Analysis on Steady-State Vibration of Nonlinear System by Convolution Integral

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ANALYTICAL METHOD FOR STEADY STATE VIBRATION OF SYSTEM WITH LOCALIZED NONLINEARITIES USING CONVOLUTION INTEGRAL AND GALERKIN METHOD

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The analytical method using transfer function or impulse response is very effective for nonlinear systems with localized nonlinearities. This is because the nonlinear problem can be reduced to the problem on points connected with nonlinear element. In the present paper, analytical method for the steady state vibration which includes subharmonic vibration is proposed using convolution integral with the impulse response. The Galerkin method is introduced to solve the nonlinear equations formulated by the convolution integral, and then the steady state vibration is obtained. The present method has an advantage that stability or instability of the obtained steady state vibration can be discriminated, where a characteristic of the convolution integral is utilized. Numerical example of three degree-of-freedom mass-spring system is shown and the validity of the method is clarified through comparison with the result by Runge-Kutta-Gill method.

1. INTRODUCTION

Nonlinearities such as the nonlinear spring often exist in supported and connected points of mechanical structures, however, those number is remarkably smaller than that of the linear elements. The analytical method using transfer function or impulse response is effective for such nonlinear system with localized nonlinearities. This is because the nonlinear problem can be reduced to the problem on the supported and the connected points with nonlinear element, hence the size of nonlinear problem becomes small regardless of the number of degree-of-freedom of the system. In addition, when the transfer function or the impulsive response are measured for the existing system, the system damping can be easily modelized.

Vibration analyses using transfer function or impulse response for such systems with localized nonlinearities have ever been reported. Hagedorn et al. [1] showed that the transient vibration of such a system could be calculated by convolution integral with linear impulsive response and nonlinear restoring force. Chiang et al. [2] proposed the nonlinear substructure synthetic method by use of convolution integral and transition matrix [3] and analyzed the transient vibration of rotor-housing system with bearing nonlinear characteristics. Ren [4, 5] derived nonlinear equation from the transfer function synthesis method with the nonlinear connecting force and obtained the steady state vibration and aperiodic vibration by harmonic balance method. Gordis et al. [6] reported nonlinear substructure synthetic method using the nonlinear Volterra integral equation. The nonlinear equation for the steady state vibration using transfer function synthetic method can be commonly solved by the harmonic balance method, however, it is not always effective on predicting vibration response of the system in a practical sense, because it is impossible to discriminate stability and instability of the vibration which is the important vibration property.

In the present paper, the analytical method using convolution integral is proposed for steady state vibration of the system with localized nonlinearity including subharmonic vibration. The method also permits the stability-instability discrimination for the steady state vibration. The numerical results for fundamental and subharmonic vibration of three-degree-of-freedom system with localized nonlinear spring are illustrated and validity of the proposed method is discussed in comparison with Runge-Kutta-Gill method.

2. ANALYTICAL METHOD

2.1. SYSTEM WITH LOCALIZED NONLINEARITIES

The analytical method proposed in the present study is very effective for the system with localized nonlinearity. In the case of mass-spring system, for example, the nonlinear system includes a system where some masses are supported at respective points by a nonlinear spring as shown in Figure 1, or a system in which masses are connected each other with nonlinear spring. In this paper, the analytical method for the former system is demonstrated for its simplicity. m_p represents a mass subjected to external force $f(t)$ as shown in Figure 1, m_q a mass supported with nonlinear spring and m_r a mass of which vibration is determined. Displacements of m_q and m_r are represented as x and y respectively. $g(x)$ denotes restoring force property of the nonlinear spring. $f(t)$ is defined as $F\sin K\omega t$ so that the steady state vibration including subharmonic vibration of order $1/K$ is considered.

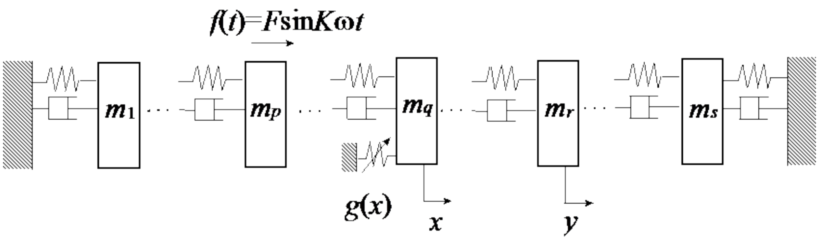


Figure 1 Multi-degree-of-freedom System with Localized Nonlinearity.

2.2 TRANSIENT VIBRATION

An outline of the method for transient vibration analysis using convolution integral is described in this section, which was reported by Hagedorn et al. and provides basis for the following steady state vibration analysis. When restoring force of the nonlinear spring supporting m_q is regarded as external force on m_q, displacement x of m_q and displacement y of m_r can be determined by convolution integral as the following equations (1) and (2), respectively,

$$x(t) = \int_0^t h^{qp}(t-\tau) f(\tau) d\tau - \int_0^t h^{qq}(t-\tau) g\{x(\tau)\} d\tau \tag{1}$$

$$y(t) = \int_0^t h^{rp}(t-\tau) f(\tau) d\tau - \int_0^t h^{rq}(t-\tau) g\{x(\tau)\} d\tau \tag{2}$$

where $h^{ab}(t)$ (a=q or r, b=p or q) is impulse response of the system without the nonlinear spring $g(x)$ and superscripts a and b represent the location of response and excitation, respectively. The initial displacement and initial velocity of the system are set to zero. Eq. (1) related to m_q represents a nonlinear equation with respect to x, on the other hand, for the case where the mass is not supported by nonlinear spring as in Eq. (2) it can be calculated linearly because $x(t)$ is already known in Eq. (1). When Eq. (1) and Eq. (2) are expressed in discrete form, they become as follows.

$$x_n = \sum_{k=0}^{n-1} h_{n-k}^{qp} f_k \Delta t - \sum_{k=0}^{n-1} h_{n-k}^{qq} g(x_k) \Delta t \tag{3}$$

$$y_n = \sum_{k=0}^{n-1} h_{n-k}^{rp} f_k \Delta t - \sum_{k=0}^{n-1} h_{n-k}^{rq} g(x_k) \Delta t \tag{4}$$

Δt is a time interval of discretization process, where $t=n\Delta t$ and $\tau=k\Delta t$. The subscript in Eq. (3) and Eq. (4) corresponds to the discretized time steps and then x_n shows displacement at $t=n\Delta t$ for example. Eq. (3) is expressed as the following matrix form along with the initial displacement $x_0=0$.

$$\begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} h_0^{qp} & 0 & \cdots & 0 \\ h_1^{qp} & h_0^{qp} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_n^{qp} & h_{n-1}^{qp} & \cdots & h_0^{qp} \end{bmatrix} \begin{Bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{Bmatrix} \Delta t - \begin{bmatrix} h_0^{qq} & 0 & \cdots & 0 \\ h_1^{qq} & h_0^{qq} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_n^{qq} & h_{n-1}^{qq} & \cdots & h_0^{qq} \end{bmatrix} \begin{Bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_n) \end{Bmatrix} \Delta t \tag{5}$$

x_n can be calculated from x_0, x_1, ...and x_{n-1} in Eq. (5), accounting that $h^{qp}{}_0=h^{qq}{}_0=0$. Therefore it is possible to obtain displacement x_n of a mass supported by nonlinear spring via forward substitution procedure increasing n successively. The displacement y_n of a mass which is not supported by the nonlinear spring can be easily calculated by substitution of x_0, x_1, ...and x_{n-1} to Eq. (4).

2.3 STEADY STATE VIBRATION

The steady state vibration including subharmonic vibration of order $1/K$ has K times period of external force. When time history response due to the periodic external force is divided into every K period, Eq. (5) is rewritten as follows.

$$\begin{Bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{Bmatrix} = \begin{bmatrix} \mathbf{h}_0^{qp} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{h}_1^{qp} & \mathbf{h}_0^{qp} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{h}_n^{qp} & \mathbf{h}_{n-1}^{qp} & \cdots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \mathbf{f} \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \end{Bmatrix} \Delta t - \begin{bmatrix} \mathbf{h}_0^{qq} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{h}_1^{qq} & \mathbf{h}_0^{qq} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{h}_n^{qq} & \mathbf{h}_{n-1}^{qq} & \cdots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \mathbf{g}(\mathbf{x}_0) \\ \mathbf{g}(\mathbf{x}_1) \\ \vdots \\ \mathbf{g}(\mathbf{x}_n) \end{Bmatrix} \Delta t \tag{6}$$

where the subscript shows the order of every K period. When a time interval corresponding to each K period of the external force is divided into M equal sections, $\mathbf{x}_i$ and $\mathbf{f}$ are expressed as column vectors which consist of M elements and $\boldsymbol{g}(\mathbf{x}_i)$ a column vector which consists of M elements of restoring force $g(x_i)$.

$$\mathbf{x}_i = \begin{Bmatrix} x_{i\times M} \\ x_{i\times M+1} \\ \vdots \\ x_{(i+1)M-1} \end{Bmatrix}, \quad \mathbf{f} = F \begin{Bmatrix} \sin 2\pi K \dfrac{0}{M} \\ \sin 2\pi K \dfrac{1}{M} \\ \vdots \\ \sin 2\pi K \dfrac{M-1}{M} \end{Bmatrix}, \quad \mathbf{g}(\mathbf{x}_i) = \begin{Bmatrix} g(x_{i\times M}) \\ g(x_{i\times M+1}) \\ \vdots \\ g(x_{(i+1)M-1}) \end{Bmatrix} \tag{7}$$

$\mathbf{h}^{ab}{}_i$ is $M\times M$ matrix which consists of the impulse response $h^{ab}{}_j$, written as

$$\mathbf{h}_0^{ab} = \begin{bmatrix} h_0^{ab} & 0 & \cdots & 0 \\ h_1^{ab} & h_0^{ab} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{M-1}^{ab} & h_{M-2}^{ab} & \cdots & h_0^{ab} \end{bmatrix}, \qquad \mathbf{h}_i^{ab} = \begin{bmatrix} h_{i\times M}^{ab} & \cdots & \cdots & h_{(i-1)M+1}^{ab} \\ h_{i\times M+1}^{ab} & h_{i\times M}^{ab} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ h_{(i+1)M-1}^{ab} & h_{(i+1)M-2}^{ab} & \cdots & h_{i\times M}^{ab} \end{bmatrix} \quad (i \geq 1) \tag{8}$$

$\mathbf{x}_i$ (i=0,1,....,n) in Eq. (6) are individually different vectors because { $\mathbf{x}_0{}^T$, $\mathbf{x}_1{}^T$, ..., $\mathbf{x}_n{}^T$ } corresponds to the transient vibration derived from Eq. (5), where (^{T}) denotes transposed matrix. When the time has sufficiently passed, it is considered that the transient vibration converges to steady state vibration. Then Eq. (6) can be rewritten under such a convergence as follows,

$$\begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{x} \\ \vdots \\ \mathbf{x} \\ \mathbf{x} \end{Bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \vdots \\ \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} & \vdots \\ \ddots & \mathbf{h}_i^{qp} & \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} \\ \ddots & \mathbf{h}_{i+1}^{qp} & \mathbf{h}_i^{qp} & \ddots & \ddots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \\ \mathbf{f} \end{Bmatrix} \Delta t - \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \vdots \\ \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} & \vdots \\ \ddots & \mathbf{h}_i^{qq} & \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} \\ \ddots & \mathbf{h}_{i+1}^{qq} & \mathbf{h}_i^{qq} & \ddots & \ddots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) \end{Bmatrix} \Delta t \tag{9}$$

where $\mathbf{x}$ is a vector corresponding to time history response of the steady state vibration. If the system has damping, the impulse response $h^{ab}(t)$ converges to zero in adequate course of time. Hence $\mathbf{h}^{ab}{}_i$ which consists of the impulse response data is regarded as zero when the subscript is larger than N, that is, $\mathbf{h}_N^{ab} = \mathbf{h}_{N+1}^{ab} = \cdots = \mathbf{0}$. Therefore the next equation (10) can be obtained by calculating bottom row of Eq. (9).

$$\mathbf{x} = \mathbf{H}^{qp}\mathbf{f} - \mathbf{H}^{qq}\mathbf{g}(\mathbf{x}) \tag{10}$$

where $\mathbf{H}^{ab}$ is represented as,

$$\mathbf{H}^{ab} = \sum_{i=0}^{N-1} \mathbf{h}_i^{ab} \Delta t \tag{11}$$

N is the smallest value of i which satisfies the following relationship.

$$\frac{\sqrt{\sum_{k=i\times M}^{(i+1)M-1}\left(h_k^{ab}\right)^2}}{\sqrt{\sum_{k=0}^{i\times M-1}\left(h_k^{ab}\right)^2}} \le \varepsilon \tag{12}$$

Eq. (12) signifies that the ratio between the magnitude (square root of the squared sum) at a single (i+1)th period of the impulse response and the magnitude up to the ith period becomes smaller than ε. Note that N for h_i^{qp} are different from N for h_i^{qq}. $\varepsilon=10^{-5}$ is adopted in the following example. The equation on **y**, which determines the steady state vibration of m_r, is obtained through the same procedure expressed as follows,

$$\mathbf{y} = \mathbf{H}^{rp}\mathbf{f} - \mathbf{H}^{rq}\mathbf{g}(\mathbf{x}) \tag{13}$$

If Eq. (10) is solved for **x**, the time history of steady-state vibration x is obtained, so that y can be calculated by substituting **x** into Eq. (13). However, it is difficult to solve such simultaneous nonlinear equations containing M unknowns as Eq. (10) generally. Therefore, the Galerkin method is adopted in order to acquire approximate solution, which is explained as below.

When the subharmonic vibration of order 1/K is assumed, the solution of Eq. (10) is expressed as follows.

$$\mathbf{x} = A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K \tag{14}$$

where $\mathbf{S}_1$, $\mathbf{C}_1$, $\mathbf{S}_K$ and $\mathbf{C}_K$ represent the following vectors, which consist of sine or cosine function divided into M terms equally.

$$\mathbf{S}_1 = \begin{Bmatrix} \sin 2\pi\frac{0}{M} \\ \sin 2\pi\frac{1}{M} \\ \vdots \\ \sin 2\pi\frac{M-1}{M} \end{Bmatrix},\quad \mathbf{C}_1 = \begin{Bmatrix} \cos 2\pi\frac{0}{M} \\ \cos 2\pi\frac{1}{M} \\ \vdots \\ \cos 2\pi\frac{M-1}{M} \end{Bmatrix},\quad \mathbf{S}_K = \begin{Bmatrix} \sin 2\pi K\frac{0}{M} \\ \sin 2\pi K\frac{1}{M} \\ \vdots \\ \sin 2\pi K\frac{M-1}{M} \end{Bmatrix},\quad \mathbf{C}_K = \begin{Bmatrix} \cos 2\pi K\frac{0}{M} \\ \cos 2\pi K\frac{1}{M} \\ \vdots \\ \cos 2\pi K\frac{M-1}{M} \end{Bmatrix} \tag{15}$$

A_1 and B_1 denote magnitude of the subharmonic component of order 1/K, and A_K and B_K magnitude of the fundamental harmonic component. Hereby the second equation of Eqs. (7) yields the expression of $\mathbf{f}=F\mathbf{S}_K$. Substituting Eq. (14) into Eq. (10) and multiplying $\mathbf{S}_1^T$, $\mathbf{C}_1^{\,T}$, $\mathbf{S}_K^{\,T}$ and $\mathbf{C}_K^{\,T}$ respectively on each term from its left hand side, the following equations are obtained.

$$\left.\begin{aligned}
\frac{M}{2}A_1 &= F\mathbf{S}_1{}^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{S}_1{}^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}B_1 &= F\mathbf{C}_1{}^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{C}_1{}^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}A_K &= F\mathbf{S}_K{}^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{S}_K{}^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}B_K &= F\mathbf{C}_K{}^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{C}_K{}^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K)
\end{aligned}\right\} \tag{16}$$

A_1, B_1, A_K and B_K can be calculated from Eqs. (16) by the Newton-Raphson Method, and the time history of x can be obtained from Eq. (14). If A_K and B_K are calculated by Eqs. (16) under the condition of A_1=B_1=0, solution on fundamental harmonic vibration is obtained.

The solution $\mathbf{y}$ of Eq. (13) is also expressed as follows.

$$\mathbf{y} = C_1\mathbf{S}_1 + D_1\mathbf{C}_1 + C_K\mathbf{S}_K + D_K\mathbf{C}_K \tag{17}$$

Substituting Eq. (17) into Eq. (13) and multiplying $\mathbf{S}_1{}^T$, $\mathbf{C}_1{}^T$, $\mathbf{S}_K{}^T$ and $\mathbf{C}_K{}^T$ respectively on each term of equation, the following equations are obtained.

$$\left.\begin{aligned}
\frac{M}{2}C_1 &= F\mathbf{S}_1{}^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{S}_1{}^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}D_1 &= F\mathbf{C}_1{}^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{C}_1{}^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}C_K &= F\mathbf{S}_K{}^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{S}_K{}^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K) \\
\frac{M}{2}D_K &= F\mathbf{C}_K{}^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{C}_K{}^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K)
\end{aligned}\right\} \tag{18}$$

A_1, B_1, A_K and B_K obtained from Eqs. (16) are substituted into Eq. (18), hence C_1, D_1, C_K and D_K, that is, the steady state vibration of y, can be determined.

3. DISCRIMINATION OF STABILITY AND INSTABILITY

Not only that the steady state vibration including subharmonic vibration is determined from Eq. (10) as shown in the previous section, but also unstable steady state vibration is obtained since the periodic solution is assumed as shown in Eq. (14). Discrimination method for stability and instability of the steady state vibration is described in this section, where the convolution integral is used to calculate the transient vibration.

The discrimination can be performed by investigating the transient response of the initial state set to the steady state solution. If it diverges from the initial steady state vibration and then converges to other steady state, the initial steady state vibration is regarded as unstable, otherwise it is discriminated as stable steady state. The transient vibration for a given initial steady state solution can be calculated by the procedure for transient vibration as already described in the section 2.2. Such calculation is expressed as follows,

$$\begin{Bmatrix} \mathbf{x} \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_i \end{Bmatrix} = \begin{bmatrix} \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \mathbf{f} \\ \vdots \\ \mathbf{f} \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \end{Bmatrix} \Delta t - \begin{bmatrix} \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \mathbf{g}(\mathbf{x}) \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}_1) \\ \vdots \\ \mathbf{g}(\mathbf{x}_i) \end{Bmatrix} \Delta t \tag{19}$$

where the equations of the transient vibration are based on the bottom row of Eq.(9), and the convergence of impulse response to become zero is considered, that is, $\mathbf{h}_n{}^{qp}=0$ ($n \geq N'$) and $\mathbf{h}_n{}^{qq}=0$ ($n \geq N$).

Since $\mathbf{x}$ in Eq. (19) is already obtained as steady state solution, the calculation of the transient vibration starts from $\mathbf{x}_1$ of the second row in Eq. (19) and successively the time history response $\mathbf{x}_i$ can be calculated by the forward substitution procedure since the diagonal elements of $\mathbf{h}_0{}^{qq}$ are zero. When A_1', B_1', A_K' and B_K' represent magnitude coefficients of $\sin\omega t$, $\cos\omega t$, $\sin K\omega t$ and $\cos K\omega t$ components of $\mathbf{x}_i$ respectively, they are obtained from the following equations.

$$A_1' = \frac{2}{M}\mathbf{S}_1{}^T\mathbf{x}_i \ , \quad B_1' = \frac{2}{M}\mathbf{C}_1{}^T\mathbf{x}_i \ , \quad A_K' = \frac{2}{M}\mathbf{S}_K{}^T\mathbf{x}_i \ , \quad B_K' = \frac{2}{M}\mathbf{C}_K{}^T\mathbf{x}_i \tag{20}$$

In the case of subharmonic vibration of order $1/K$, A_1' and B_1' are compared with A_1 and B_1 of the steady state vibration, hereby it is regarded as stable if the following relationship is satisfied.

$$\frac{\sqrt{(A_1 - A_1')^2 + (B_1 - B_1')^2}}{\sqrt{A_1{}^2 + B_1{}^2}} \leq \frac{1}{10} \tag{21}$$

Eq. (21) denotes that both the difference between A_1 and A_1' and also the difference between B_1 and B_1' are small. The stability condition in right side of Eq. (21) seems comparatively moderate, since the difference between the approximate solution of A_1 and B_1 obtained by the Galerkin method and the exact solution $\mathbf{x}_i$ by the transient calculation frequently becomes large. In the case of the fundamental harmonic vibration, the following condition with A_K, A_K', B_K and B_K' is adopted.

$$\frac{\sqrt{(A_K - A_K')^2 + (B_K - B_K')^2}}{\sqrt{{A_K}^2 + {B_K}^2}} \le \frac{1}{10} \tag{22}$$

4. EXAMPLE FOR THREE DEGREE-OF-FREEDOM MASS-SPRING SYSTEM

A numerical example on fundamental harmonic vibration and subharmonic vibration of order 1/3 in three degree-of -freedom mass-spring system of Figure 2 is shown. m_1 is subjected to sinusoidal wave force and m_2 is supported by a nonlinear spring. The property of nonlinear spring is defined as $g(x)=\beta x^3$. The system parameters used in the example are shown in Table 1. The case K=3 is illustrated so that the subharmonic vibration of order 1/3 is considered. The number of division in three periods is set to M=60. The impulse response of each mass is determined by modal analysis on free vibration for corresponding given initial velocity.

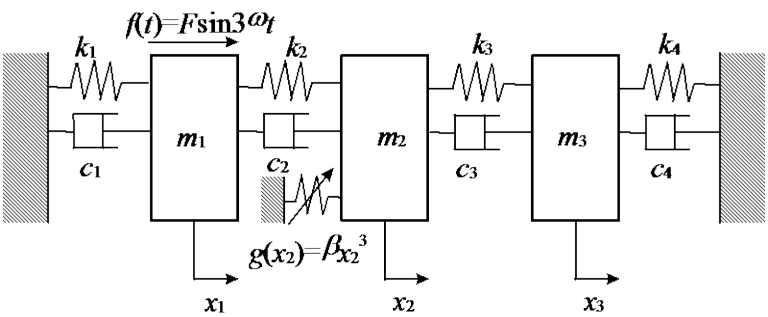


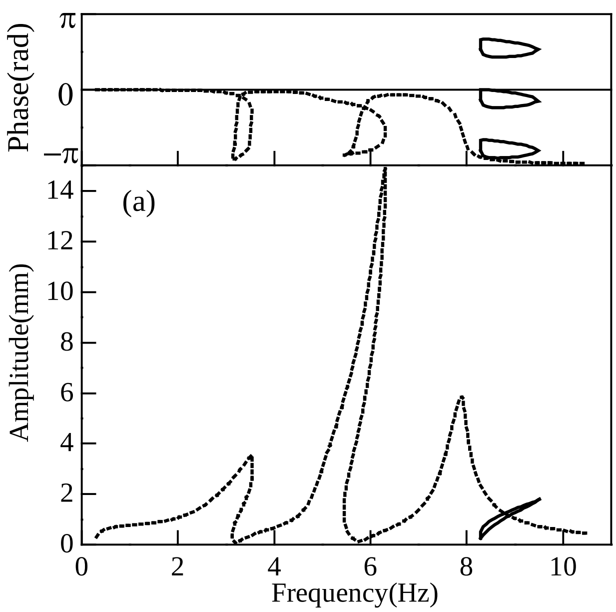
Figure 2 Three-degree-of-freedom mass-spring system.

TABLE 1

System Parameters

Mass (kg)	Damping Coefficient (Ns/m)	Spring Constant (N/m)	Nonlinear Spring Constant (N/m^3)	Amplitude of Excitation (N)
m_1=1.0	c_1=1.0	k_1=1000	β=1.0×10^8	F=1.0
m_2=2.0	c_2=1.0	k_2=1000		
m_3=3.0	c_3=1.0	k_3=1000		
	c_4=1.0	k_4=1000		

Response curves of the fundamental harmonic vibration and the subharmonic vibration of order 1/3 on m_1, m_2 and m_3 are shown in Figure 3(a), 3(b) and 3(c) respectively. The dotted line denotes amplitude $\sqrt{A_3^2 + B_3^2}$ and phase $\tan^{-1}(B_3 / A_3)$ of the fundamental harmonic vibration, and also the solid line denotes amplitude $\sqrt{A_1^2 + B_1^2}$ and phase $\tan^{-1}(B_1 / A_1)$ of the subharmonic vibration. The amplitude and the phase of the fundamental component in the subharmonic vibration almost coincide with those of the fundamental harmonic vibration, hence they are omitted in the figures. There are three solutions for the subharmonic vibration which have the same amplitude, whereas the phase has difference of 2π/3 radian respectively. In Figure 4, the calculation result of m_2 using the convolution integral is compared with the result of the Runge-Kutta-Gill method for equations of motion of the system shown in Figure 2, where the amplitude is represented as RMS value. Furthermore stability and instability of the steady state vibration are discriminated in Figure 4. It is found that both results sufficiently coincide on the stable vibration.



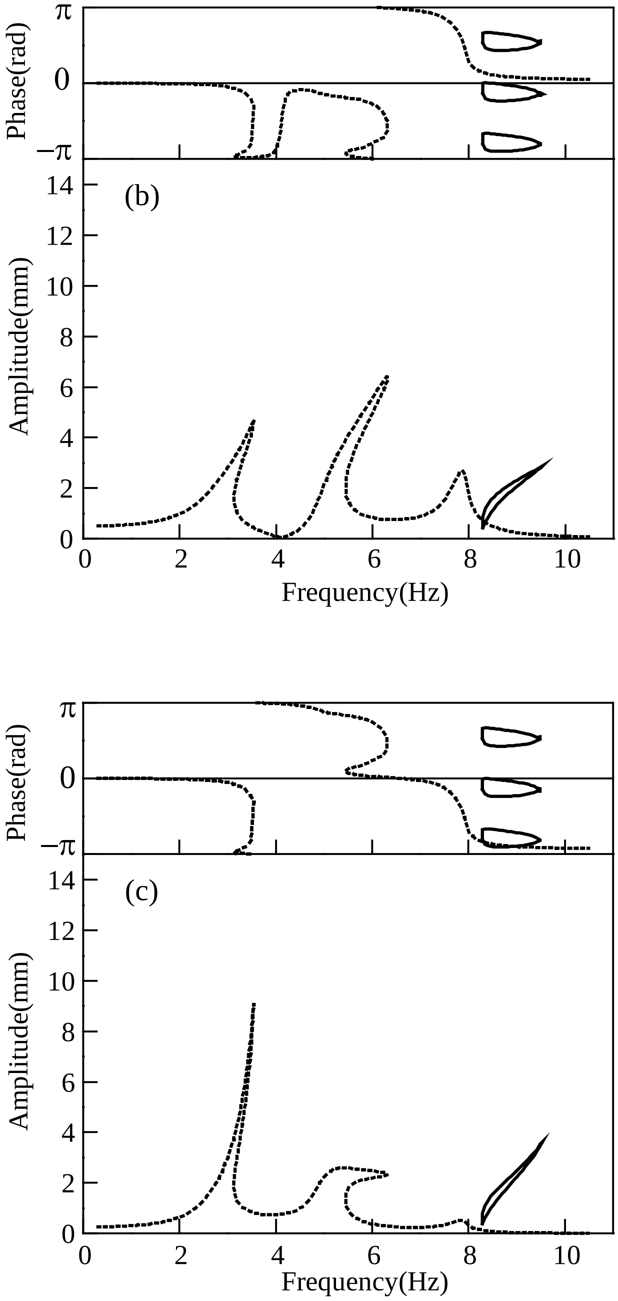


Figure3 Response curve. (a) m_1 , (b) m_2 , (c) m_3 . ··················· ; fundamental harmonic vibration. ———— ; 1/3 subharmonic vibration component.

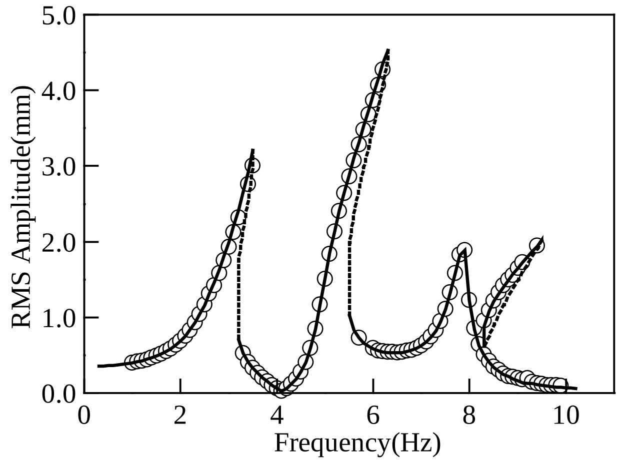


Figure 4 Comparison with RKG method. ——— ; stable vibration by convolution integral, ··········; unstable vibration by convolution integral, ○; stable vibration by RKG method.

Calculation result of stability analysis for the subharmonic vibration is represented schematically in Figure 5. Coordinates of A_1' and B_1' are the coefficients of sinωt and cosωt terms calculated in Eq. (20), and those values of transient vibration are plotted as dotted line in Figure 5 along with the increase of time. If the steady state vibration is stable, location (A_1', B_1') does not move, while it leaves the steady state point if unstable. However, even if stable, the convergent point of the transient vibration differs from the stable steady state vibration point obtained by the convolution integral for the reason as already described in section 3. The calculation result for the case 8.29Hz is shown in Figure 5(a), where the unstable subharmonic vibration of order 1/3 converges to another stable one. In the case of 9.00Hz which is shown in Fig 5(b), the unstable vibration converges to the fundamental vibration.

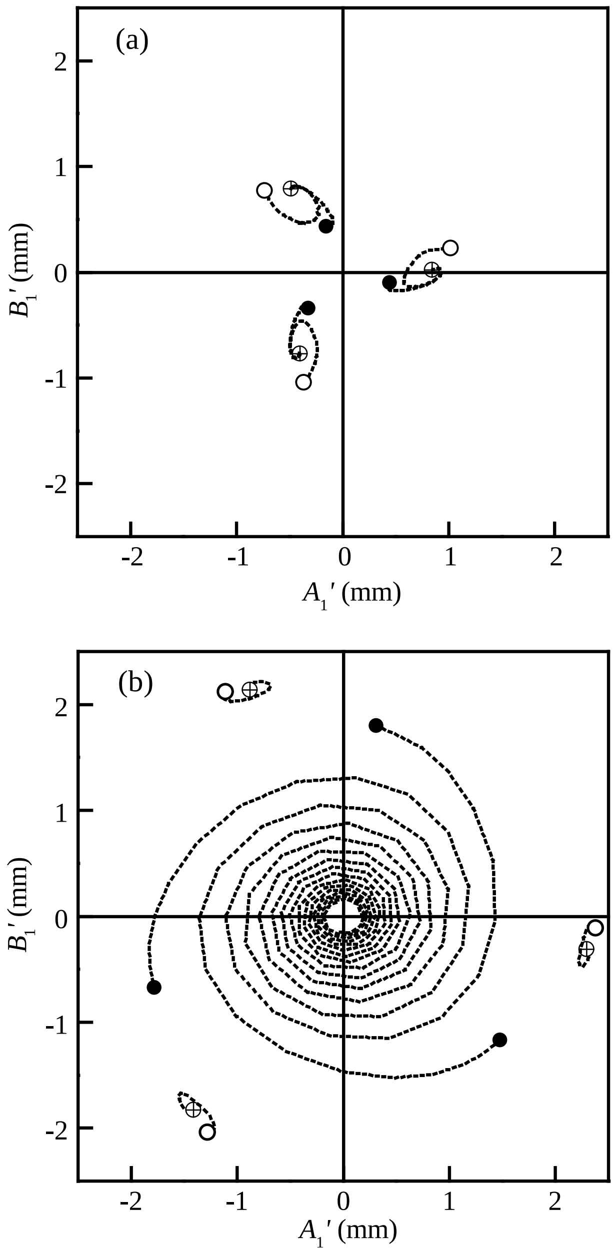


Figure 5 Stability and instability in subharmonic vibration of order 1/3. (a) 8.29Hz, (b) 9.00Hz. ··················; orbit of transient vibration, –; stable vibration, ,; unstable vibration, /; convergent point.

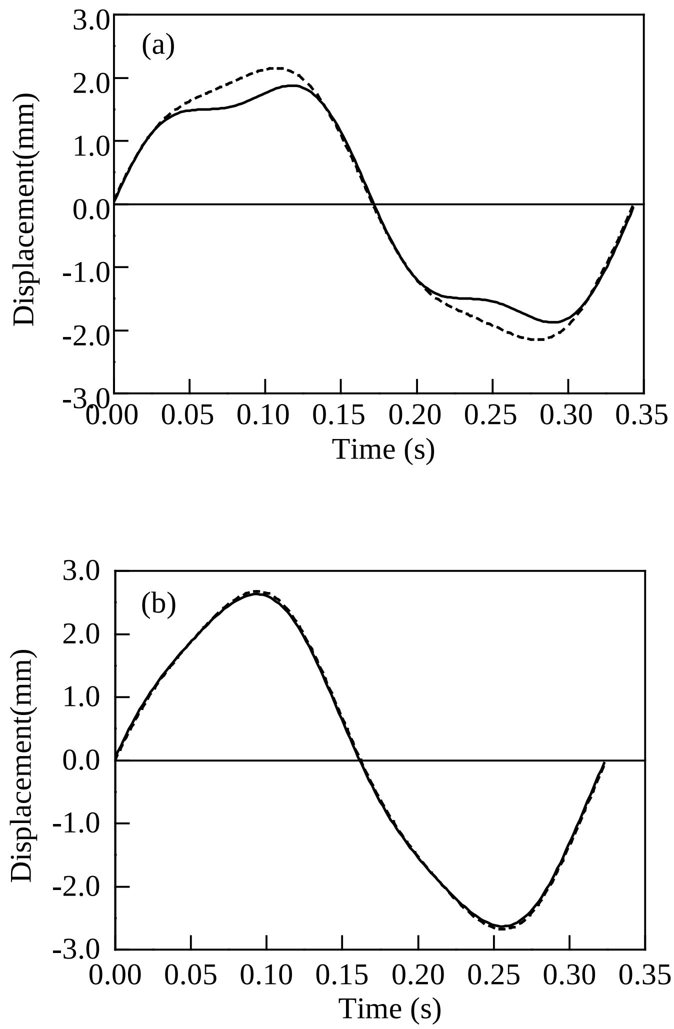


Figure 6 Comparison of direct method and Galerkin approximation. (a) 8.70Hz; (b) 9.24Hz.

——————; direct method, ························ ; Galerkin method.

To discuss the accuracy of the approximate solution using the Galerkin method, Eq. (10) is solved directly by the Newton-Rapson method (which is called the direct method henceforth) and the result is compared with that of the Galerkin method. The time history response obtained by the Galerkin method is given as initial value for **x** in the direct method. Comparison between 1/3 order subharmonic vibrations of both waves is shown in Figure 6. For the case 8.70Hz which is shown in Figure 6(a), the Galerkin method(dotted line) is slightly different from the direct method(solid line). However, the both lines almost coincide in Figure 6(b) of 9.24Hz case. Both waves coincide on most region of the response curve, while the case where both waves do not coincide is limited in the vicinity of an edge of the subharmonic region. It is considered that the overall system with localized nonlinearity exhibits a weak nonlinearity, although the localized nonlinearity is strong. Therefore valid solution of Eq. (10) can be determined by the Galerkin method with assumption of the simple solution such as Eq. (14).

5. CONCLUSIONS

The analysis of steady state vibration using convolution integral is proposed for the system with localized nonlinearities and it is shown that the nonlinear equation formulated by suggested analytical process can be solved by the Galerkin method. Subharmonic vibration is also included in the steady state vibration. Furthermore, the stability of each solution can be discriminated by the transient analysis using the convolution integral. Numerical example of three degree-of-freedom mass-spring system is demonstrated and the following results are obtained. (1)Response curves of the fundamental harmonic vibration and the subharmonic vibration of order 1/3 can be easily calculated by the present method. (2)Stability or instability of the steady state vibration can be discriminated by the transient analysis using convolution integral. (3)The result obtained from the convolution integral coincides well with the result by the RKG method. Therefore the present method is efficient for the steady state vibration analysis of nonlinear system with localized nonlinearities.

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