeneity in the error-term variance.

There is a possibility of integrating the logit and weibit models under a closed-form formulation, since both the Gumbel and Weibull distributions are in a family of extreme value distributions. A generalized extreme value distribution in probability and statistics could play an important role in the integration. The generalized extreme value (GEV) distribution consists of the Gumbel-, Fréchet-, and Weibull-type extreme value distributions and has a greater variety in shape than either the Gumbel or Weibull distribution. The Gumbel-type extreme distribution is the Gumbel distribution. Note that the GEV distribution above is different from the GEV distribution that is the basis for deriving the nested logit and other more elaborate logit models (e.g., cross-nested logit model) in travel behavior analysis. Recently, to avoid confusion, the latter GEV distribution has been referred to as the multivariate extreme value distribution.

Nakayama (2013) proposed a discrete choice model with a GEV-distributed utility. This previous model has a complicated utility function, and it results in parameter estimation instability. In this study, we improve the previous model and propose a more simplified formulation of the generalized logit model with GEV-distributed utility. The generalized logit model includes the (multinomial) weibit and multinomial logit models as special cases, because the GEV distribution combines the Gumbel-, Fréchet-, and Weibull-type extreme value distributions. Thus, it is a unified model of logit and weibit in a single closed-form expression. In contrast, the weibit models proposed previously do not include the logit model. Furthermore, the generalized logit model can avoid one of the limitations of the logit model, viz., the homogeneity of the utility's variance.

The generalized logit model is incorporated into a transportation network equilibrium model as a route choice model formulated as an optimization problem under uncongested networks with an objective function that includes Tsallis entropy, a type of generalized entropy. Finally, the relationship between Tsallis entropy and the generalized logit model with GEV-distributed utility is examined, and its mathematical framework is elucidated.

2. GEV distribution and *q*-exponential function

The GEV distribution is important for deriving the generalized logit model. GEV is explained again in this section to help readers understand it more readily, even though Nakayama (2013) has already explained it. The logit model has a Gumbel-distributed utility (or error term). The Gumbel distribution is a type of extreme value distribution, and the GEV distribution includes the Gumbel distribution. The cumulative distribution function (CDF), $\tilde{G}(x)$, of GEV is expressed as

$$\tilde{G}(x) = \exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\theta}\right)\right]^{-\frac{1}{\gamma}}\right\} \tag{1}$$

where μ, θ (> 0), and γ are parameters (e.g., Johnson *et al.*, 1995). When $\gamma = 0$, $\tilde{G}(x) = \exp[-\exp\{-(x-\mu)/\theta\}]$, because $\lim_{\rho\to 0}(1 + \rho x)^{1/\rho} = \exp(x)$. This is the CDF of the Gumbel distribution. Thus, the GEV distribution includes the Gumbel distribution as a special case.

Tsallis (1994, 2009) proposed a type of generalization of Boltzmann–Gibbs statistical mechanics and thermodynamics. A core concept in his study is Tsallis entropy, a generalization of Boltzman–Gibbs (or Shannon) entropy. Such a generalization is sometimes called *q*-generalization. The basic operations of *q*-analysis appear in *q*-generalized statistical mechanics. Tsallis (1994) generalized the exponential function as follows:

$$\exp_q(x) := [1+(1-q)x]^{\frac{1}{1-q}}, \tag{2}$$

with the domain $\{x \mid 1+(1-q)x \geq 0\}$. Recently, this generalized exponential function has been called the *q*-exponential function (e.g., Umarov *et al.*, 2008). When $q = 1$, $\exp_1(x) = \exp(x)$, because $\lim_{\rho\to 0}(1 + \rho x)^{1/\rho} = \exp(x)$, as stated above. Thus, we confirm that the *q*-exponential function is a type of generalization of the exponential function. The *q*-logarithm function is also defined as follows:

$$\ln_q(x) := \frac{x^{1-q}-1}{1-q} \tag{3}$$

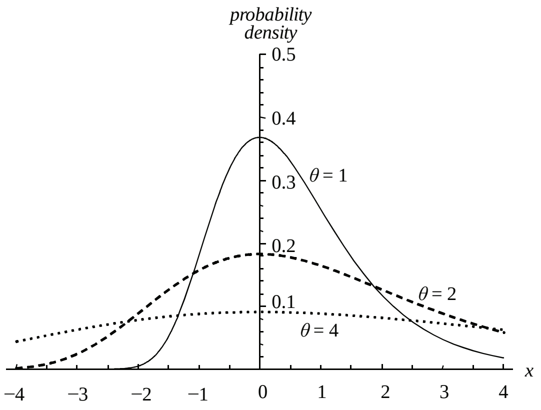

Fig. 1. PDF of GEV distribution with q = 1 and μ = 0

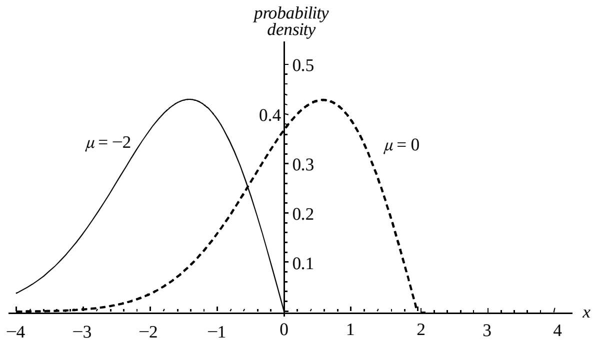

Fig. 2. PDF of GEV distribution with q = 1/2 and θ = 1

for $x > 0$. When $q = 1$, $\ln_1(x) = \ln(x)$. Therefore, the q-logarithm function includes the (standard) logarithm function as a special case. Furthermore, $\ln_q(\exp_q[x]) = x$.

Let $q = \gamma + 1$. Using the q-exponential function in Eq. (2), the CDF of the GEV distribution is rewritten as

$$\hat{G}(x) = \exp\left[-\exp_q\left(-\frac{x-\mu}{\theta}\right)\right] = \exp\left\{-\left[1-(1-q)\frac{x-\mu}{\theta}\right]^{\frac{1}{1-q}}\right\} \tag{4}$$

with the domain $\left\{ x \middle| 1-(1-q)(x-\mu)/\theta \geq 0 \right\}$. The CDF of $\hat{G}(x)$ must be within the range [0.0, 1.0] and increasing. Therefore, $\theta \geq 0$ is required. When q = 1, μ = 0, and θ = 1, this GEV distribution is the standard Gumbel distribution, whose CDF is $\exp[-\exp(-x)]$. As stated above, the CDF of the Gumbel distribution is $\exp[-\exp\{-(x-\mu)/\theta\}]$. Replacing one of the exponential functions of the Gumbel distribution's CDF with the q-exponential function yields that of the GEV distribution. Thus, the GEV distribution is a type of q-generalization of the Gumbel distribution.

Fig. 1 shows the probability density function (PDF) of the (q-form) GEV distribution with q = 1 (and μ = 0), that is, the Gumbel distribution. The domain of the Gumbel distribution is from negative infinity to positive infinity. The distribution becomes flatter as θ increases. As will be stated below, θ determines the distribution's variance.

Fig. 2 presents the PDF of the (q-form) GEV distribution with q = 1/2 and θ = 1. The distribution with q = 1/2 leans to the right, while the Gumbel distribution (GEV with q = 1) leans to the left. The domain with q = 1/2 and θ = 1 is $x \leq 2 + \mu$. When $\mu = -2$, the domain is $x \leq 0$. A change in μ translates the distribution in the x direction.

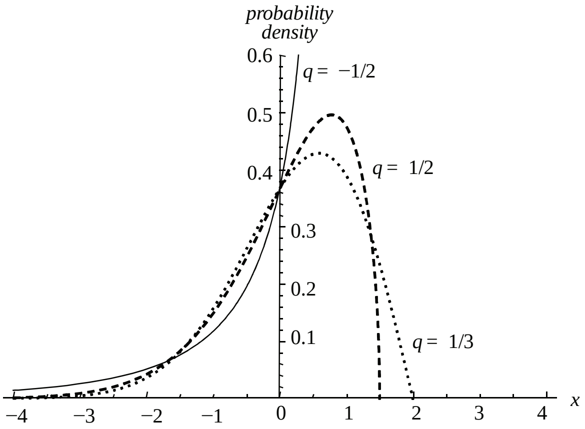

Fig. 3. PDF of GEV distribution with $q < 1$, $\theta = 1$, and $\mu = 0$

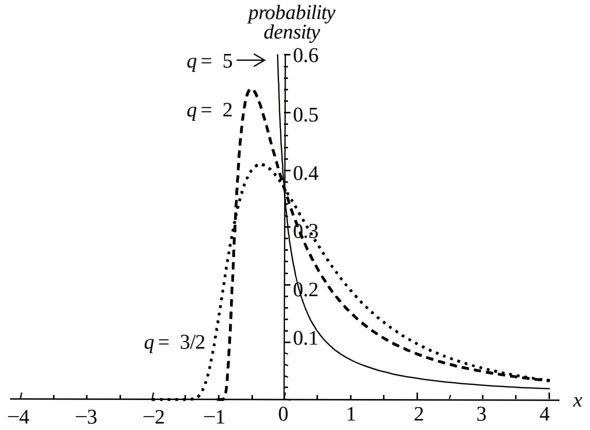

Fig. 4. PDF of GEV distribution with $q > 1$, $\theta = 1$, and $\mu = 0$

Figs. 3 and 4 show the PDFs of the (q-formed) GEV distribution with $\theta = 1$ and $\mu = 0$. The distribution with $q < 1$ leans to the right, as shown in Fig. 3, while that with $q > 1$ leans to the left, as shown in Fig. 4. Thus, the (q-formed) GEV distribution has various shapes according to the value of q and other parameters. This flexibility helps in fitting the distribution to the data.

The mean of the (q-form) GEV distribution is

$$\begin{cases} \mu + \dfrac{\Gamma(2-q)-1}{q-1}\theta & q < 2 \text{ and } q \neq 1 \\ \mu + \eta\theta & q = 1 \end{cases} \tag{5}$$

where η = 0.572216 (Euler constant) and $\Gamma(\cdot)$ is the gamma function. When $q > 2$, the distribution has no mean. Note that the (q-form) GEV distribution is not symmetric, and the mean does not generally equal the mode. The mode of the (q-form) GEV distribution is μ, and its variance is

$$\begin{cases} \dfrac{\Gamma(3-2q)-\left[\Gamma(2-q)\right]^2}{\left(q-1\right)^2}\theta^2 & q < \dfrac{3}{2} \text{ and } q \neq 1 \\ \dfrac{\pi^2\theta^2}{6} & q = 1 \end{cases} \tag{6}$$

When $q \geq 3/2$, the distribution has no variance. As the value of q increases, the tail of the distribution becomes fatter. In the field of finance, a fat-tailed distribution is important for considering risks. Similarly, it is useful in transportation. One of the well-known issues involving a fat-tailed distribution is modeling route choice behavior. Sheffi (1985) pointed out that the normality assumption might not be appropriate for modeling the distribution of perceived travel times, and a distribution with a long tail to the right (positive skewness) could be more appropriate. Note that a distribution with a long tail to the left (negative skewness) must be applied for the error term in route choice models, since the utility is usually defined as a negative value of travel time. The (q-form) GEV distribution is negatively skewed with a long left tail when $q < 1$ as shown in Fig. 3. Thus, it would be useful as a utility distribution for route choice.

3. *q*-generalized logit model

In the multinomial logit model, the random utility consists of the systematic utility and the error term. The separativeness of systematic utility and error term of an independent utility results in homogeneity in the variance of the error terms. On the other hand, the random utility in the proposed generalized logit model cannot be decomposed into a systematic utility and error term. Thus, the key point of this current model development is to derive a model with the expression of the entire random utility, U_{ij}, without decomposition into a deterministic utility and error term, where U_{ij} is the random utility of route j (= 1, 2,…, J_i) for origin-destination (OD) pair i (= 1, 2,…, I). Let v_{ij} be the mean of U_{ij}, that is, $\mathrm{E}[U_{ij}] = v_{ij}$, where $\mathrm{E}[\cdot]$ is the expectation operator. Furthermore, assume $v_{ij} = \Sigma_{k=1}^{K} \alpha_{ik} y_{ijk}$, where α_{ik} is parameter k for OD pair i, y_{ijk} is explanatory variable k on route j for OD pair i, and K is the total number of explanatory variables. In the case in which the parameters are common among the OD pairs, v_{ij} is $\Sigma_{k=1}^{K} \alpha_k y_{ijk}$. Thus, the mean utility, instead of the systematic utility, is given by the utility function in the generalized logit model.

To meet the assumption that the mean of random utility of route j for OD pair i, which follows the (q-form) GEV distribution, is equal to v_{ij}, that is, $\mathrm{E}[U_{ij}] = v_{ij}$, the CDF, $G_{ij}(x)$, of U_{ij} must be the following:

$$
\begin{aligned}
G_{ij}(x) &= \exp\left[-\frac{\exp_{2-q_i}(v_{ij})}{\left\{\Gamma(2-q_i)\right\}^{1/(q_i-1)}} \exp_{q_i}(-x) \right] \\
&= \exp\left[-\exp_{q_i}\left(-\frac{x - \dfrac{1}{1-q_i}\left\{1 - \dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}\right\}}{\dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}} \right) \right],
\end{aligned}
\tag{7}
$$

Substituting $\mu = \dfrac{1}{1-q_i}\left[1 - \dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}\right]$ and $\theta = \dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}$ into Eq. (5), we can confirm that $\mathrm{E}[U_{ij}] = v_{ij}$ in the distribution of Eq. (7). As Eq. (5) states, $q_i < 2$ is required in order for the mean of random utility distribution to exist. Furthermore, it must be the case that

$$1+(q_i-1)v_{ij} \geq 0 \tag{8}$$

to define $\exp_{2-q_i}(v_{ij})$, if $q_i \neq 1$. When $q_i = 1$, such a condition is not required.

The probability of choosing route j for OD pair i is the probability that the utility on route j for OD pair i is greater than those on any other routes for OD pair i. That is, the utility on route j for OD pair i is the maximum for OD pair i. Therefore, the probability of choosing route j for OD pair i is given by

$$\begin{aligned} p_{ij} &= \Pr[U_{ij} > \max_{j'(\neq j)}(U_{ij'})] \\ &= \int_{x\in\Omega_i} G_{i1}(x)\cdots G_{ij-1}(x) g_{ij}(x) G_{ij+1}(x)\cdots G_{iJ_i}(x)\,dx \end{aligned} \tag{9}$$

where p_{ij} is the probability of choosing route j for OD pair i, Pr[·] and max(·) are the operators that determine the probability and the maximum, respectively, $g_{ij}(x)$ is the PDF of the utility on route j for OD pair i, J_i is the number of routes for OD pair i, and Ω_i is the domain of the CDF for OD pair i.

To simplify the following equation expansion, let

$$z_i := \exp_{q_i}(-x). \tag{10}$$

The PDF of the utility on route j for OD pair i is

$$g_{ij}(x) = \frac{d}{dx} G_{ij}(x) = -\frac{\exp_{2-q_i}(v_{ij})}{\left[\Gamma(2-q_i)\right]^{1/(q_i-1)}} G_{ij}(x) \frac{dz_i}{dx} \tag{11}$$

because $dz_i/dx = -[\exp_{q_i}(-x)]^{q_i}$. Substituting the above into Eq. (9) yields

$$\begin{aligned} p_{ij} &= -\frac{\exp_{2-q_i}(v_{ij})}{\left[\Gamma(2-q_i)\right]^{1/(q_i-1)}} \int_{z_i\in\hat{\Omega}_i} \prod_{j=1}^{J_i} G_{ij}(x)\,dz_i \\ &= -\frac{\exp_{2-q_i}(v_{ij})}{\left[\Gamma(2-q_i)\right]^{1/(q_i-1)}} \int_{z_i\in\hat{\Omega}_i} \exp\left[-\frac{z_i}{\left[\Gamma(2-q_i)\right]^{1/(q_i-1)}} \sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})\right] dz_i \\ &= \frac{\exp_{2-q_i}(v_{ij})}{\sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})} \left[\exp\left\{-\frac{z_i}{\left[\Gamma(2-q_i)\right]^{1/(q_i-1)}} \sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})\right\}\right]_{\infty}^{0} \\ &= \frac{\exp_{2-q_i}(v_{ij})}{\sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})} \end{aligned} \tag{12}$$

where $\hat{\Omega}_i$ is the domain of z_i. Thus, the q-generalized logit model is given by

$$p_{ij} = \frac{\exp_{2-q_i}(v_{ij})}{\sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})} \tag{13}$$

This closely resembles the choice probability equation of the multinomial logit model. When $q_i = 1$, the route choice probability of Eq. (13) is

$$p_{ij} = \frac{\exp(v_{ij})}{\sum_{j=1}^{J_i} \exp(v_{ij})} \tag{14}$$

because $\exp_1(x) = \exp(x)$. This is the multinomial logit model equation. The above q-generalized logit model, a discrete choice model with a (q-form) GEV-distributed utility, includes the multinomial logit model as a special case.

Eq. (13) can also be expressed as

$$p_{ij} = \frac{\exp_{2-q_i}(v_{ij})}{\sum_{j=1}^{J_i} \exp_{2-q_i}(v_{ij})} = \frac{\left[1+(q_i-1)v_{ij}\right]^{\frac{1}{q_i-1}}}{\sum_{j=1}^{J_i}\left[1+(q_i-1)v_{ij}\right]^{\frac{1}{q_i-1}}} = \frac{\left(v_{ij}+\dfrac{1}{q_i-1}\right)^{\frac{1}{q_i-1}}}{\sum_{j=1}^{J_i}\left(v_{ij}+\dfrac{1}{q_i-1}\right)^{\frac{1}{q_i-1}}} = \frac{\tilde{v}_{ij}^{-\xi_i}}{\sum_{j=1}^{J_i}\tilde{v}_{ij}^{-\xi_i}}, \tag{15}$$

where $\tilde{v}_{ij} = v_{ij} + 1/(q_i - 1) = \alpha_{i0} + 1/(q_i - 1) + \alpha_{i1} y_{ij1} + \alpha_{i2} y_{ij2} + \cdots + \alpha_{iK} y_{ijK}$, $\xi_i = -1/(q_i - 1)$, y_{ijk} is a variable, K is the number of variables and α_{ik} is a parameter. This is the weibit model (see e.g., Castillo *et al.*, 2008). The q-generalized logit model thus involves both logit and weibit as special cases.

The CDF, $G_i^{\max}(x)$, of the maximum of U_{ij} (j=1, 2,…, J_i) is given by

$$G_i^{\max}(x) = \prod_{j=1}^{J_i} G_{ij}(x) = \exp\left[-\frac{\sum_{j=1}^{J_i}\exp_{2-q_i}(v_{ij})}{\left\{\Gamma(2-q)\right\}^{1/(q_i-1)}}\exp_{q_i}(-x)\right]. \tag{16}$$

Thus, $\max[U_{ij}\,|\,\forall j]$ follows the GEV distribution. Measuring the change in consumer surplus is useful for policy analysis. The expected consumer surplus can be given by the mean maximum utility in random utility theory (e.g., Train, 2003, p.59), and, thus, the distribution of maximum utility is important. Comparing the above equation with Eq. (7), we obtain the following as the mean maximum utility:

$$\mathrm{E}\left[\max\left(U_{ij}\,|\,\forall j\right)\right] = \ln_{2-q_i}\left[\sum_{j=1}^{J_i}\exp_{2-q_i}(v_{ij})\right]. \tag{17}$$

We can confirm that substituting $\ln_{2-q_i}[\sum_j \exp_{2-q_i}(v_{ij})]$ into Eq. (7) yields Eq. (16). When $q_i = 1$, $\max[U_{ij}\,|\forall j] = \ln[\sum_j \exp(v_{ij})]$. Thus, the above equation is the generalization of log-sum mean maximum utility of the logit model. Eq. (15) shows that the q-generalized logit model reduces to

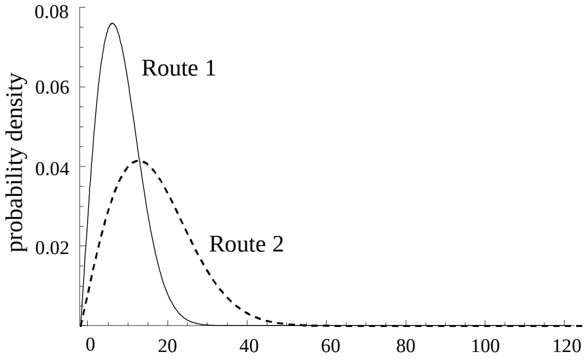

Fig. 5. PDFs of route disutilities between OD pair 1

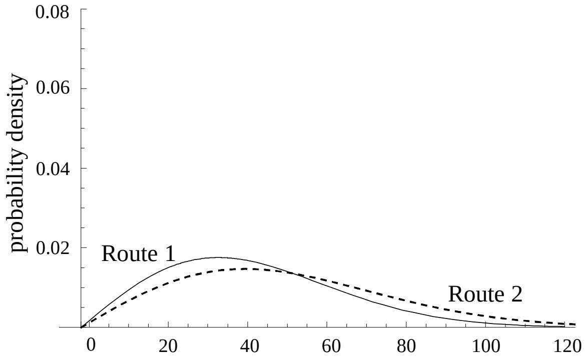

Fig. 6. PDFs of route disutilities between OD pair 2

the weibit model for $q_i \neq 1$. The mean maximum utility of the weibit model expressed by Eq. (15) is given by Eq. (17) for $q_i \neq 1$. Note that some simulation method might be needed to obtain consumer surplus in monetary terms, since the marginal utility of travel cost is not independent of travel cost (McFadden, 1999; Herriges and Kling, 1999; Fosgerau and Bierlaire, 2009).

It is natural that the variance of the utility becomes large as the lengths of routes increase. This is satisfied in the q-generalized logit model as follows. The CDF of random utility is given by Eq. (7). From Eq. (6), the variance of the above random utility is expressed as

$$\begin{cases} \dfrac{\Gamma(3-2q_i)-\left[\Gamma(2-q_i)\right]^2}{(q_i-1)^2}\left(\dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}\right)^2 & q_i < \dfrac{3}{2} \text{ and } q_i \neq 1 \\ \dfrac{\pi^2}{6}\left(\dfrac{1+(q_i-1)v_{ij}}{\Gamma(2-q_i)}\right)^2 & q_i = 1 \end{cases} . \tag{18}$$

As the above equation shows, the variance links to v_{ij}. Therefore, as the absolute value of the mean, v_{ij}, increases, the variance increases. When $q_i < 1$ and $v_{ij} = -c_{ij}$, the variance of the utility on route j for OD pair i becomes large as its travel time increases, where c_{ij} is the travel cost on route j for OD pair i. Thus, the homogeneity of the utility variance is relaxed in the q-generalized logit model.

Suppose that two OD pairs in the network are connected by two pairs of non-overlapping routes. The route utility follows the (q-form) GEV distribution in the q-generalized logit model. The route disutility, that is, $-U_{ij}$, can be interpreted as the random variable of generalized travel cost. Set $c_{11} = 10$, $c_{12} = 20$, $c_{21} = 50$, and $c_{22} = 60$. The difference between the two travel costs is ten for each OD pair. Therefore, the route choice probabilities between the two OD pairs are equal in the standard logit model. Figs. 5 and 6 show the distributions of the disutilities (or generalized travel costs), $-U_{11}$, $-U_{12}$, $-U_{21}$, and $-U_{22}$, for the two OD pairs when $q_1 = q_2 = 0.5$ in the q-generalized logit model. The figures show that the variances of the route disutilities (or generalized travel costs) for OD pair 2 are much larger than those for OD pair 1. The probability of choosing route 1 for OD pair 1, p_{11}, is 0.771, and that of route 2, p_{12}, is 0.229. On the other hand, the probability of choosing route 1 between OD pair 2, p_{21}, is 0.587, and that of route 2, p_{22}, is 0.413. Thus, the homogeneity of the utility variance is relaxed, and the q-generalized logit model provides more intuitive route choice probabilities for different OD pairs.

4. Parameter estimation of *q*-generalized logit model

In this section, some properties of parameter estimation in the q-generalized logit model are discussed.

4.1. Alternative view: logit model with a flexible utility function

For simplicity, q_i is assumed to be common to all OD pairs, that is, $q_i = q$, in this section. As stated in the previous section, the q-generalized logit model, which assumes the GEV distribution, is given by Eq. (13). Here, we introduce an alternative view of the model, i.e., a logit model with flexible utility function. This alternative view is useful for practical applications, i.e., empirical estimation of the model. The q-generalized logit model can be rewritten as

$$p_{ij} = \frac{\exp[f(v_{ij})]}{\sum_{j'=1}^{J_i} \exp[f(v_{ij'})]} \tag{19}$$

$$\text{where } f(v_{ij}) = \ln\left[\exp_{2-q}(v_{ij})\right] \tag{20}$$

Thus, the q-generalized logit model can also be understood under the standard logit model framework, i.e., the logit model with the non-linear utility function defined as Eq. (20), where the standard Gumbel distribution is assumed for the error term. A similar transformation can be performed for the weibit model as well.

In the previous model of Nakayama (2013), $f(v_{ij}) = \ln\left[\exp_q\{v_{ij}/(s-[1-q]v_{ij})\}\right]$, where s is a parameter to be specified or estimated. Table 1 summarizes the differences between the two models. The difference is only in the definition of systematic utility, and the behavior of $f(v_{ij})$ is quite similar. In fact, when s is assumed to be one, the two models are identical. In other words, by using the mean of the GEV (instead of its mode) as a systematic utility, we can reduce one parameter that we were required to specify in the previous model. The important point is that this contributes to simplifying the model structure without loss of theoretical foundation. In the

Table 1. Differences in model formulas between the current and previous q-generalized logit models

	Distribution assumption	Systematic utility	Utility function after logit transformation $f(v_{ij})$	$f'(v_{ij})$	$f''(v_{ij})$
Proposed model	U_{ij} ~ GEV	v_{ij} = mean of the GEV	$\ln\left[\exp_{2-q}(v_{ij})\right]$	$\frac{1}{1+(q-1)v_{ij}}$	$\frac{1-q}{\left\{1+(q-1)v_{ij}\right\}^2}$
Nakayama (2013)	U_{ij} ~ GEV	v_{ij} = mode of the GEV	$\ln\left[\exp_q\left\{v_{ij}/(s-\left[1-q\right]v_{ij})\right\}\right]$	$\frac{1}{s+(q-1)v_{ij}}s$	$\frac{1-q}{\left\{s+(q-1)v_{ij}\right\}^2}$

function $f(v_{ij})$ in the previous model, the systematic utility is divided by the function of systematic utility, causing instability of parameter estimation. Thus, some simplifications were required in the previous model, but we had no theoretical rationale for assuming that s is equal to one. From an empirical perspective, our current work provides the theoretical rationale for the simplification of the previous model.

In this section, the systematic utility, v_{ij}, is defined as $-\alpha(y_{ij1} + \beta y_{ij2})$, where α and β are unknown parameters, and y_{ij1} and y_{ij2} are travel cost and time, respectively. Under such settings, β can be understood directly as the value of travel time. Let τ_{ij} denote the generalized cost of route j for OD pair i, that is, $\tau_{ij} = y_{ij1} + \beta y_{ij2}$. Fig. 7 shows how the utility function changes with changes in parameter q. As the figure illustrates, a smaller q shows higher concavity of utility $f(v_{ij})$.

The parameters q and α in the proposed model could play a similar role under certain conditions, as shown in Fig. 8, possibly causing a model identification issue. This problem is due primarily to the fact that both q and α are related to both the scale and the degree of concavity of the utility $f(v_{ij})$: the scale of $f(v_{ij})$ is monotonically increasing with respect to q, while monotonically decreasing with respect to α, and the degree of concavity of $f(v_{ij})$ is monotonically decreasing with respect to q, while monotonically increasing with respect to α. To avoid this identification issue, an alternative model introduced in Chikaraishi and Nakayama (2015) can be used, since only parameter q is related to the elasticity of marginal utility with respect to the generalized cost τ_{ij} in the model. The alternative model shown in Chikaraishi and Nakayama (2015) is the case in which $f(v_{ij}) = -\alpha\ln_q(\tau_{ij})$. This model is also a natural extension of the conventional logit model and includes the logit and weibit models as special cases: $f(v_{ij}) = -\alpha(\tau_{ij} - 1)$, when $q = 0$ (logit model), and $f(v_{ij}) = -\alpha\ln(\tau_{ij})$, when $q = 1$ (weibit model). However, its utility does not follow GEV. A comparative analysis of different generalized logit models can be found in Chikaraishi and Nakayama (2015).

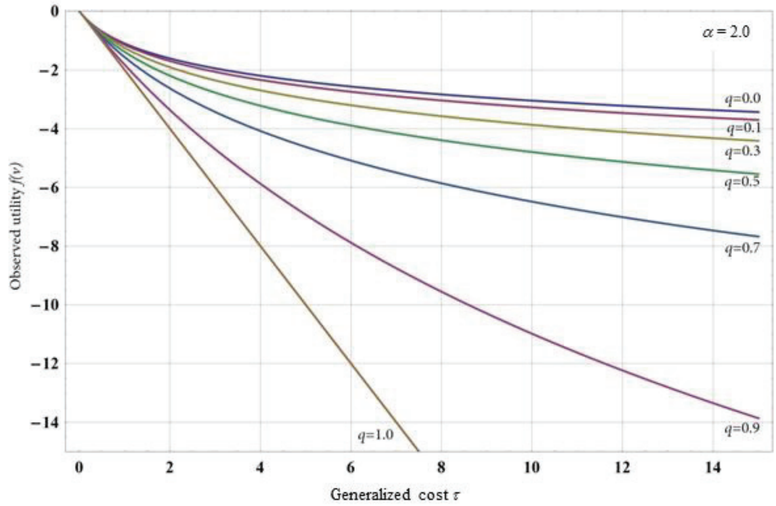

Fig. 7. Effect of q value on the utility function $f(v_{ij})$

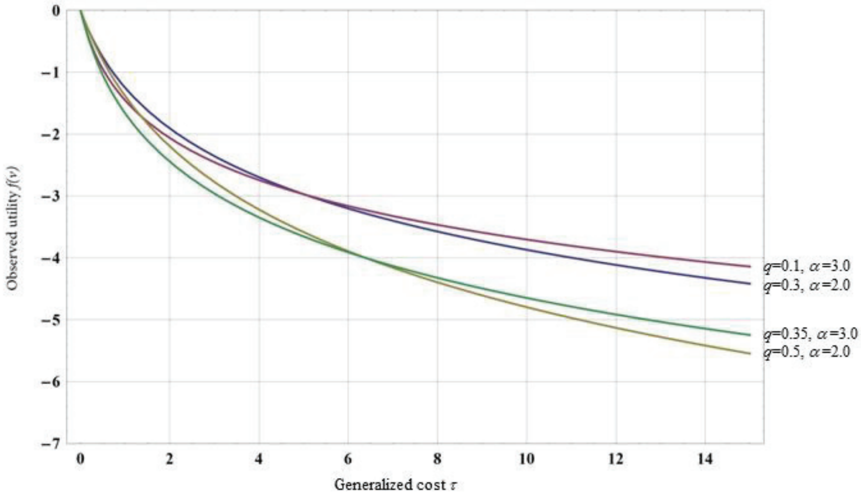

Fig. 8. Combination effects of q and α on the utility function $f(v_{ij})$

4.2. Simulation analysis

In the simulation analysis, a simple route choice problem is considered, as shown in Fig. 9. In total, we prepare seven datasets for simulation analysis as shown in Table 2. We first set the true parameter values for α, β, and q, and then generate the utility value on each route for 10,000 drivers by generating random numbers for (uniformly distributed) y_{ij1}, y_{ij2}, and the Gumbel-distributed error, ε_{ij}. We assume that drivers choose the route that has the maximum utility.

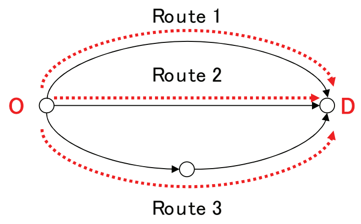

Fig. 9. Route choice problem considered in the simulation analysis

Table 2. Simulation datasets used in this study

	Utility function	Parameter value			y_{ij1}	y_{ij2}	ε_{ij}	Sample size
		α	β	q				
Dataset 1	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.0	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 2	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.1	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 3	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.3	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 4	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.5	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 5	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.7	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 6	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	0.9	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000
Dataset 7	Proposed: $\ln[\exp_{2-q}(v_{ij})]$	−2.0	1.5	1.0	Uniform[0.1, 1.0]	Uniform[0.1, 1.0]	Standard Gumbel	10,000

The model in this study is much simpler than the previous model of Nakayama (2013), and we can estimate the parameters in all datasets. Tables 3 and 4 show the estimation results of three different models, the logit, weibit, and proposed q-generalized logit models. Table 3 indicates that the model performances of the logit and weibit models measured by final log-likelihood are somewhat similar to those of the proposed model under particular values of q (particularly when q = 0.0 for weibit, and when q = 1.0 for logit), but the proposed model is superior to those models, especially when q is between 0.3 and 0.7. However, we also confirm that when q is underestimated, α is consistently overestimated, as shown in Table 4, implying that these two parameters can substitute for each other.

5. *q*-generalized logit traffic assignment and Tsallis entropy

5.1. q-generalized logit traffic assignment

The q-generalized logit route choice can be applied to traffic assignment. This is called the q-generalized logit traffic assignment in this study. That traffic assignment is formulated as a fixed-point problem in which the following equation is satisfied for any OD pair and any route:

Table 3. Estimation results of logit, weibit, and proposed models

	logit model				weibit model				proposed model (q-generalized logit model)						
	α		β		α		β		α		β		q[1]		
	param	t-value	param	t-value	param	t-value	param	t-value	param	t-value	param	t-value	param	t-value[2]	t-value[3]
Dataset 1	−0.54	−11.29	1.65	9.66	−0.69	−21.59	1.65	9.39	−2.06	−5.59	1.62	9.81	0.00	0.00	0.05
Dataset 2	−0.58	−12.09	1.64	10.34	−0.73	−23.01	1.64	10.07	−2.61	−4.90	1.62	10.46	0.00	0.00	0.35
Dataset 3	−0.68	−14.07	1.65	12.08	−0.86	−26.54	1.66	11.69	−4.49	−1.28	1.65	11.94	0.04	0.05	1.00
Dataset 4	−0.85	−17.25	1.61	14.82	−1.04	−31.45	1.65	14.06	−3.14	−2.27	1.62	14.55	0.33	2.46	4.91
Dataset 5	−1.12	−22.20	1.56	19.12	−1.32	−38.31	1.61	17.71	−2.33	−4.44	1.57	18.81	0.64	8.11	4.56
Dataset 6	−1.58	−29.51	1.52	25.94	−1.81	−47.63	1.58	23.75	−2.28	−7.55	1.53	25.70	0.85	11.38	2.05
Dataset 7	−1.99	−35.09	1.51	31.68	−2.24	−53.51	1.56	28.71	−2.09	−11.38	1.51	31.63	0.98	2.82	0.05

1) $q = \exp(qq)/(1+\exp(qq))$ and qq was estimated; 2) standard deviation was calculated based on delta method (null: $q = 0$); 3) standard deviation was calculated based on delta method (null: $q = 1$)

Table 4. Final log-likelihoods of logit, weibit, and proposed models

	Final log-likelihood		
	logit	weibit	proposed
dataset 1	–10742.7	–10733.6	–10733.8
dataset 2	–10708.6	–10698.4	–10698.4
dataset 3	–10608.0	–10596.4	–10595.6
dataset 4	–10435.0	–10425.6	–10422.4
dataset 5	–10112.3	–10116.4	–10102.7
dataset 6	–9470.9	–9506.6	–9465.5
dataset 7	–8844.2	–8935.3	–8844.1

$$p_{ij} = \frac{\exp_{2-q_i}\left[v_{ij}\right]}{\sum_{j'=1}^{J_i}\exp_{2-q_i}\left[v_{ij'}\right]} = \frac{\exp_{2-q_i}\left[-\alpha\, c_{ij}(\mathbf{p})\right]}{\sum_{j'=1}^{J_i}\exp_{2-q_i}\left[-\alpha\, c_{ij'}(\mathbf{p})\right]} \tag{21}$$

where $\mathbf{p}$ is the vector of route choice probabilities, $c_{ij}(\mathbf{p})$ is the travel cost function on route j for OD pair i, $v_{ij} = -\alpha c_{ij}(\mathbf{p})$, and α is a positive parameter. Clearly, $c_{ij}(\mathbf{p}) > 0$, and $v_{ij} < 0$ in the traffic assignment of Eq. (21). Therefore, $q_i \le 1$ is assumed, according to Eq. (8).

5.2. Non-congested network case

The multinomial logit equation of Eq. (14) can be obtained by maximizing the Shannon (or Boltzman–Gibbs) entropy. The Shannon entropy for route choice probabilities for OD pair i is $-\sum_j p_{ij} \ln p_{ij}$. As stated in section 2, the q-logarithm function is given as $\ln_q(x) = (x^{1-q} - 1)/(1 - q)$, and $\ln_1(x) = \ln(x)$ when $q = 1$. Using the q-logarithm function, the Tsallis entropy (e.g., Tsallis, 2009) is defined as

$$S_{q_i}(\mathbf{p}_i) = -\frac{1-\sum_{j=1}^{J_i} p_{ij}^{q_i}}{q_i - 1} = -\sum_{j=1}^{J_i} p_{ij}^{q_i} \ln_{q_i}(p_{ij}) \tag{22}$$

where $\mathbf{p}_i$ is the vector whose component is p_{ij} (j = 1, 2,…, J_i), and $S_{q_i}(\mathbf{p}_i)$ is the Tsallis entropy. When q_i = 1, $S_1(\mathbf{p}_i) = -\Sigma_j\, p_{ij} \ln p_{ij}$, the standard entropy (Shannon entropy). Thus, the Tsallis entropy is the q-generalized entropy, and includes the standard entropy. The following constrained maximization problem of the Tsallis entropy yields the q-generalized logit equation of Eq. (21):

$$\max_{\mathbf{p}_i}.\ S_{2-q_i}(\mathbf{p}_i) \tag{23}$$

$$s.t. \sum_{j=1}^{J_i} p_{ij} = 1 \tag{24}$$

In the case of a non-congested network, the q-generalized logit traffic assignment can be formulated as follows:

$$\begin{aligned} &\min. \sum_{i=1}^{I}\sum_{j=1}^{J_i} p_{ij}^{2-q_i}\left[-v_{ij} + \ln_{2-q_i}(p_{ij})\right] \\ &s.t. \sum_{j=1}^{J_i} p_{ij} = 1 \quad (i = 1, 2, ..., I) \end{aligned} \tag{25}$$

When q_i = 1, the objective function of the above problem is $\alpha\, \Sigma_i\, \Sigma_j\, p_{ij}\, c_{ij} + \Sigma_i\, \Sigma_j\, p_{ij} \ln p_{ij}$, because $v_{ij} = -\alpha\, c_{ij}$. This is identical to the objective function of Fisk’s optimization problem (Fisk, 1980) for logit-type stochastic user equilibrium under noncongested networks (e.g., see Oppenheim, 1995, p. 170 for Fisk’s problem). Thus, the above problem is a generalized optimization problem of traffic assignment with multinomial logit route choice (multinomial logit-based stochastic user equilibrium).

Define the following Lagrangean function:

$$L(\mathbf{p}) = \sum_{i=1}^{I}\sum_{j=1}^{J_i} p_{ij}^{2-q_i}\left[-v_{ij} + \ln_{2-q_i}(p_{ij})\right] - \sum_{i=1}^{I} \lambda_i \left(\sum_{j=1}^{J_i} p_{ij} - 1\right) \tag{26}$$

where λ_i is the Lagrangean multiplier for OD pair i. The condition necessary for solving the above minimization problem is to find the solution of $\partial L/\partial p_{ij} = 0$ for any i and j. Then,

$$\frac{\partial L}{\partial p_{ij}} = (2-q_i)\, p_{ij}^{1-q_i}\left[-v_{ij} + \ln_{2-q_i}(p_{ij})\right] + 1 - \lambda_i = 0 \tag{27}$$

because $d\ln_q(x)/dx = x^{-q}$, since $\ln_q(x) = (x^{1-q} - 1)/(1-q)$. The above is organized as

$$p_{ij}^{1-q_i} = \frac{1+(1-q_i)\lambda_i}{(2-q_i)\left[1+(q_i-1)v_{ij}\right]} \tag{28}$$

According to Eq. (27), λ_i must be greater than zero, because $\ln_q(p_{ij}) > 0$, $v_{ij} < 0$, and $q_i \le 1$. Therefore, $2-q_i$, $1+(1-q_i)\lambda_i$, and $1+(q_i-1)v_{ij}$ are all positive, and then,

$$p_{ij} = \frac{\left[1+(q_i-1)v_{ij}\right]^{\frac{1}{q_i-1}}}{\left[\frac{1+(1-q_i)\lambda_i}{2-q_i}\right]^{\frac{1}{q_i-1}}} = \frac{\exp_{2-q_i}(v_{ij})}{\left[\frac{1+(1-q_i)\lambda_i}{2-q_i}\right]^{\frac{1}{q_i-1}}} \tag{29}$$

Summing the above equation with respect to the routes yields

$$\sum_{j\in J_i} p_{ij} = 1 = \frac{\sum_{j\in J_i}\exp_{2-q_i}(v_{ij})}{\left[\frac{1+(1-q_i)\lambda_i}{2-q_i}\right]^{\frac{1}{q_i-1}}} \tag{30}$$

Combining Eqs. (29) and (30) gives

$$p_{ij} = \frac{\exp_{2-q_i}(v_{ij})}{\sum_{j'=1}^{J_i}\exp_{2-q_i}(v_{ij'})} \tag{31}$$

Thus, the minimization problem of Eq. (25) solves the q-generalized logit traffic assignment with Eq. (21).

5.3. Congested network case

Fisk’s optimization problem with logit route choice network equilibrium can also be applied to a congested network (Fisk, 1980). Note that Daganzo (1982) and Sheffi & Powell (1982) also examined the unconstrained formulation. Prashker & Bekhor (1999), Bekhor & Prashker (1999), Bekhor & Prashker (2001), and Chen *et al.* (2012) formulated the network equilibrium problems with cross-nested logit, paired combinatorial logit, generalized nested logit, and path-size logit, respectively. More recently, Zhou *et al*. (2012) considered the C-logit route choice in traffic assignment. Kitthamkesorn & Chen (2013, 2014) formulated an optimization problem of network equilibrium with weibit route choice, but they assumed that the route travel cost is the product of link costs or the sum of logarithms of link costs. However, it is difficult to formulate a q-generalized logit traffic assignment for a congested network as an optimization problem with a single integral and standard route travel cost structure. Aashtiani & Magnanti (1981) introduced a nonlinear complementarity problem for the traffic assignment problem, and we adopt that problem in this paper. Compared with the weibit network equilibrium formulation of Kitthamkesorn & Chen (2013, 2014), our formulation uses the nonlinear complementarity problem but assumes the standard travel cost structure. Let

$$f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = (2-q_i)\,p_{ij}^{\;1-q_i}\left[-v_{ij}(\mathbf{p}) + \ln_{2-q_i}(p_{ij})\right] + 1 - \lambda_i \tag{32}$$

and

$$f_i^{\lambda}(\mathbf{p}) = \sum_{j=1}^{J_i} p_{ij} - 1 \tag{33}$$

where $\boldsymbol{\lambda}$ is the vector of Lagrangean multipliers λ_i and v_{ij} is a function of $\mathbf{p}$ for a congested network, $v_{ij}(\mathbf{p})$. The complementarity problem of a q-generalized logit traffic assignment is to find $\mathbf{p}$ and $\boldsymbol{\lambda}$ subject to

$$\left\langle \mathbf{p}, \mathbf{f}^{p}(\mathbf{p},\lambda)\right\rangle + \left\langle \lambda, \mathbf{f}^{\lambda}(\mathbf{p})\right\rangle = 0,\ \ \mathbf{p} \ge \mathbf{0},\ \ \mathbf{f}^{p}(\mathbf{p},\lambda) \ge \mathbf{0},\ \ \lambda \ge \mathbf{0},\ \ \mathbf{f}^{\lambda}(\mathbf{p}) \ge \mathbf{0}, \tag{34}$$

where $\mathbf{f}^{p}(\mathbf{p},\lambda)$ and $\mathbf{f}^{\lambda}(\mathbf{p})$ are the vector-valued functions whose component functions are $f_{ij}^{p}(\mathbf{p},\lambda)$ and $f_i^{\lambda}(\mathbf{p})$, respectively, $\langle\cdot,\cdot\rangle$ is the inner product, and **0** is the zero vector.

As stated in the previous section, if Eq. (27), that is, $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = 0$ and $f_i^{\lambda}(\mathbf{p}) = 0$, holds, then the generalized logit model is obtained. Clearly, the necessary condition of $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = 0$ and $f_i^{\lambda}(\mathbf{p}) = 0$ under $\mathbf{p}, \boldsymbol{\lambda} \ge 0$ is the above problem. The sufficient condition is proven using reductio ad absurdum. If there exists $\mathbf{p}_i$ such that $\sum_{j=1}^{J_i} p_{ij} - 1 > 0$, then $\lambda_i = 0$. Clearly, there exists $p_{ij'} > 0$. $f_{ij'}^{p}(\mathbf{p},\boldsymbol{\lambda}) > 0$, even if $p_{ij'} > 0$, because $v_{ij} < 0$ and $\lambda_i = 0$. Then, $p_{ij'} f_{ij'}^{p}(\mathbf{p},\boldsymbol{\lambda}) > 0$. This contradicts the above complementarity problem. Thus, $\sum_{j=1}^{J_i} p_{ij} - 1 = 0$ for any OD pair. If $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) > 0$, $p_{ij} = 0$. Then, $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = 1 - \lambda_i > 0$, that is, $\lambda_i < 1$. There must be (at least one) $p_{ij'} > 0$ because of $\sum_{j=1}^{J_i} p_{ij} - 1 \ge 0$ and $\mathbf{p} \ge \mathbf{0}$, even if $p_{ij'} > 0$, $p_{ij'}^{\;q_i-1}\ln_{q_i}(p_{ij'})q_i - p_{ij'}^{\;q_i-1}q_i\hat{v}_{ij'} > 0$, and $\lambda_i < 1$, as stated above. Then, $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = 1 - \lambda_i > 0$ and $p_{ij} f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) > 0$. This also contradicts the above complementarity problem. Therefore, $f_{ij}^{p}(\mathbf{p},\boldsymbol{\lambda}) = 0$ for any i and j. Consequently, the sufficient condition is proved, and the above complementarity problem solves the q-generalized logit traffic assignment.

The q-generalized logit traffic assignment model can also be formulated as a fixed-point problem of Eq. (21). The problem is to find p_{ij} ($0 \le p_{ij} \le 1$) $\forall i$, with $\forall j$ subject to Eq. (21). The vector $\mathbf{p}$ is in the finite closed convex set for which $0 \le p_{ij} \le 1$ ($\forall i$, $\forall j$). The right-hand side of Eq.

(21) is also in the same closed convex set. Clearly, it is continuous in the set. According to Brouwer's fixed-point theorem, the existence of a solution to the problem is guaranteed.

If the Jacobian of $[\mathbf{f}^p(\mathbf{p}, \boldsymbol{\lambda}), \mathbf{f}^\lambda(\mathbf{p})]^T$ is positive definite, the solution is unique, where ^{T} is the transpose of a matrix or vector. For any $[\mathbf{p}', \boldsymbol{\lambda}']^T$,

$$\begin{bmatrix} \mathbf{p}' \\ \boldsymbol{\lambda}' \end{bmatrix}^T \begin{bmatrix} \nabla_{\mathbf{p}} \mathbf{f}^p & \nabla_\lambda \mathbf{f}^p \\ \nabla_{\mathbf{p}} \mathbf{f}^\lambda & \nabla_\lambda \mathbf{f}^\lambda \end{bmatrix} \begin{bmatrix} \mathbf{p}' \\ \boldsymbol{\lambda}' \end{bmatrix} = \mathbf{p}'^T \nabla_{\mathbf{p}} \mathbf{f}^p \mathbf{p}' \tag{35}$$

because $\nabla_\lambda \mathbf{f}^\lambda = \mathbf{0}$ and $\mathbf{p}'^T \nabla_\lambda \mathbf{f}^p \boldsymbol{\lambda}' = \boldsymbol{\lambda}'^T \nabla_{\mathbf{p}} \mathbf{f}^\lambda \mathbf{p}'$. If $\mathbf{p}'^T \nabla_{\mathbf{p}} \mathbf{f}^p \mathbf{p}' > 0$ for any $\mathbf{p}' \neq \mathbf{0}$, the q-generalized traffic assignment has a unique network flow. The Jacobian, $\nabla_{\mathbf{p}} \mathbf{f}^p$, is given by

$$\frac{\partial f_{ij}^p(\mathbf{p}, \boldsymbol{\lambda})}{\partial p_{i'j'}} = \begin{cases} (2-q) p_{ij}^{-q_i} \left[1 + (q_i - 1) v_{ij} - p_{ij} \dfrac{\partial v_{ij}}{\partial p_{ij}} \right] & if\, i' = i and\ j' = j \\ -(2-q_i) p_{ij}^{1-q_i} \dfrac{\partial v_{ij}}{\partial p_{ij'}} & if\, i' = i and\ j' \neq j \\ 0 & if\, i' \neq i \end{cases} \tag{36}$$

If $[c_{ij}(\mathbf{p}) - c_{ij}(\mathbf{p}')](\mathbf{p} - \mathbf{p}') > 0$, then $-[v_{ij}(\mathbf{p}) - v_{ij}(\mathbf{p}')](\mathbf{p} - \mathbf{p}') > 0$. In this case, $\nabla_{\mathbf{p}} \mathbf{f}^p$ is positive definite, and the uniqueness of the solution of the complementarity problem is guaranteed. However, the condition $[c_{ij}(\mathbf{p}) - c_{ij}(\mathbf{p}')]\ (\mathbf{p} - \mathbf{p}') > 0$ might be restrictive. The condition is not necessary and sufficient, but just necessary. There is some room for relaxing the uniqueness condition, and the relaxing of this condition will be a goal of our future study.

There are various ways of solving the complementarity problem. One is to reformulate the problem using quadratic Fischer–Burmeister functions. The Fischer–Burmeister function, $\phi(x,y)$, is $x + y - \sqrt{x^2 + y^2}$ (Fischer & Jiang, 2000), and Lo & Chen (2000) introduced it to traffic assignment formulation. The function is (always) non-negative, $\phi(x,y) \geq 0$, and $\phi(x,y) = 0$ is equivalent to $x \geq 0$, $y \geq 0$, and $x\,y = 0$. Therefore, the complementarity problem of solving $x\, f(x) = 0$ s.t. $x \geq 0$ and $f(x) \geq 0$ is reformulated as min $\phi(x, f(x))$. The solution of minimizing $\phi(x, f(x))$ without constraints is identical to that of the original complementarity problem. However, the Fischer–Burmeister function, $\phi(x,y)$, is not differentiable at $(x, y) = (0, 0)$. In this study, the following quadratic Fischer–Burmeister function is adopted:

$$L(\mathbf{p}, \boldsymbol{\lambda}) = \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \phi\left[p_{ij}, f_{ij}^p(\mathbf{p}, \boldsymbol{\lambda})\right]^2 + \frac{1}{2} \sum_{i=1}^{I} \phi\left[\lambda_i, f_i^\lambda(\mathbf{p})\right]^2 \tag{37}$$

Clearly, $L(\mathbf{p}, \boldsymbol{\lambda}) \geq 0$. A solution of the unconstrained optimization problem of minimizing $L(\mathbf{p}, \boldsymbol{\lambda})$ is identical to that of the above complementarity problem of Eq. (34). Many algorithms for

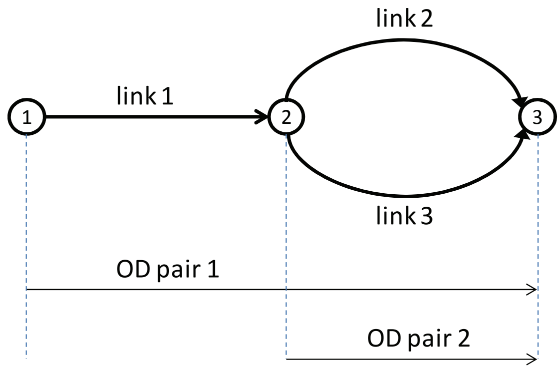

Fig. 10. Example network

unconstrained optimization problems have been developed. For example, a conjugate gradient method with the Polak–Ribiere formula, which guarantees its convergence, solves the above optimization problem.

5.4. Example

The q-generalized logit traffic assignment is applied to the simple example network shown in Fig. 10. This is one of the simplest networks with multiple OD pairs and multiple routes. The network has two OD pairs, and each OD pair has two routes. OD pair 1 contains nodes 1 and 3, and OD pair 2 contains nodes 2 and 3. Route 1 for OD pair 1 consists of links 1 and 2, and route 2 consists of links 1 and 3. On the other hand, route 1 for OD pair 2 consists of link 2 and route 2 consists of link 3. The demands of OD pairs 1 and 2 are both 150. Set $q_1 = q_2 = q$ and $\alpha = 2$. The travel time functions on the three links are

$$t_1(x) = t_3(x) = 15\left[1 + \left(\frac{x}{200}\right)^2\right] \tag{38}$$

$$t_2(x) = 10\left[1 + \left(\frac{x}{100}\right)^2\right] \tag{39}$$

where $t_a(\cdot)$ is the travel time function on link a (a = 1, 2, 3).

When q = 1, the q-generalized logit traffic assignment becomes the standard (multinomial) logit traffic assignment. Then,

$$p_{11} = \frac{\exp(-2c_{11})}{\exp(-2c_{11}) + \exp(-2c_{12})} = \frac{\exp(-2t_2)}{\exp(-2t_2) + \exp(-2t_3)} \tag{40}$$

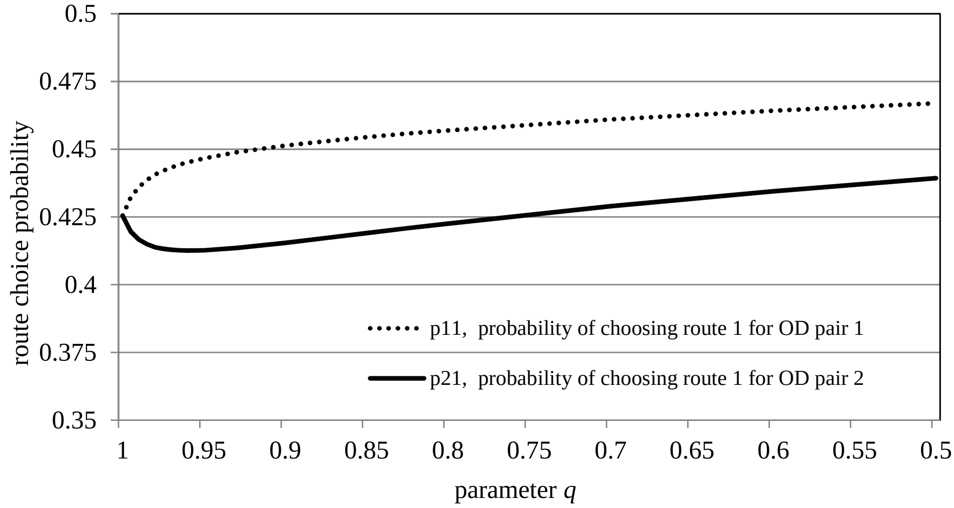

Fig. 11. Route choice probabilities and parameter *q*

$$p_{21} = \frac{\exp(-2c_{21})}{\exp(-2c_{21}) + \exp(-2c_{22})} = \frac{\exp(-2t_2)}{\exp(-2t_2) + \exp(-2t_3)} \tag{41}$$

Thus, $p_{11} = p_{21}$ and $p_{12} = p_{22}$, when $q = 1$.

As stated above, this example considers the case of $q \leq 1$. Then, the route choice probabilities are given by

$$p_{11} = \frac{\exp_{2-q}(-2c_{11})}{\exp_{2-q}(-2c_{11}) + \exp_{2-q}(-2c_{12})} \tag{42}$$

$$p_{21} = \frac{\exp_{2-q}(-2c_{21})}{\exp_{2-q}(-2c_{21}) + \exp_{2-q}(-2c_{22})} \tag{43}$$

In general, $p_{11} \neq p_{21}$ and $p_{12} \neq p_{22}$. Fig. 11 illustrates p_{11} and p_{21} of the solved *q*-generalized logit traffic assignment problem with different values of *q*. Note that the parameter *q* in the horizontal axis decreases in Fig. 11. When $q = 1$, $p_{11} = p_{21} = 0.425$. Because of homogeneity of variance in the standard multinomial logit model, $p_{11} = p_{21}$, if $q = 1$. Although $c_{11} - c_{12} = c_{21} - c_{22}$, c_{11} or $c_{12} >$ c_{21} or c_{22}, that is, the two route travel times for OD pair 1 are longer than those for OD pair 2. It is natural that the influence of the travel cost difference for OD pair 1 is smaller and $p_{11} > p_{21}$, as discussed above, even if the differences of the two route travel times are equal for the two OD pairs. As Fig. 11 shows, $p_{11} \neq p_{21}$ when $q \neq 1$, and we can confirm that the *q*-generalized logit model alleviates the homogeneity of variance. Initially, as *q* decreases until approximately 0.9, the variance of the utility on route 1 for OD pair 1 becomes increasingly different from that of OD pair 2. In other words, p_{11} and p_{21} differ. Then, the difference decreases gradually.

6. Conclusions

The multinomial logit model has a closed-form expression and is mathematically convenient. However, the Gumbel-distributed utility in the multinomial logit model is restrictive, especially in route choice behavior and network equilibrium analysis, owing to the homogeneity of variance. Especially in application to the modeling of route choice behavior and network equilibrium analysis, the homogenous variance of the logit model can cause serious biases in the analysis. In this study, the GEV distribution, which includes the Gumbel and Weibull distributions as special cases, was incorporated into the discrete choice model. The CDF of the GEV distribution was given by replacing the standard exponential function with the q-exponential function (a type of generalized exponential function) in the CDF of the Gumbel distribution. The q-generalized logit model with a GEV-distributed utility allows heteroscedastic variance and flexible shape and includes the multinomial logit and weibit models as special cases.

The parameter estimation of the q-generalized logit model was also examined. An identification problem in parameter estimation might occur under particular limited conditions. The results of an example of parameter estimation using simulated data indicated its applicability.

The generalized logit model with a GEV-distributed utility was incorporated into the transportation network equilibrium model. The network equilibrium model with generalized logit route choice was formulated as an optimization problem under uncongested networks. The objective function included the Tsallis entropy, a type of generalized entropy. For congested networks, it was formulated as the complementarity problem. The existence of equilibrium flows was proved, and a uniqueness condition was examined.

In this study, the Gumbel distribution, logit model, and network equilibrium model were considered in a unified framework of q-generalization with q-analysis, which included the operation of q-exponential and q-logarithm functions, or Tsallis statistics. In our future study, more relaxed conditions for unique equilibrium network flow will be examined. In addition, more empirical work is necessary to understand fully the properties of the proposed model. To do this, an efficient link-based algorithm must be developed for large-scale networks. To alleviate the overlapping problem in route choice, a commonality factor could be introduced into the model, for example by including it in Eq. (20) in an additive manner. This will be discussed in a later paper.

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