# N dependence of upper bounds of critical temperatures of 2D O(N) spin models

# $N$ Dependence of Upper Bounds of Critical Temperatures of 2D $O(N)$ Spin Models

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## Abstract

We investigate critical temperature of the classical $O(N)$ spin model in two dimensions. We show that if $N$ is large and there is a phase transition in the system, the critical inverse temperature $\beta_c$ obeys the bound $\beta_c(N) >$ const. $N \log N$.

**Running Head: Critical Temperature of 2D $O(N)$ Spin Model**

### I. INTRODUCTION

Quark confinement in 4 dimensional non-abelian lattice gauge thoeries and spontaneous mass generations in two dimensional (2D) non-abelian sigma models are widely believed

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[18]. These models exhibit no phase transitions in the hierarchical model approximation of Wilson-Dyson type or Migdal-Kadanov type [10], but we still do not have a rigorous proof for the real system.

We recently considered a block-spin-type transformation of random walk which appears in the $O(N)$ spin models [3,4], and showed that [11] the correlation functions are represented by self-avoiding walks on $\mathbf{Z}^\nu$. This considerably improves our previous estimates for the inverse critical temperature $\beta_c$ of the system

$$\frac{\beta_c}{N} \geq \frac{\mu_\nu}{\mu_\nu^2 - 1}, \text{ as } N \to \infty \tag{1.1}$$

where $\mu_\nu \in (\nu, 2\nu - 1)$ is the connective constant of self-avoiding walk on $\mathbf{Z}^\nu$ ($\mu_2 = 2.653\cdots$). In this paper, we amalgamate our previous methods with the idea of the $N^{-1}$ expansion [14,15] and the cluster expansion [5,9,13,16], the technology to represent quantities of infinite volume limit by finite volume quantities. In a spirit, our single block cluster expansion is similar to that in [1]. Our main conclusion in this paper is

**Main Theorem** *The critical inverse temperature $\beta_c(N)$ of the two-dimensional $O(N)$ Heisenberg Model obeys the following bound for large $N$:*

$$\beta_c(N) > \text{const.}\, N \log N \tag{1.2}$$

*where* const. $> 0$ *is independent of $N$.*

This result is announced in [12]. As will be discussed, for the dimension $\nu > 2$, we have

$$G_0(0) \geq \frac{\beta_c(N)}{N} \geq \frac{1}{\mu_\nu} \tag{1.3}$$

where $G_0(x)$ is the lattice Green's function on the $\nu$ dimensional lattice $\mathbf{Z}^\nu$. Therefore a strong deviation exists in the $N$ dependence of the critical temperature of the 2D $O(N)$ Heisenberg model. We expect a combination of the present method and renormalization group type argumemts will establish our longstanding conjecture on the 2D sigma model.

The $\nu$ dimensional $O(N)$ spin (Heisenberg) model is defined by the Gibbs measure

$$< F > \equiv \frac{1}{Z_\Lambda(\beta)} \int F(\phi) \exp[-H_\Lambda(\phi)] \prod_i \delta(\phi_i^2 - 1) d\phi_i. \tag{1.4}$$

Here $\Lambda \subset \mathbf{Z}^\nu$ is the large square with its center at the origin. Moreover $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$, $Z_\Lambda$ is the partition function defined so that $< 1 >= 1$. $H_\Lambda$ is the Hamiltonian given by

$$H_\Lambda \equiv -\frac{\beta(N)}{2} \sum_{|x-y|_1=1} \phi(x)\phi(y), \tag{1.5}$$

where $|x-y|_1 = \sum_i |x_i - y_i|$ and $\beta(N)$ is the inverse temperature. To appeal to the $1/N$ expansion [15], we set

$$\beta(N) = N\beta. \tag{1.6}$$

We organize the paper as follows: in Sect.2, we represent the theory in terms of a determinant by introducing an auxialiary field $\psi$ and integrating out the spin variables. We discuss the reason why phase transitions may not occur in two-dimensional systems which have $O(N)$ symmetries. In Sect.3, we argue the polymer expansion when $|\psi(x)|$ are all small.

Sect.4 is the main part of this paper in which we prove that the contributions from large fields are small and negligible. Since $\psi(x)$ can get large, we decompose $\Lambda$ into two regions, the large and the small field regions and we estimate their contributions separately. The polymer expansion will be done combining these two regions. In Sect. 5, we represent the free energy by the convergent polymer expansion, from which analyticity of the free energy follows. We discuss some related problems in Sect. 6.

In Appendixes, we calculate decay rates and inverses of Green's functions used in this paper. We also discuss polymer expansions of Green's functions and Gaussian measures restricted to subsets of $\mathbf{Z}^2$.

## II. DETERMINANT REPRESENTATION

We substitute the identity $\delta(\phi^2 - 1) = \int \exp[-ia(\phi^2 - 1)] da/2\pi$ into eq.(1.4) with the condition [3,4] that $\mathrm{Im} a_i \le -\nu N\beta$. We set

$$\mathrm{Im}\, a_i = -N\beta(\nu + \frac{m^2}{2}),\ \mathrm{Re}\, a_i = \sqrt{N}\beta\psi_i, \tag{2.1}$$

where $m^2 \geq 0$ will be determined soon. Thus we have

$$
\begin{aligned}
Z_\Lambda &= c^{|\Lambda|} \int \cdots \int \exp[-\frac{N\beta}{2} < \phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)\phi > + \sum_j i\sqrt{N}\beta\psi_j] \prod \frac{d\phi_j d\psi_j}{2\pi} \\
&= c^{|\Lambda|} \int \cdots \int \det(m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j] \prod \frac{d\psi_j}{2\pi} \\
&= c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi}, \qquad (2.2)
\end{aligned}
$$

where $c$ are constants which may be different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|_1,1}$ is the lattice laplacian and

$$
F(\psi) = \det(1 + \frac{2iG}{\sqrt{N}}\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j]. \qquad (2.3)
$$

Moreover $G = (m^2 - \Delta)^{-1}$ is Green's function (matrix) discussed later. In the same way, the two point functions are given by

$$
< \phi_0 \phi_x > = \frac{1}{\tilde{Z}} \int \cdots \int (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)^{-1}_{0x} F(\psi) \prod \frac{d\psi_j}{2\pi}, \qquad (2.4)
$$

where $\tilde{Z}$ is the obvious normalization constant. We choose $m \geq 0$ so that $G(0) = \beta$, where

$$
\begin{aligned}
G(x) &= \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} g(p) e^{ipx} \prod_{i=1}^{\nu} \frac{dp_i}{2\pi}, \qquad (2.5) \\
g(p) &\equiv \frac{1}{m^2 + 2\sum(1 - \cos p_k)} \in [\frac{1}{m^2 + 4\nu}, \frac{1}{m^2}]. \qquad (2.6)
\end{aligned}
$$

This choice is possible for any $\beta$ ( and $N$ ) if and only if $\nu \leq 2$, that is, if and only if $G_0(0) \equiv G(0)|_{m^2=0} = \infty$. In other words, we can rewrite eq.(2.3) as

$$
F(\psi) = \det{}_3(1 + \frac{2iG}{\sqrt{N}}\psi)^{-N/2} \exp[-\mathrm{Tr}(G\psi)^2] \qquad (2.7)
$$

for any $\beta$, only for $\nu \leq 2$, where $\det{}_3(1 + A) = \det[(1 + A)e^{-A + A^2/2}]$.

The factor $\exp[i\sqrt{N}\beta \sum \psi_x]$ in (2.3) is the reminiscence of the double-well potential $\prod \delta(\phi_x^2 - 1)$ which is responsible for phase transitions. Then roughly speaking, the disappearance of $\exp[i\sqrt{N}\beta \sum \psi_x]$ in ( 2.7) means absence of the effect of the double-well potential and is consistent with absence of phase transitions [2].

An explicit calculation shows that $m^2 = \beta^{-1}(\sqrt{1+4\beta^2} - 2\beta)$ for $\nu = 1$. For $\nu = 2$, $G(0)$ is expressed by the complete elliptic integral of the first kind $F(k, \pi/2) = \int_0^{\pi/2} d\varphi(1 - k^2 \sin^2\varphi)^{-1/2}$:

$$G(0) = \frac{1}{2\pi}\int_0^{\pi} \frac{dp}{\sqrt{(1+2\varepsilon-\cos p)(3+2\varepsilon-\cos p)}}$$
$$= \frac{k}{2\pi}F(k,\pi/2) = \frac{1}{2\pi}[O(\varepsilon) + \frac{3}{2}\log 2 + \frac{1}{2}\log\frac{1}{\varepsilon}],$$

where $\varepsilon = m^2/4$ and $k = (1+\varepsilon)^{-1}$. Then the condition $G(0) = \beta$ implies that

$$m^2 \sim 32 e^{-4\pi\beta} \text{ as } \beta \to \infty \tag{2.8}$$

which is consistent with the renormalization group arguments, see [6] and references therein.

If $\nu \geq 3$, such an $m \geq 0$ exists if $\beta \leq G_0(0)$. If $\beta > G_0(0)$, there exists spontaneous magnetization in the system [7]. That is $NG_0(0) > \beta_c(N) > N/\mu_\nu$ for $\nu > 2$.

If $m$ is chosen so that $G(0) = \beta$, $\det_3(1 + 2iG\psi/\sqrt{N})^{-N/2}$ is almost equal to $\exp[4i\mathrm{Tr}(G\psi)^3/(3\sqrt{N})]$ and is regarded as a small perturbation to the Gaussian measure $\sim \exp[-\mathrm{Tr}(G\psi)^2]\prod d\psi$. Namely $F(\psi)$ looks like $|F(\psi)| = \det(1 + 4G\psi G\psi/N)^{-N/4}$ which is strictly positive. If this is justified, then from eq.(2.4), we have exponential decay of the correlation functions :

$$< \phi_0\phi_x > \sim \frac{1}{\tilde{Z}}\int\cdots\int (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)^{-1}_{0x}|F(\psi)|\prod\frac{d\psi_j}{2\pi}$$
$$\leq |\sup_\psi (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)^{-1}_{0x}|$$
$$\leq (m^2 - \Delta)^{-1}_{0x} \sim e^{-m|x|}.$$

## III. POLYMER (CLUSTER) EXPANSION IN SMALL FIELD

### A. Polymer Expansion

Let

$$d\mu_\Lambda(\psi) = \det{}^{1/2}[C^{-1}]\exp[- < \psi, C^{-1}\psi >]\prod\frac{d\psi(x)}{\sqrt{\pi}} \tag{3.1}$$

be the Gaussian probability measure of mean zero and covariance $\frac{1}{2}C$ where $C^{-1} \equiv G^{\circ 2}$ and $G^{\circ 2}$ is the matrix given by $G^{\circ 2}(x,y) = G(x-y)^2$. The partition function $Z_\Lambda$ is given by

$$Z_\Lambda \;=\; Z_\infty \int \det{}_3^{-N/2}(1 + \frac{2i}{\sqrt{N}} G\psi) d\mu_\Lambda(\psi), \tag{3.2}$$

$$Z_\infty \;\equiv\; \det{}^{-1/2}[C^{-1}] = \det{}^{1/2}[C], \tag{3.3}$$

up to a non-important multiplicative factor. Our purpose is to discuss analyticity of the free energy $\alpha_F = -\lim \log Z_\Lambda/|\Lambda|$ in $\beta$. Since $m$ is analytic in $\beta \geq 0$, the assertion is trivial if there is no determinant. In the present case where we have the determinant, which is quite non-linear and non-local in $\psi(x)$, we represent $Z_\Lambda$ in terms of polymers:

**Theorem 1** *The partition function $Z_\Lambda$ is represented by polymers $\rho_X$, $X \subset \Lambda$:*

$$Z_\Lambda \;=\; Z_\infty \left[ \sum_p \frac{1}{p!} \sum_{\cup_1^p X_i = \Lambda} \prod_i \rho_{X_i} \right], \tag{3.4}$$

*where $X_i$ are unions of squares $\Delta \subset \Lambda$ of size $L \times L$ ($L >> 1$ is determined later ) and $X_i \cap X_j = \emptyset$, $(i \neq j)$. Given $\beta > 0$, if $N$ is chosen large, $N \geq \exp[\mathrm{const.}\beta]$, there exist strictly positive constants $\delta_c$ and $m_c$ such that*

$$|\rho_X| \;\leq\; \exp[-\delta_c n_X \log N - m_c \mathcal{L}(X)], \tag{3.5}$$

*where $n_X$ is the number of squares $\Delta_i$ in $X$ and $\mathcal{L}(X)$ is the length of the shortest connected tree graph over centers of $\Delta_i \subset X$. The free energy is the convergent series of $\rho_X$.*

Each $\rho_X$ is analytic in $\beta$. Thus the Main Theorem follows from Theorem 1 since $\alpha_F$ is represented by the convergent series of $\rho_X$. The proof of this theorem is, however postponed until Sect.5. Here we restrict ourselves to the small field case where the expansion can be easily done by the $N^{-1}$ expansion.

### B. Small and Large Fields

We let $\tilde{G} \equiv [G^{\circ 2}]^{1/2}$. Then $C$ and $\tilde{G}$ have the following Fourier expansions:

$$C \;=\; \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{ip(x-y)} \tilde{g}^{-2}(p) \prod_{i=1}^{2} \frac{dp_i}{2\pi}, \tag{3.6}$$

$$\tilde{G} = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} e^{ip(x-y)}\tilde{g}(p)\prod_{i=1}^{2}\frac{dp_i}{2\pi}, \tag{3.7}$$

$$\tilde{g}(p) = \left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(p-k)g(k)\prod_{i=1}^{2}\frac{dk_i}{2\pi}\right]^{1/2} \in [\frac{c_1}{m^2+8}, \frac{c_2}{m}]. \tag{3.8}$$

Here and below, $c$ stands for generic constant independent of $\beta$ which may change from place to place even in the same equations, and $c_0$, $c_1$, $\cdots$ stand for similar constants which are kept in the same equations. The following lemma is proved in Appendix A:

**Lemma 2** *For $m < 1$, the kernels $G$, $\tilde{G}$, $\tilde{G}^{-1}$ and $C$ exhibit the followng exponential decay:*

$$G(x,y) \le c\log(1+\frac{1}{m})\exp[-m_*|x-y|], \tag{3.9}$$

$$|\tilde{G}(x,y)| \le c\log(1+\frac{1}{m})\exp[-m|x-y|], \tag{3.10}$$

$$|\tilde{G}^{-1}(x,y)| \le c(1+m^2)\exp[-m|x-y|], \tag{3.11}$$

$$|C(x,y)| \le c(1+m^2)\exp[-m|x-y|] \tag{3.12}$$

*where $|x| = \sqrt{x_1^2+x_2^2}$ and $m_* > 0$ is a constant defined by $2\cosh(m_*) = 2+m^2$.*

We introduce the notion of large field region $R$ and small filed region $K$:

$$R = \{x; N^\delta \le |\psi(x)|\}, \quad K = \Lambda - R \tag{3.13}$$

where $N = N(\beta)$ and a positive constant $\delta < 1/2$ is chosen so that if $|\psi(x)| \le N^\delta$ for all $x$, then $N^{-1/2}||G^{1/2}\psi G^{1/2}|| << 1$. Then the determinant is perturbatively expanded and the higher order terms are negligible. Since $\mathrm{spec}\, G \in [(8+m^2)^{-1}, m^{-2}]$ and $m^{-2} \sim (32)^{-1}e^{4\pi\beta}$, these conditions are satisfied if $\exp[12\pi\beta] < N$ for large $\beta$. The following is one of the most typical choices satisfying these conditions (though they are not optimal ) :

$$\delta = \frac{1}{12}, \quad N(\beta) = \exp[400\pi\beta]. \tag{3.14}$$

**Remark 1** *For matrices $A$ and $B$, we define $A \circ B$ by $(A\circ B)(x,y) = A(x,y)B(x,y)$. This is called the Hadamard product of $A$ and $B$. It is easy to see that $A \circ B \ge 0$ if $A \ge 0$ and $B \ge 0$.*

**Remark 2** *The kernel functions* $C(x)$*,* $\tilde{G}(x)$ *and* $\tilde{G}^{-1}(x)$ *decay faster than* $\exp[-\sqrt{2}m|x|]$*, see Appendix. Of course,* $m_* < m$*,* $m_* = m - O(m^2)$*. However since* $m_*$ *is almost equal to* $m$ *in the present problem where* $m << 1$*, we use* $m$ *for* $m_*$ *for notational simplicity in the remaining part of the paper. If* $\beta < O(1)$*, it is enough to choose* $L$ *(the size for the expansion) and* $N$ *larger than some constants for the convergence. So it suffices to consider the case* $\beta >> 1$*.*

**Remark 3** *In this paper, we use free boundary conditions for Green's function* $G$ *and its inverse, and we assume that the* $\psi$ *field distributes only in the large square region* $\Lambda \subset \mathbf{Z}^2$*. Other boundary conditions can be easily adopted without changing the main estimates in the present paper.*

### C. Polymer Expansion in Small Field Region

We first consider the case of $R = \emptyset$. In this case, we decompose $\Lambda \subset \mathbf{Z}^2$ into squares (denoted $\Delta$ or $\Delta_i$ below ) of size $L \times L$ whose centers are at $\Lambda \cap L\mathbf{Z}^2$. Collections of these squares are called paved sets. We also define $L_0 << L$, where $L$ and $L_0$ are chosen so that

$$L << N << e^{mL}, \quad G(L_0) = N^{-2}. \tag{3.15}$$

For this to be satisfied, we take $L$ slightly larger than $m^{-1}$. Typically we may take $L = 20m^{-1}\log N$ so that $e^{mL} = N^{20}$, in which case $L_0 = L/10$. These satisfy the conditions on $L$ and $N$.

Let $\tau(\psi)$ be an even, positive and decreasing (in $|\psi|$ ) $C^\infty$ function such that

$$\tau(\psi) = \begin{cases} 1 & \text{for } |\psi| < N^\delta \\ 0 & \text{for } |\psi| > N^\delta + h \end{cases} . \tag{3.16}$$

We may take the limit $h \to 0$ after all calculations ($\lim_{h\to 0}\tau(\psi) = \theta(N^\delta - |\psi|)$), but we can keep $h$ as a non-zero constant (say 1).

We multiply

$$1 = \sum_{K\subset\Lambda} \tau(\psi_K)\tau^c(\psi_R) \tag{3.17}$$

to $d\mu_\Lambda$, where $\tau^c(\psi) = 1 - \tau(\psi)$, $R = K^c = \Lambda - K$ and $\tau(\psi_K) \equiv \prod_{x \in K} \tau(\psi(x))$, $\tau^c(\psi_R) \equiv \prod_{x \in R} \tau^c(\psi(x))$. We call $K$ the small field region and $R = K^c$ the large field region. Then

$$Z_\Lambda \equiv Z_\infty \sum_R Z(R), \tag{3.18}$$

$$Z(R) \equiv \int \det{}_3^{-N/2}(1 + \frac{2i}{\sqrt{N}} G\psi) \tau^c(\psi_R) \tau(\psi_K) d\mu_\Lambda(\psi). \tag{3.19}$$

We put $Z_\Lambda(R) = Z_\infty Z(R)$ and we first consider the case $R = \emptyset$:

$$Z_\Lambda(R = \emptyset) \equiv Z_\infty \int \eta_\Lambda d\mu_\Lambda(\psi), \tag{3.20}$$

$$\eta_\Lambda \equiv \det{}_3^{-N/2}(1 + \frac{2i}{\sqrt{N}} G\psi) \prod_{x \in \Lambda} \tau(\psi(x)), \tag{3.21}$$

We introduce interpolation parameters $s_i$ into $d\mu_\Lambda(\psi)$ to expand the measure [5,16]. Let $Y \subset \Lambda$ be a paved set consisting of $p$ squares $\{\Delta_1, \cdots, \Delta_p\}$. Let $\{\Delta_{j_1}, \cdots, \Delta_{j_p}\}$ be any permutation of them such that $\Delta_{j_1} = \Delta_1$ and let $a$ be a map from $\{1, \cdots, p-1\}$ into itself such that $a(k) \leq k$. Then we have a set of ordered links $\{(j_{a(i)}, j_{i+1}); i = 1, \cdots, p-1\}$ which is regarded as a tree graph $T'$ over $\{\Delta_i\}$ with root $\Delta_1$. Let

$$C_Y = \chi_Y C \chi_Y, \tag{3.22}$$

where $\chi_Y$ is the charcteristic function of $Y$. For a given permutation and a function $a = a_{T'}$, we define

$$C_Y(\{s\}) = [\prod_{i=1}^{p-1}((1 - s_i)\mathcal{P}_i + s_i)] C_Y, \tag{3.23}$$

$$M_{T'} = \prod_{k=1}^{p-1} \prod_{i=a(k)}^{k-1} s_i, \tag{3.24}$$

where $\mathcal{P}_i$ are operators which bisect paved sets: $\mathcal{P}_i C_X = C_{X \setminus X_i} + C_{X \cap X_i}$, $X_i \equiv \cup_{k=1}^i \Delta_{j_k}$. See Appendix C for the construction and for the proof of next theorem, see [5,16]:

**Theorem 3** $Z_\Lambda(R = \emptyset)$ *have the cluster expansion*

$$Z_\Lambda(R = \emptyset) = Z_\infty \left[ \sum_n \frac{1}{n!} \sum_{\cup_1^n Y_i = \Lambda} \prod_i S_{Y_i} \right] \eta_\Lambda, \tag{3.25}$$

*where $Y_i$ are paved sets such that $\cup_1^n Y_i = \Lambda$ and $Y_i \cap Y_j = \emptyset$ for $i \neq j$. Let $Y = \cup_{k=1}^p \Delta_k$ be one of $Y_i$. Then $S_Y$ is the differential and integral operator given by*

$$\begin{aligned} S_Y &= \sum_{T'} \int_0^1 ds_1 \cdots ds_{p-1} M_{T'}(s) \int d\mu_Y(\{s\}, \psi) \\ &\times \prod_{k=1}^{p-1} \left[ \sum_{x_k \in \Delta_{j_{a(k)}}} \sum_{y_{k+1} \in \Delta_{j_{k+1}}} \frac{1}{2} C(x_k, y_{k+1}) \frac{\partial^2}{\partial\psi(x_k)\partial\psi(y_{k+1})} \right], \end{aligned} \tag{3.26}$$

*where $\sum_{T'}$ is the sum over all tree graphs $T' = \{(j_{a(k)}, j_k)\}$ over $\{j_1, j_2, \cdots, j_p\}$ ($j_1 = 1$) and*

$$d\mu_Y(\{s\}, \psi) = \det{}^{-1/2}[C_Y(s)] \exp[- < \psi, C_Y^{-1}(\{s\})\psi >] \prod_{x \in Y} \frac{d\psi(x)}{\sqrt{\pi}}. \tag{3.27}$$

*Here $C_Y(\{s\})$ is given by (3.23) and depends on permutations only.*

There are many graphs $T'$ which have the same links and vertices but belong to different permutations $\{j_1, j_2, \cdots, j_p\}$ of $\{1, \cdots, p\}$. The following lemma is well known [5,16]:

**Lemma 4** *The measure $M_T \prod ds_i$ is the probability measure in the following sense:*

$$\sum_{T':T(T')=T} \int_0^1 M_{T'} \prod_1^{p-1} ds_i = 1, \tag{3.28}$$

*where $\sum_{T':T(T')=T}$ means the sum over tree graphs $T'$ which have the same links with $T$.*

Let

$$A_\Lambda = \frac{2i}{\sqrt{N}} G\psi \tag{3.29}$$

for simplicity, and let $\Lambda = \cup_{i=1}^p Y_i$ be one of the partitions which appear in eq.(3.25). Since $\{\psi_{Y_i}\}$ are coupled in the determinant, we introduce interpolation parameters $s_{ij}$ and set

$$A_\Lambda = \sum A_{Y_i} + \sum_{i<j} (A_{Y_i,Y_j} + A_{Y_j,Y_i}) \to A + B(s), \tag{3.30}$$

$$A \equiv \sum A_{Y_i},\ B(s) \equiv \sum_{i<j} s_{ij}(A_{Y_i,Y_j} + A_{Y_j,Y_i}), \tag{3.31}$$

in the determinant, where

$$A_{Y_i} = \chi_{Y_i} A_\Lambda \chi_{Y_i},\ A_{Y_i,Y_j} = \chi_{Y_i} A_\Lambda \chi_{Y_j}. \tag{3.32}$$

We iteratively apply the identity $f(1) = \int_0^1 ds \partial_s f(s) + f(0)$ to $\det_3(1 + A + B(s))$. If all $s_{ij}$ are set zero, then the determinant is factorized with respect to $\psi_{Y_i}$. We thus have :

$$Z_\Lambda(R = \emptyset) \; = \; Z_\infty \left[ \sum_n \frac{1}{n!} \sum_{\cup_1^n X_i = \Lambda} \prod_i \rho_{X_i} \right].$$

Here $\{X_i\}_1^n$ are partitions of $\Lambda$ into polymers, $X_i \cap X_j = \emptyset$, $(i \neq j)$, $\cup X_i = \Lambda$ and

$$\rho_X \; = \; \sum_p \frac{1}{p!} \sum_{Y_1 \cup \cdots \cup Y_p = X} \prod S_{Y_i} \left[ \sum_{\gamma \in \tilde{T}(\{Y_i\})} \int ds_\gamma \partial_\gamma \right] \eta_X(\{Y_i\}), \tag{3.33}$$

$$\eta_X(\{Y_j\}) \; = \; \det{}_3^{-N/2}(1 + \sum_i A_{Y_i} + \sum_{i<j} s_{ij}(A_{Y_i,Y_j} + A_{Y_j,Y_i}))\tau(\psi_X), \tag{3.34}$$

where $S_Y$ is the interpolation operators on $Y$ defined by (3.26) and

1. $\cup Y_i = X$ and $Y_i$ are mutually disjoint paved sets,

2. $\tilde{T}(\{Y_i\})$ is the set of connected graphs (not necessarily trees) over $\{Y_i\}_{i=1}^p$,

3. $ds_\gamma = \prod_{(ij)\in\gamma} ds_{ij}$ and $\partial_\gamma = \prod_{(ij)\in\gamma}(\partial/\partial s_{ij})$, (put $s_{ij} = 0$ if $(i,j) \notin \gamma$ ).

In the rest of this section, we prove the following theorem which ensures that the free energy $\log Z_\Lambda(R = \emptyset)$ is the convergent series of $\rho_X$ [13], if $N$ is chosen large:

**Theorem 5** *Assume that $R = \emptyset$ and let $n$ be the number of $\Delta$ in $X \subset \Lambda$. If $N \geq N(\beta)$, there exist strictly positive constants $\delta_0$ and $m_0$ such that*

$$|\rho_X| \; \leq \; \exp[-n\delta_0 \log N - m_0 \mathcal{L}(X)], \;\; n \geq 2 \tag{3.35}$$

$$\rho_\Delta \; = \; \exp[-W_\Delta], \;\; n = 1 \tag{3.36}$$

*where $\mathcal{L}(X)$ is the length of the shortest tree graph connecting all centers of squares $\Delta_i \subset X$, and $W_\Delta$ is the single square activity defined later.*

To prove this, we first set

$$\eta_X(\{Y_i\}) \; \equiv \; \exp[-\frac{N}{2}\sum_{i=1}^{2} V_i(A,B)] \prod_{x\in X} \tau(\psi(x)), \tag{3.37}$$

$$V_1(A,B) \; = \; \frac{1}{2}\mathrm{Tr}\left(B^2 - (B\frac{1}{1+A})^2\right) + \frac{1}{3}\mathrm{Tr}\left(\frac{1}{1+A}B\right)^3, \tag{3.38}$$

$$V_2(A,B) \; = \; \log\det{}_3(1+A) + \log\det{}_4(1 + \frac{1}{1+A}B). \tag{3.39}$$

The derivatives of $V_i$ with respect to $s_{ij}$ can be done by the contour integrals:

$$\left(\prod \frac{\partial}{\partial s_{ij}}\right) \eta_X(s) = \int_{\mathcal{C}} \frac{\eta_X(t)}{\prod (t_{ij} - s_{ij})^2} \prod \frac{dt_{ij}}{2\pi i}$$

where $\mathcal{C}$ is the product of the circles $|t_{ij} - s_{ij}| = r_{ij}$ on $\mathbf{C}$ with their radiuses $r_{ij}$ given by

$$r_{ij} = N^{\tilde{\delta}} \exp[\frac{4}{5} m \mathrm{dist}(Y_i, Y_j)], \text{ where } \tilde{\delta} > 0. \tag{3.40}$$

Put $B_{ij} = 2it_{ij}(G_{Y_iY_j}\psi_{Y_i} + G_{Y_jY_i}\psi_{Y_j})/\sqrt{N}$. Then for $|t_{ij}| < r_{ij} + 1$ , we find that

$$\begin{aligned} |B_{ij}(x,y)| &\leq \text{const.} \log(1+m^{-1}) N^{-1/2+\delta+\tilde{\delta}} \exp[-\frac{m}{5}|x-y|], \\ N|\mathrm{Tr}\chi_X B^3 \chi_X| &\leq N^{-1/2+3\delta+3\tilde{\delta}+2\varepsilon_0}|X|, \\ \varepsilon_0 &\equiv -2.1 \times \log m / \log N \ \ (\sim 1/100 \text{ if } N \sim e^{400\pi\beta}), \end{aligned} \tag{3.41}$$

where $\varepsilon_0$ is chosen slightly larger than $-2\log m/\log N$ so that $N^{\varepsilon_0} > cm^{-2}\log(1+m^{-1})$ and some trivial constants can be absorbed by $N^{\varepsilon_0}$. We choose $\tilde{\delta} > 0$ so that

$$\hat{\delta} \equiv \frac{1}{2} - 3(\delta + \tilde{\delta}) - 2\varepsilon_0 > 0. \tag{3.42}$$

For example, we can choose as $\delta = 1/12$, $\tilde{\delta} = 1/16$, $\hat{\delta} = 1/16 - 2\varepsilon_0$. Thus we have:

**Lemma 6** *If $N$ is chosen so large that (3.42) holds, then*

$$|\prod_{(ij)\in\gamma} \partial/\partial s_{ij} \eta_X| \leq \exp[-n\tilde{\delta}\log N - m_2 \sum_{(ij)\in\gamma} \mathrm{dist}(Y_i, Y_j)]||\eta_X|| \tag{3.43}$$

*where $m_2 = 4m/5$ and $\gamma$ are connected tree graphs over $\{Y_i \subset X\}$, and $n$ is the number of the bonds in the graph $\gamma$. Moreover*

$$||\eta_X|| \equiv \sup_{\{|t_{ij}| \leq r_{ij}+1\}} |\eta_X(t)| \leq \exp[N^{-\hat{\delta}}|X|]. \tag{3.44}$$

**Lemma 7** *Let $\cup_{i=1}^n \Delta_i = Y$ and let $x_i \in \Delta_i$. Then*

$$|\int d\mu_Y(s,\psi) \prod_1^n \frac{\partial}{\partial \psi(x_i)} \eta_Y(\psi)| < \exp[-n\tilde{\delta}\log N + N^{-\hat{\delta}}|Y|]. \tag{3.45}$$

*Proof.* Each derivative acts either on $\det_3^{-N/2}(\cdots)$ or on $\tau(\psi)$. If it acts on $\det_3^{-N/2}(\cdots)$, it yields the factor bounded by $N^{-\tilde{\delta}}$. (We can get a much smaller factor $N^{-1/6+\varepsilon_0}$ this case. ) On the other hand, if $\partial/\partial\psi(x)$ acts on $\tau(\psi(x))$,

$$\frac{\partial}{\partial\psi(x)}\tau(\psi(x)) = 0 \text{ unless } N^\delta < |\psi(x)| < N^\delta + h.$$

Note that

$$d\mu_Y(s) \to \prod \exp[-\mathrm{z}(x)^2]\frac{d\mathrm{z}(x)}{\sqrt{\pi}}$$

by the linear transformation $\psi(x) = (\tilde{G}_Y^{-1}\mathrm{z})(x)$, where $\tilde{G}_Y^{-1} = \sqrt{C_Y}$. Since $C^{-1} = G^{\circ 2}$ and $C_Y(s)$ is a convex linear combination of $\{C_{Y_i}\}$, we see

$$\sum_{x\in Y} \mathrm{z}_x^2 =< \psi, \chi_Y C_Y(s)^{-1}\chi_Y\psi >\geq \frac{1}{(8+m^2)^2}\sum_{x\in Y}\psi^2(x).$$

If $|\psi(x)| > N^\delta$, then $\{y : |\mathrm{z}(y)| > N^{\delta-\varepsilon_0}, |x-y| < L_0\} \neq \emptyset$ since $|\psi(x)| = |\sum_y \tilde{G}_Y^{-1}(x,y)\mathrm{z}(y)|$ and $|\tilde{G}^{-1}(x)| < c(1+m^2)e^{-m|x|}$. Thus the contributions from the derivatives of $\tau$ are exponentially smaller than those from the derivatives of $\det_3^{-N/2}(\cdots)$. Q.E.D.

The single square activity $\rho_\Delta = e^{-W_\Delta}$ is defined by

$$\rho_\Delta = \int \det{}_3^{-N/2}(1+A_\Delta)\tau(\psi_\Delta)d\mu_\Delta(\psi). \tag{3.46}$$

Since $|\log\det_3^{-N/2}(1+A_\Delta)| = O(N|\operatorname{Tr}A_\Delta^3|)$, we have $W_\Delta = O(N^{-1/2+3\delta+3\varepsilon_0})$ which is independent of locations of $\Delta$ ($|\Delta| = L^2 < N^{\varepsilon_0}$).

Let $d_i$ be the number of lines which connect $\Delta_i$ with other $\Delta_j$ in the tree graph, i.e. $d_i$ the incidence number. Then $\sum_{i=1}^n d_i = 2n-2$, where $n$ is the number of squares $\Delta_i$ in $Y$. In this case there can appear $d_i$ derivatievs $\partial^{d_i}/\partial\psi(x)^{d_i}$, $x \in \Delta_i$ in eq.(3.26). By integration by parts, we can shift the action of $\partial/\partial\psi$ from $\tau$ to $\det_3^{-N/2}(\cdots)$ or to $\exp[-<\psi, C_Y^{-1}\psi>]$.

**Lemma 8** *[16] With the notation of (3.26) in Theorem 3 (with p replaced by n), let*

$$\mathcal{F}(x_1,y_2,\cdots,y_n) \equiv |\prod_{i=1}^{n-1} C(x_i,y_{i+1})\int d\mu_Y(s,\psi)\prod_1^{n-1}\frac{\partial^2}{\partial\psi(x_i)\partial\psi(y_{i+1})}\eta_Y(\psi)|$$

*where* $x_k \in \Delta_{j_{a(k)}}$, $y_{k+1} \in \Delta_{j_{k+1}}$. *Let* $\gamma$ *is the tree graph defined by* $a(\cdot)$. *Then*

$$\sum_{\{x_k,y_{k+1}\}}\mathcal{F}(x_1,y_2,\cdots,y_n) \leq \exp[-n(\tilde{\delta}-4\varepsilon_0)\log N - \frac{4m}{5}\mathcal{L}_0(X) + N^{-\hat{\delta}}|X|] \tag{3.47}$$

*where* $x_k \in \Delta_{j_{a(k)}}$, $y_{k+1} \in \Delta_{j_{k+1}}$ *and* $\mathcal{L}_0(X) = \sum_{(i,j)\in\gamma}\operatorname{dist}(\Delta_i,\Delta_j)$.

*Proof.* Without loss, we assume $\{j_k = k\}_{k=1}^n$. Let $d_i \geq 1$ be the incidence number of the vertex $\Delta_i$. Since $\#\{\Delta_j : \mathrm{dist}(\Delta_i, \Delta_j) < 2, i \neq j\} = 8$, $\sum_i |x_i - y_{i+1}|$ is larger than

$$\frac{4}{5}\sum_i |x_i - y_{i+1}| + \frac{1}{10}\sum_i \sum_{x\in\Delta_i} \sum_{y:(x,y)\in\gamma} |x-y| \geq \frac{4}{5}\sum_i |x_i - y_{i+1}| + \frac{L}{10}\sum_i [\frac{d_i}{9}]^{3/2},$$

where $[x]$ = the maximal integer not larger than $x$. By integration by parts, we see that

$$|\mathcal{F}(x_1, y_2, \cdots)| = |\prod_{i=1}^{n-1} C(x_i, y_{i+1}) \int d\mu_Y(s,\psi)\Phi\Psi| \tag{3.48}$$

where relabelling $\{x_i, y_{i+1}\}$ as $\{x_i, x_{i,1}, \cdots, x_{i,d_i-1}\}_1^n$, $x_{i,k} \in \Delta_i$,

$$\Psi = \prod_{i=1}^n \frac{\partial}{\partial\psi(x_i)}\eta_Y(\psi), \tag{3.49}$$

$$\Phi = (-1)^{\sum d_i - n} e^H \prod_{i=1}^n \prod_{j=1}^{d_i-1} \frac{\partial}{\partial\psi(x_{i,j})} e^{-H}, \tag{3.50}$$

$$H = <\psi_Y, C_Y^{-1}(s)\psi_Y> . \tag{3.51}$$

Rewriting $\{x_{i,j}\}$ as $\{\xi_i\}_1^{n-2}$, we put

$$\Phi = e^{-H}\prod_{i=1}^{n-2}\frac{\partial}{\partial\psi(\xi_i)}e^{-H} = \sum_I (-1)^{|I|}\prod_{i\in I} H_{\xi_i}(\sum_{P\subset I^c}\prod_{(j,k)\in P} H_{\xi_j,\xi_k}), \tag{3.52}$$

where $I$ are subsets of $\{1, \cdots, n-2\}$, $P$ are sets of unordered pairs of elements in $I^c$ and

$$H_\xi = 2\sum_\zeta C^{-1}(\xi,\zeta)\psi(\zeta),\ H_{\xi_1\xi_2} = 2C^{-1}(\xi_1,\xi_2). \tag{3.53}$$

The number of partitions $I \subset \{1, \cdots, n-2\}$ is $2^{n-3}$ ($|I^c|$ must be even) and note that

$$\sum_{P\subset I^c}\prod_{(j,k)\in P} H_{\xi_j,\xi_k} = \int \prod_{i\in I^c}\phi(\xi_i)d\nu_H(\phi)$$

where $d\nu_H(\phi)$ is the Gaussian measure of mean zero and covariance $H = 2G^{\circ 2}$.

We first estimate the first term of $\Phi$, $I = \{1, \cdots, n-2\}$:

$$\sum_{\{\zeta_i\}}\prod 2|C(x_i, y_{i+1})| \prod_i |C^{-1}(\xi_i,\zeta_i)| \left[\int d\mu_Y(s,\psi)\prod_i |\psi(\zeta_i)||\Psi|\right] \leq M\left[\int d\mu_Y(s,\psi)\Psi^2\right]^{\frac{1}{2}}$$

where the integral of $\Psi^2$ is bounded by Lemma 7 ( easily extended to $\Psi^2$ ) and

$$M \equiv \sum_{\zeta_i}\prod 2|C(x_i, y_{i+1})| \prod |C^{-1}(\xi_i,\zeta_i)| \left[\int d\mu_Y(s,\psi)\prod\psi(\zeta_i)^2\right]^{\frac{1}{2}}. \tag{3.54}$$

We take the sum over $\{\xi_i\}_1^{n-2} \subset \{x_k, y_{k+1}\}_1^{n-1}$ and put

$$\sum_{\xi \in \Delta_{a(k)}} \sum_{\xi' \in \Delta_{k+1}} 2|C(\xi,\xi')||C^{-1}(\xi,\tilde{x}_k)||C^{-1}(\xi',\tilde{y}_{k+1})| \equiv m^{-4}\delta f(\Delta_{a(k)},\Delta_{k+1})(\tilde{x}_k,\tilde{y}_{k+1}).$$

Then $\delta f(\Delta_{a(k)},\Delta_{k+1})(\tilde{x}_k,\tilde{y}_{k+1})$ is bounded by

$$\exp[-m\{\mathrm{dist}(\Delta_{a(k)},\Delta_{k+1}) + \mathrm{dist}(\Delta_{a(k)},\tilde{x}_k) + \mathrm{dist}(\Delta_{k+1},\tilde{y}_{k+1})\}] \tag{3.55}$$

except for a coefficient $O(\log^4(1+m^{-1}))$ which originates from $C^{-1} = G^{\circ 2}$. Here the constraints $\tilde{x}_k \in \Delta_{a(k)}$ and $\tilde{y}_{k+1} \in \Delta_{k+1}$ do not hold anymore. For $x_k$ or $y_{k+1}$ not contained in $\{\xi\}_1^{n-1}$, we put $\tilde{x}_k = x_k$ or $\tilde{y}_{k+1} = y_{k+1}$ and put $\delta f(\Delta_{a(k)},\Delta_{k+1})(\tilde{x}_k,\tilde{y}_{k+1}) = 2C(\tilde{x}_k,\tilde{y}_{k+1})\chi_{\Delta_{a(k)}}(\tilde{x}_k)\chi_{\Delta_{k+1}}(\tilde{y}_{k+1})$. This again satisfies the bound (3.55).

Assume that $\tilde{\Delta}_i \subset \Lambda$ contains $\tilde{d}_i$ points of $\{\zeta_i\}$. If $d_{ij}$ points in $\tilde{\Delta}_i$ couple with $d_{ij}$ points in $\tilde{\Delta}_j$ (the same points appear twice in $\prod \psi(\zeta)^2$), $\sum_j d_{ij} = 2\tilde{d}_i$ and we have the factor $\binom{2\tilde{d}_i}{d_{ij}}\binom{2\tilde{d}_j}{d_{ij}} d_{ij}!$ ( $2d_{ii}$ for $(i,i)$.) Since $\prod_j (d_{ij})! < (2\tilde{d}_i)!$, we find that

$$\begin{aligned}
\int d\mu \prod \psi(\zeta_i)^2 &\leq \prod [(2\tilde{d}_i)!]^{\frac{1}{2}} \prod_i \left[ \sum_{\{d_{ij}\}_j} \frac{(2\tilde{d}_i)!}{d_{i,1}! \cdots d_{i,n}!} \prod_j |C(\mathrm{dist}(\tilde{\Delta}_i,\tilde{\Delta}_j))|^{\frac{1}{2}d_{ij}} \right] \\
&\leq \prod [(2\tilde{d}_i)!]^{\frac{1}{2}} \prod_i \left[ \sum_j |C(\mathrm{dist}(\tilde{\Delta}_i,\tilde{\Delta}_j))|^{\frac{1}{2}} \right]^{2\tilde{d}_i} \\
&\leq c_0^{2(n-2)} \prod [(2\tilde{d}_i)!]^{\frac{1}{2}}
\end{aligned} \tag{3.56}$$

where $c_0 = O(1)$. Since $(2d)! \leq e^{2d \log 2d}$ and $\prod \delta f(\Delta_{a(k)},\Delta_{k+1})(\tilde{x}_k,\tilde{y}_{k+1})$ is bounded by

$$\exp[-\frac{4m}{5} \sum_k \{\mathrm{dist}(\Delta_{a(k)},\Delta_{k+1}) + \mathrm{dist}(\Delta_{a(k)},\tilde{x}_k) + \mathrm{dist}(\Delta_{k+1},\tilde{y}_{k+1})\} - \frac{mL}{10} \sum_i [\frac{\tilde{d}_i}{9}]^{\frac{3}{2}}]],$$

we see that $(2\tilde{d}_i)!$ are compensated and the sum over $\{\tilde{x}_k,\tilde{y}_{k+1}\}$ yields $m^{-4(n-1)}$.

The coefficients $\int \prod_{\xi \in I^c} \phi(\xi) d\nu_H$ of $\prod_{\xi \in I} H_\xi$ are again bounded by (3.56) by replacing $c_0$ by $c_0 \log(1+m^{-1})$ and $2\tilde{d}_i$ by corresponding incidence numbers. Thus the total contribution of $\Phi$ is bounded by $2^{n-3}$ times of the result of $I = \{1,\cdots,n-2\}$. Q.E.D.

We introduce mass parameters $m_i$ for later conveniences :

$$0 < m_0 < \tilde{m}_0 < m_1 = \frac{m}{10} < m_2 = \frac{4m}{5} < m. \tag{3.57}$$

where $Lm_0 \sim O(\beta) >> 1$. The following lemmas are well-known to experts [5,8,16]:

**Lemma 9 ( [16], Lemma A.5 )** *For a paved set $X$ consisting of $n$ squares $\{\Delta_i\}$, let $T(X)$ denote the set of tree graphs $\gamma$ over $\Delta_i$ and $\mathcal{L}(X)$ denote the length of the shortest tree graph over centers of $\Delta_i \subset X$. Let $\mathrm{dist}_c(\Delta_i, \Delta_j)$ be the distance from the center of $\Delta_i$ to that of $\Delta_j$. Then there exist constants $K_1 = o(1)$ and $K_2 = o(1)$ such that*

$$(1) \quad \sum_{X \ni 0} \sum_{\gamma \in T(X)} \exp[-\tilde{m}_0 \sum_{(ij) \in \gamma} \mathrm{dist}_c(\Delta_i, \Delta_j)] < K_1^n, \tag{3.58}$$

$$(2) \quad \sum_{X \ni 0} \exp[-\tilde{m}_0 \mathcal{L}(X)] < K_2^n. \tag{3.59}$$

*Proof.* (1) Interchange the order of $\sum_X$ and $\sum_\gamma$, and take the sum over positions of $\Delta_i$ for each $\gamma$. If $\Delta_i$ are distinguishable, the result is bounded by $K^{n-1}$ where $K = o(1)$ since $\Delta_i$ are squares of size $L \times L$ and $e^{-\tilde{m}_0 L} << 1$. However the same configuration is counted $n!$ times. Then

$$\sum_{X \ni 0} \exp[-\tilde{m}_0 \sum_{(ij) \in \gamma} \mathrm{dist}_c(\Delta_i, \Delta_j)] < \frac{K'^n}{n!}.$$

We finally note that the number of tree graphs is $n^{n-2} < n! e^n$ to take the sum over $\gamma$.
(2) This is clear from $\exp[-\tilde{m}_0 \mathcal{L}(X)] \leq \sum_{\gamma \in T(X)} \exp[-\tilde{m}_0 \sum_{(ij) \in \gamma} \mathrm{dist}_c(\Delta_i, \Delta_j)]$. Q.E.D.

**Lemma 10 ( [5], Appendix C)** *Let $X$ be a paved set consisting of $n_X$ squares $\Delta_i \subset X$. Let $f(Y)$ be functions satisfying the bounds*

$$|f(Y)| \leq \exp[-n_Y \tilde{\delta}_0 \log N - \tilde{m}_0 \mathcal{L}(Y)],$$

*where $n_Y$ is the number of squres $\Delta_i$ in $Y$ and $\mathcal{L}(Y)$ is the length of shortest tree graph over centers of $\Delta_i \subset Y$. Then there exist strictly positive constants $\delta_0$ ($\sim \tilde{\delta}_0$) and $m_0$ ($\sim \tilde{m}_0$) such that*

$$|\frac{1}{p!} \sum_{Y_1 \cup \cdots \cup Y_p = X} \prod f(Y_i)| \leq \exp[-n_X \delta_0 \log N - m_0 \mathcal{L}(X)], \tag{3.60}$$

*where $\{Y_i : i = 1, \cdots, p\}$ are paved sets such that $X$ cannot be devided into two disconnected parts without bisecting some $Y_i$.*

*Proof.* We first extract the tree decay factor $\exp[-n_X\delta_0 \log N - m_0\mathcal{L}(X)]$ from $\prod f(Y_i)$ choosing $\delta_0$ and $m_0$ slightly less than $\tilde{\delta}_0$ and $\tilde{m}_0$. We show that the remaining sum converges. By Cayley's theorem on the number of the tree graphs with fixed incidence numbers $d_1, \cdots, d_p$, we have

$$|\sum_T(\cdot)| = |\sum_{\{d_i\}} \sum_{T,\{d_i\}\text{fixed}} (\cdot)| \le \sum_{d_1,\cdots,d_p} \frac{(p-2)!}{\prod(d_i-1)!} \sup_{(T,d):\text{fixed}} |(\cdot)|,$$

and take the sum over the $Y_i$'s starting from the end branches of the tree. Let $Y_p$ be one of the end branches and let $Y_j$ be the ancestor. Fix $\Delta_j \subset Y_p \cap Y_j$ and take the sum over $Y_p$. The sum is convergent and is bounded by $\sum_{Y_p \ni 0} |f(Y_p)|$. Next take the sum over $\Delta_j \subset Y_j$, which yields $(n_{Y_j})^{d_j-1}$. Repeating this, we see that the sum is bounded by $n_X[\sum_{Y\ni 0} |f(Y)| e^{n_Y}]^p$ since $\sum n_Y^d/d! \le e^{n_Y}$. $e^{n_Y}$ is compensated by a fraction of $\exp[-n_Y\tilde{\delta}_0 \log N]$ in $f(Y)$. See also [5,16] for the detail. Q.E.D.

*Proof of Theorem 5.* We obtain $f(Y)$ in Lemma 10 from Lemma 8 by taking the sum over $T'$ in (3.26). This yields a constant less than 1. Thus we obtain $f(Y)$ in Lemma 10.

We determine the parameters $\tilde{\delta}_0$ and $\tilde{m}_0$. In Lemma 8, $X$ may be single squares $\Delta$, and they do not have tree decay factors. Moreover $\Delta_i$ and $\Delta_j$ may be nearest neighbour each other and $\text{dist}(\Delta_i, \Delta_j) = 1$. Then we put $\tilde{\delta}_0 \equiv (\tilde{\delta} - 4\varepsilon_0)/2$ and borrow $N^{-\tilde{\delta}_0}$ from $N^{-2\tilde{\delta}_0}$ in eq.(3.47) in Lemma 8 to extract the factor $\exp[-\tilde{m}_0\mathcal{L}(\Delta_i \cup \Delta_j)] = e^{-\tilde{m}_0 L}$ this case. Namely

$$\tilde{m}_0 \equiv \tilde{\delta}_0 \frac{\log N}{L} \quad (\sim \tilde{\delta}\frac{m}{40} \text{ if } L = 20 \log N/m). \tag{3.61}$$

Let $T(\{Y_i\})$ be the set of tree graphs (no loops) over $\{Y_i\}$ such that $\cup Y_i = X$. Thus applying (3.43) and (3.47) to (3.33), we have from (3.33) that

$$|\rho_X| \le \sum_{p=1}^{n} \frac{1}{p!} \sum_{\cup Y_i = X} \prod_1^p \left(A(Y_i)e^{\tilde{\varepsilon}_i}\right) \left(\sum_T \prod_{(ij)\in T} b_{ij}\right),$$

where $A(Y) \le \exp[-n_Y\tilde{\delta}_0 \log N - \tilde{m}_0\mathcal{L}(Y) + c_1 N^{-\hat{\delta}}|Y|]|$, $(c_1 = O(1))$, $Y_i \cap Y_j = \emptyset$ for $i \ne j$, and $b_{ij} \equiv \exp[-\tilde{\delta}_0 \log N - \tilde{m}_0 \text{dist}_c(Y_i, Y_j)]$ comes from $\partial/\partial s_{ij}$ and

$$\text{dist}_c(Y_i, Y_j) = \min_{\Delta_i \subset Y_i, \Delta_j \subset Y_j} \text{dist}_c(\Delta_i, \Delta_j). \tag{3.62}$$

Moreover we have put ( effects of loops in $\tilde{T}(\{Y_i\})$ )

$$1+\sum_{\ell\neq i} b_{i\ell}+\sum_{\ell<m} b_{i\ell}b_{im}+\cdots<\exp[\sum_{j:j\neq i} b_{ij}]<e^{\tilde{\epsilon}_i},\quad \tilde{\epsilon}_i=O(N^{-\tilde{\delta}_0}).$$

Then we can extract $\exp[-n_X\delta_0\log N-m_0\mathcal{L}(X)]$ choosing $\delta_0$ and $m_0$ slightly smaller than $\tilde{\delta}_0$ and $\tilde{m}_0$, respectively, to compensate $N^{-\hat{\delta}}|X|\leq n_X N^{-\hat{\delta}+\varepsilon_0}$. Finally we use lemma 10 to prove that the ramaining terms converge. Q.E.D.

**Remark 4** *We may choose* $\delta=\frac{1}{12}$ *and* $\tilde{\delta}=\frac{1}{16}$ *so that* $\hat{\delta}=\frac{1}{2}-3(\delta+\tilde{\delta})-2\varepsilon_0=\frac{1}{16}-2\varepsilon_0$. *Then* $\tilde{\delta}_0\sim\frac{1}{2}\tilde{\delta}=\frac{1}{32}$. *For large* $N$, $\delta_0\sim\tilde{\delta}_0$ *and* $m_0\sim\tilde{m}_0$.

## IV. POLYMER EXPANSION WITH LARGE FIELDS

We here show that the contributions from large field regions are small and that the dominant contributions come from small field regions we discussed. The analysis is easy in two extremal cases where $|\psi(x)|$ are very small or very large. If $|\psi|$ are small, we expand the determinant using the $N^{-1}$ expansion, and we extract small fields as $\exp[-<\psi,C^{-1}\psi>]$, leaving large fields untouched. Very large fields are easily estimated by the $|\psi|^{-N/2}$ behaviour of the determinant (thus the contribution is small). But it is hard to estimate contributions from $N^\delta<|\psi(x)|<N^{1/2+\delta}$ and from $|\psi(x)|<N^\delta$ near $R$. We bound their contributions by the stability. This makes our analysis complicated ( crude ).

For the large field region $R$ introduced by $\prod_{x\in R}\tau^c(\psi(x))$, we define another large field region $R^0=R(L_0)$ which includes points of $K=\Lambda-R$ near $R$:

$$R^0=R(L_0)\equiv\{x\in\Lambda;\mathrm{dist}(x,R)\leq L_0\}. \tag{4.1}$$

Let $\tilde{D}$ be the smallest paved set containing $R^0$. We denote $D$ the union of $\tilde{D}$ and those $\Delta\subset K$ nearest to $\tilde{D}$. We set $\partial D=D-\tilde{D}$, and we call it a collar [8] or a corridor [16]. Decompose $D$ into connected components $D_i$, and set $R_i=D_i\cap R$ and $R_i^0=D_i\cap R^0$. Then

$$D\ =\ \cup D_i,\ \mathrm{dist}(D_i,D_j)\geq L, i\neq j. \tag{4.2}$$

$$R_i\ =\ D_i\cap R,\ \mathrm{dist}(R_i,R_j)>3L+2L_0, i\neq j. \tag{4.3}$$

It is convenient to define two types of small field region:

$$K^0 = \Lambda - D, \ \tilde{K} = \Lambda - R^0. \tag{4.4}$$

In the following, we may write

$$A_i = \frac{2i}{\sqrt{N}} \chi_i G \psi \chi_i, \ A_{ij} = \frac{2i}{\sqrt{N}} \chi_i G \psi \chi_j, \tag{4.5}$$

where $\chi_0 = \chi_{\tilde{K}}, \chi_1 = \chi_{R^0}$, i.e., $A_0 = A_{\tilde{K}}$, $A_1 = A_{R^0}$, $A_{01} = A_{\tilde{K},R^0}$ and so on when there is no danger of confusion. Then we can factorize the determinant (see Remark below):

$$\det(1 + A_\Lambda) \ = \ \det(1 + A_{R^0}) \det(1 + A_{\tilde{K}} - W_{\tilde{K}}), \tag{4.6}$$

$$W_{\tilde{K}} \ \equiv \ A_{\tilde{K},R^0} \frac{1}{1 + A_{R^0}} A_{R^0,\tilde{K}}. \tag{4.7}$$

Here and hereafter we regard $A_R$, $G_R$ and so on as operators on $\mathbf{C}^R$, and $A_{R_1,R_2}$, $G_{R_1,R_2}$ and so on as operators $\mathbf{C}^{R_1} \to \mathbf{C}^{R_2}$, where $R, R_1, R_2 \subset \Lambda$.

**Theorem 11** *Let $D_i$ be any connected paved set and let $R_i$ be a large field region consistent with $D_i$. Put $R_i^0 = \{x \in D_i; \mathrm{dist}(x, R_i) \le L_0\}$. Then the following (stability) bound holds:*

$$\int |\det{}^{-N/2}(1 + A_{R_i^0})| \prod_{x \in R_i} d\psi(x) \ = \ \exp[- < \psi_{\tilde{R}_i}, \mathcal{T}_i \psi_{\tilde{R}_i} > - E(\psi_{\tilde{R}_i})] \tag{4.8}$$

$$E(\psi_{\tilde{R}_i}) \ \ge \ \frac{\beta}{10} |R_i| N^{\delta_2} \tag{4.9}$$

*where $\tilde{R}_i \ \equiv \ R_i^0 \backslash R_i$, $< \psi_{\tilde{R}_i}, \mathcal{T}_i \psi_{\tilde{R}_i} >$ is a positive bilinear form of $\psi_{\tilde{R}_i}$ defined later and $\delta_2 = O(1)$ $(= 1/24)$ is a strictly positive constant discussed later.*

**Theorem 12** *The small field contribution is represented by the polymer expansion:*

$$\begin{aligned} Z_{\tilde{K}} \ &\equiv \ \int \exp[- \sum_i < \psi_{\tilde{R}_i}, \mathcal{T}_i \psi_{\tilde{R}_i} > + \frac{N}{2} \mathrm{Tr} A_{\tilde{K}}] \det{}^{-N/2}(1 + A_{\tilde{K}} - W_{\tilde{K}}) \prod_{x \in K} \frac{d\psi(x)}{\sqrt{\pi}} \\ &= \ Z_\infty \left( \frac{\det^{\frac{1}{2}}(C_{\tilde{K}})}{Z_\infty} \right) \left[ \sum \frac{1}{n!} \sum_{\cup_1^n X_i = K} \prod \tilde{\rho}_{X_i} \right] \end{aligned} \tag{4.10}$$

*where $C_{\tilde{K}} = [\chi_{\tilde{K}} G^{\circ 2} \chi_{\tilde{K}}]^{-1}$. $\tilde{\rho}_X$ satisfies the following bound uniformly in $\psi_R(x)$:*

$$|\tilde{\rho}_X| \le \exp[-m_0 \mathcal{L}(X \wedge D) - \delta_0 n_X \log N + \pi L_0^2 |R_X| \delta \log N], \ \ \textit{for } n_X \ge 2. \tag{4.11}$$

*Here $R_X = R \cap X$, $n_X$ the number of unit squares $\Delta \subset X$ such that $\Delta \cap R^0 = \emptyset$, and $\mathcal{L}(X \wedge D)$ is the length of the shortest tree graph over $D_\ell \subset X$ and centers of $\Delta \subset X$.*

The reader should note that these theorems mean that

$$Z_\infty Z(R) \sim Z_{\tilde{K}} \exp[-\sum_i \min_{\psi_{\tilde{R}_i}} E(\psi_{\tilde{R}_i})]$$

and $\rho_X \sim \tilde{\rho}_X \exp[-\min_{\psi_{\tilde{R}_X}} E(\psi_{\tilde{R}_X})]$. (The estimate of $\psi_{R_i}$ in $Z_{\tilde{K}}$ remains.) Since the factor $E(\psi_R)$ compensates $\pi L_0^2 |R_X| \delta \log N$ in $\tilde{\rho}_X$ which originates from small fields near $R_X$, we obtain sufficiently small $\rho_X$. We prove these theorems in the rest of this section.

**Remark 5** *For matrices A,B, C and D of sizes $\ell \times \ell$, $m \times m$, $m \times \ell$ and $\ell \times m$ rspectively, we have (blockwise diagonalization [11]):*

$$\begin{pmatrix} A & D \\ C & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B - CA^{-1}D \end{pmatrix} \begin{pmatrix} 1 & A^{-1}D \\ 0 & 1 \end{pmatrix}. \tag{4.12}$$

### A. Polymer Expansion of $A_i$ with Large Fields

#### *1. Properties and Expansions of $J(R)$*

We note that

$$W_{\tilde{K}} = G_{\tilde{K},R^0} J(R^0, \psi_{R^0}) G_{R^0,\tilde{K}} \frac{2i}{\sqrt{N}} \psi_{\tilde{K}}, \tag{4.13}$$

$$J(R^0, \psi_{R^0}) = \frac{1}{G_{R^0} - i\frac{\sqrt{N}}{2}\psi_{R^0}^{-1}}, \tag{4.14}$$

where $G_{R^0} = \chi_{R^0} G \chi_{R^0}$, $G_{\tilde{K},R^0} = \chi_{\tilde{K}} G \chi_{R^0}$ and so on, and $\psi_{R^0}$ is regarded as the diagonal matrix: $\psi_{R^0} = \mathrm{diag}(\psi(x), x \in R^0)$. We first study properties of the operator $J(R, \psi)$.

**Lemma 13** *The following relations hold:*
*(1) $||J(R, \psi_R)|| \leq ||G_R^{-1}|| \leq 8 + m^2$ uniformly in $R \neq \emptyset$ and $\psi_R$.*
*(2) $[G_R - i\psi_R^{-1}]^{-1}(x, y)$ decays exponentially fast uniformly in $R \neq \emptyset$ and $\psi_R$ :*

$$|[G_R - i\psi_R^{-1}]^{-1}(x, y)| \leq \mathrm{const.}\, G_R(x, y). \tag{4.15}$$

*Proof.* (1) Since $m^{-2} > G_R > (m^2 + 8)^{-1} > 0$ uniformly in $R \neq \emptyset$, $G_R^{1/2}$ and $G_R^{-1/2}$ satisfy inequalities of the same type. Moreover since $G_R^{-1/2}\psi_R^{-1}G_R^{-1/2}$ is self-adjoint, we see that

$||1 - i\frac{\sqrt{N}}{2} G_R^{-1/2} \psi_R^{-1} G_R^{-1/2}|| \geq 1$. Then the conclusion follows from

$$J(R,\psi) = G_R^{-1/2} \frac{1}{1 - \frac{i\sqrt{N}}{2} G_R^{-1/2} \psi_R^{-1} G_R^{-1/2}} G_R^{-1/2}.$$

(2) We first note that

$$\frac{1}{G_R - i\psi_R^{-1}} = i\psi_R \frac{1}{G_R^{-1} + i\psi_R} G_R^{-1},$$

where ( [17], Theorem VIII.1, or use (4.12))

$$G_R^{-1} = \chi_R(-\Delta + m^2)\chi_R - B_{\partial R}, \tag{4.16}$$

$$B_{\partial R} = E(\chi_{R^c}(-\Delta + m^2)\chi_{R^c})^{-1} E^*, \tag{4.17}$$

$$E = \chi_R(-\Delta)\chi_{R^c}. \tag{4.18}$$

Here $B_{\partial R}$ is a positive operator bounded by $\chi_R(-\Delta + m^2)\chi_R$ (by the positivity) and has non-negative matrix elements. $B_{\partial R}(x,y) \neq 0$ if and only if $(x,y) \in \partial R \times \partial R$ where $\partial R = \{x \in R; {}^\exists y \in R^c, |x-y| = 1\}$. Then we have the convergent Neumann expansion

$$i\psi_R \frac{1}{\chi_R(-\Delta + m^2 + i\psi)\chi_R - B_{\partial R}} = i\psi_R G_R^D(\psi) \left[\sum_{n=0}^{\infty} (B_{\partial R} G_R^D(\psi))^n\right]$$

where $G_R^D(\psi) = [\chi_R(-\Delta + m^2 + i\psi)\chi_R]^{-1}$ and

$$|G_R^D(\psi)_{xy}| \leq G_R^D(\psi = 0)_{xy}$$

$$|\psi_x G_R^D(\psi)_{xy}| \leq (4 + m^2) G_R^D(\psi = 0)_{xy}$$

as is proved by the random walk representation of $G_R^D(\psi)$. Putting all $\psi = 0$, we find that

$$|\frac{1}{G_R - i\psi_R^{-1}}(x,y)| \leq (4 + m^2) \sum_\zeta G_R(x,\zeta) |G_R^{-1}(\zeta,y)|$$

Then (2) follows since $|G_R^{-1}(\zeta,y)| = 2(m^2+4)\delta_{\zeta y} - G_R^{-1}(\zeta,y)$ by (4.16). Q.E.D.

**Lemma 14** *$J(R,\psi)$ admits the following cluster (random walk) expansion:*

$$J(R,\psi) = \sum_{X \subset R} \delta J(X,\psi) \tag{4.19}$$

*where $X$ are intersections of $R$ with paved sets ($X = \cup_i(\Delta_i \cap R)$). Moreover $\delta J(X,\psi)$ depends only on $\psi(x)$, $x \in X$. If* $\mathrm{diam}(X) > \sqrt{2}(2L+1)$*, then*

$$||\delta J(X,\psi)|| \le \exp[-m_1\mathcal{L}(X)], \tag{4.20}$$

$$|\delta J(X,\psi)_{xy}| \le \exp[-m_1\mathcal{L}(X,x,y)], \tag{4.21}$$

*where $\delta J(X,\psi)_{xy}$ is the $(x,y)$ component of $\delta J(X,\psi)$ ( $x,y \in X$ ) and $\mathcal{L}(X,x,y)$ is the length of the shortest walk from $x$ to $y$ through all centers of $\Delta_\ell \subset X$, $x \notin \Delta_\ell$, $y \notin \Delta_\ell$.*

*Proof.* We apply the expansion procedure by Federbush and Brydges to $G_R^{-1}$. For any $X \subset R$, $X = \cup_1^n(\Delta_i \cap R)$, we choose $\Delta_1 \cap R \subset X$ and $s_1 \in [0,1]$ and define

$$\begin{aligned} G(X,s_1) &= [(1-s_1)(G^{-1}_{X\backslash\Delta_1} + G^{-1}_{\Delta_1}) + s_1 G_X^{-1}]^{-1}, \\ J(X,s_1) &= [G(X,s_1) - ih_X]^{-1}, \end{aligned}$$

where $h = \sqrt{N}\psi^{-1}/2$, $G_X^{-1} \equiv \chi_X G_R^{-1}\chi_X$ and $\Delta_i \cap R$ is denoted as $\Delta_i$ for simplicity. Then $J(X) = J(X, s_1 = 1)$ and $J(X,s_1)$ is bounded uniformly in $h$ and $s_1$, and we have

$$\begin{aligned} J(X) &= J(X, s_1 = 0) + \int_0^1 J'(X,s_1)ds_1 \\ &= J(X\backslash\Delta_1) \oplus J(\Delta_1) - \sum_{\Delta_2 \neq \Delta_1} \int_0^1 J(X,s_1)G(X,s_1)\delta G_{12}^{-1} G(X,s_1)J(X,s_1)ds_1 \end{aligned}$$

where $\delta G_{ij}^{-1} = G^{-1}_{\Delta_i\Delta_j} + G^{-1}_{\Delta_j\Delta_i}$ and we have used

$$\frac{\partial}{\partial s_1} G(X,s_1) = -\sum_{\Delta_2 \subset X\backslash\Delta_1} G(X,s_1)\left[G^{-1}_{\Delta_1,\Delta_2} + G^{-1}_{\Delta_2,\Delta_1}\right] G(X,s_1)$$

and so on. We choose $\Delta_2 \neq \Delta_1$ and $s_2$ in the next step and continue the process inductively. (See Appendix and the proof of Theorem 3).

Let $J(R)_{xy}$ be the $(x,y)$ component of $J(R)$. Then we have

$$J(R)_{x,y} = \sum_{X \subset R} \delta J(X)_{x,y}, \ \delta J(X)_{xy} = \sum_T \delta J(X)_T(x,y)$$

where $T$ are tree graphs over $\{\Delta_1 \cap R, \cdots, \Delta_n \cap R\}$ with root $\Delta_1$ and $\delta J(X)_T(x,y)$ is given by

$$\begin{aligned} &\sum_{\gamma:T(\gamma)=T} \sum_\pi (-1)^{n-1} \int M_\gamma(s) \prod_1^{n-1} ds_i \sum_{k_i=0,1} [J(X,s_\gamma)G(X,s_\gamma)]_{x,\ell_{\pi(1)}} \delta G^{-1}_{\ell_{\pi(1)},m_{\pi(1)}} \\ &\quad \times G^{(k_1)}_{m_{\pi(1)},\ell_{\pi(2)}} \delta G^{-1}_{\ell_{\pi(2)},m_{\pi(2)}} \cdots G^{(k_{n-1})}_{m_{\pi(n-2)},\ell_{\pi(n-1)}} \delta G^{-1}_{\ell_{\pi(n-1)},m_{\pi(n-1)}} [G(X,s_\gamma)J(X,s_\gamma)]_{m_{\pi(n-1)},y} \end{aligned}$$

with $G^{(0)} = G(X, s_\gamma)$ and $G^{(1)} = G(X, s_\gamma)J(X, s_\gamma)G(X, s_\gamma)$. Here $\gamma$ are tree graphs over $\{\Delta_{j_1}, \cdots, \Delta_{j_n}\}$ $(j_1 = 1)$ and for given tree $\gamma = \{b_1, b_2, \cdots, b_{n-1}\}$, $b_k = (\ell_k, m_k)$ $(\ell_k,\, m_k \in \{j_1, \cdots, j_n\})$, $\pi$ stands for permutations of $\{b_k = (\ell_k, m_k)\}_1^{n-1}$. Moreover $s_i$ are introduced following the tree graph $\gamma$. (See Theorem 2 for the notation.)

$G^{-1}(X, s_\gamma)$ is a convex linear combination of $\chi_Y(-\Delta + m^2 - B_{\partial R})\chi_Y$, $Y \subset X$. Then the non-diagonal terms of $G^{-1}(X, s_\gamma)$ are negative (ferromagnetic), and we have

$$|G^{(i)}(X, s_\gamma)_{x,y}| \;\leq\; c_1 m^{-2} \exp[-m_2|x - y|],$$

uniformly in $\{s_i\}$ and $X$, where $i = 0, 1$, $m_2 = 4m/5$ and $c_1$ is a positive constant.

If $\Delta_i$ and $\Delta_j$ are nearest neighbour and $x \in \Delta_i$ and $y \in \Delta_j$ are close to each other, some of the matrix elements $(\delta G^{-1}_{ij})_{xy}$ may be large. Since $e^{-mL} << 1$, this happens only for blocks of form $\cup_{i=1}^{p}\Delta_i$ with $\mathrm{diam}(\cup\Delta_i) \leq \sqrt{2}(2L + 1)$ (thus $p \leq 4$). Then for $n > 4$

$$|G(s)_{p,\ell_{\pi(1)}}\delta G^{-1}_{\ell_{\pi(1)},m_{\pi(1)}} G^{(k_1)}(s)_{p,\ell_{\pi(1)}} \cdots \delta G^{-1}_{\ell_{\pi(n-1)},m_{\pi(n-1)}} G(s)_{m_{\pi(n-1)},q}| \leq \exp[-\frac{1}{5} m_2 \mathcal{L}_{\pi(\gamma)}(p, q)],$$

$$\mathcal{L}_{\pi(\gamma)}(p, q) = \mathrm{dist}_c(p, \ell_{\pi(1)}) + \mathrm{dist}_c(\ell_{\pi(1)}, m_{\pi(1)}) + \mathrm{dist}_c(m_{\pi(1)}, \ell_{\pi(2)}) + \cdots + \mathrm{dist}_c(m_{\pi(n-1)}, q),$$

where $\mathrm{dist}_c(i, j) \equiv \mathrm{dist}_c(\Delta_i, \Delta_j)$. We can then extract either the tree decay factor of $\gamma$

$$\exp[-m_1^{'}(\mathrm{dist}_c(p, \ell_{\pi(1)}) + \mathrm{dist}_c(m_{\pi(n-1)}, q))] \prod_{(ij)\in\gamma} \exp[-m_1^{''}\mathrm{dist}_c(i, j)], \tag{4.22}$$

or the decay factor proportional to the length of walk, $\exp[-m_1^{''}\mathcal{L}(\Delta_p, \{\Delta\}, \Delta_q)]$ with the remainder bounded by $\sum_\pi \exp[-m_1^{'}\mathcal{L}_{\pi(\gamma)}(p, q)]$ where $m_1^{'} + m_1^{''} = m_2/5$. We complete the proof by Lemma 9, by replacing $m_1^{'}$ by $m_1 \equiv m/10 < m_1^{'}$ to compensate $K_2^n$. Q.E.D.

**Remark 6** *In the proof of Lemma 14, we may introduce interpolation parameters $s_i$ in such a way that*

$$G_R \to G_R(s) \equiv (1 - s)(\chi_{R\setminus\Delta} G_R \chi_{R\setminus\Delta} + \chi_\Delta G_R \chi_\Delta) + sG_R$$

*in the denominator of $J(R, \psi)$, though $G_R^{-1}(s)$ may not be ferromagnetic this case. See Appendix B. Moreover if $R = \cup R_i$ and $\{R_i\}$ distribute dilutely, we can just Taylor-expand the off-daigonal terms $G_{R_i,R_j}$ $(i \neq j)$. This is the standard random walk expansion.*

Let us consider the contribution from the large field region $R^0 = \cup R_i^0$, $R_i^0 = R^0 \cap D_i$:

$$\det{}_2^{-\frac{N}{2}}(1 + A_{R^0}) = \left[\prod_i \det{}_2^{-\frac{N}{2}}(1 + A_{R_i^0})\right] \det{}^{-\frac{N}{2}}\left(1 + \sum_{i\neq j} \delta A_{ij}\right), \tag{4.23}$$

$$\delta A_{ij} = A_{R_i^0,R_j^0}\frac{1}{1 + A_{R_j^0}} = G_{R_i^0,R_j^0}\frac{1}{G_{R_j^0} - \frac{i\sqrt{N}}{2\psi_{R_j^0}}}. \tag{4.24}$$

Since $R_i^0$ and $R_j^0$ are separated by distance more than $3L$, we see that

$$\begin{aligned} ||\delta A_{ij}||_1 &\leq m^{-4}\exp[-m\mathrm{dist}(R_i^0, R_j^0)] \times \min\{|R_i^0|, |R_j^0|\} \\ &\leq \min\{|R_i^0|, |R_j^0|\}\exp[-\frac{4m}{5}\mathrm{dist}(R_i^0, R_j^0)] \end{aligned} \tag{4.25}$$

uniformly in $\psi(x), x \in R^0$, where $||A||_p^p = \mathrm{Tr}|A|^p$ $(p \geq 1)$. (Note that $||A||_1 = \mathrm{Tr}|A| \leq \sum |A(x,y)|$ and $||A||_2^2 = \sum |A(x,y)|^2$. ) Then it is enough to consider $\det(1 + A_{R_i^0})$.

Let $\delta_1$ be a positive constant such that $0 < 2\delta - 3\delta_1$, and set $R_i = L_i \cup M_i$ where

$$L_i = \{x \in R_i; |\psi(x)| > N^{\frac{1}{2}+\delta_1}\},\ M_i = \{x \in R_i; |\psi(x)| \leq N^{\frac{1}{2}+\delta_1}\}. \tag{4.26}$$

($L$ stands for Large, and $M$ stands for Medium. Only in this subsection, $L$ and $L_i$ stand for regions of very large fields $\psi$. We apologize for the abuse of notation.) We also introduce

$$L_i(L_0) = \{x \in R_i^0; \mathrm{dist}(x, L_i) \leq L_0\}, \tag{4.27}$$

$$M_i(L_0) = \{x \in R_i^0; \mathrm{dist}(x, M_i) \leq L_0\}, \tag{4.28}$$

and set $\tilde{M}_i = R_i^0 - L_i = M_i \cup \tilde{R}_i$. For notational simplicity, we omit the subscript $i$ for a while and we denote $R_i^0$ by $R^0$, $R_i$ by $R$ and $L_i$ by $L$ and so on. We first extract $\psi_L = \chi_L\psi\chi_L$:

$$\det(1 + A_{R^0}) = \det(1 + A_L)\det\left[1 + (T_{\tilde{M}} - \delta T_{\tilde{M}})\frac{2i}{\sqrt{N}}\psi_{\tilde{M}}\right], \tag{4.29}$$

where we have used the following abbreviations:

$$A_L = \chi_L G \chi_L \frac{2i}{\sqrt{N}}\psi_L, \tag{4.30}$$

$$T_{\tilde{M}} = G_{\tilde{M}} - G_{\tilde{M},L}G_L^{-1}G_{L,\tilde{M}}, \tag{4.31}$$

$$\delta T_{\tilde{M}} = G_{\tilde{M},L}\left[(G_L - \frac{i\sqrt{N}}{2\psi_L})^{-1} - G_L^{-1}\right]G_{L,\tilde{M}}. \tag{4.32}$$

**Lemma 15** *If* $\{c_1 < |\psi(x)| < c_2\,; x \in A\}$ , $0 < c_i$ , *then*

$$\mathrm{spec}|G_A^{1/2}\psi_A G_A^{1/2}| \subset [\frac{c_1}{m^2+8}, \frac{c_2}{m^2}]. \tag{4.33}$$

*Proof.* Since $(8+m^2)^{-1} < f, f > \le < f, G_A f > \le m^{-2} < f, f >$ for $f \in \mathbf{C}^A$, we have

$$\begin{aligned} ||G_A^{1/2}\psi_A G_A^{1/2} f||^2 &= < \psi_A G_A^{1/2} f, G_A \psi_A G_A^{1/2} f > \\ &\ge (8+m^2)^{-1} < \psi_A G_A^{1/2} f, \psi_A G_A^{1/2} f > \\ &\ge (8+m^2)^{-2} (\inf_{x\in A} |\psi(x)|^2) \ < f, f > . \end{aligned}$$

The other inequality is also immediate. Q.E.D.

**Lemma 16** *The matrices* $T_{\tilde{M}}$ *and* $\delta T_{\tilde{M}}$ *have the following properties:*

$$T_{\tilde{M}}^{-1} = (G_{R^0}^{-1})_{\tilde{M}} \equiv \chi_{\tilde{M}} G_{R^0}^{-1} \chi_{\tilde{M}}, \tag{4.34}$$

$$T_{\tilde{M}}^{1/2} = G_{\tilde{M}}^{1/2} + t_{\tilde{M}}^{1/2}, \tag{4.35}$$

$$|t_{\tilde{M}}^{1/2}(x,y)| \le cm^{-4} \exp\left[-\frac{m}{2}\left(\mathrm{dist}(x,L) + \mathrm{dist}(y,L) + |x-y|\right)\right], \tag{4.36}$$

$$||\delta T_{\tilde{M}}||_1 \le |L| N^{-\delta_1 + \varepsilon_0}. \tag{4.37}$$

*Proof.* To show (4.34), we take the inverses of the both sides of the block-diagonalization of $G_{R^0} > 0$:

$$G_{R^0} = U \begin{pmatrix} G_L & 0 \\ 0 & T_{\tilde{M}} \end{pmatrix} U^*,\ U = \begin{pmatrix} 1 & 0 \\ G_{\tilde{M}L} G_L^{-1} & 1 \end{pmatrix}.$$

To show the second, using $T^{-1/2} = 2\int_0^\infty (T+u^2)^{-1} du/\pi$, we have

$$\begin{aligned} T_{\tilde{M}}^{-1/2} &= G_{\tilde{M}}^{-1/2} + \hat{t}_{\tilde{M}}^{-1/2}, \\ \hat{t}_{\tilde{M}}^{-1/2} &= 2\int \frac{1}{G_{\tilde{M}} + u^2} G_{\tilde{M}L} \frac{1}{F_L(u)} G_{L\tilde{M}} \frac{1}{G_{\tilde{M}} + u^2} \frac{du}{\pi}, \\ F_L(u) &= G_L - G_{L\tilde{M}} (G_{\tilde{M}} + u^2)^{-1} G_{\tilde{M}L} \end{aligned}$$

where $|G_{\tilde{M}}^{-1}(x,y)| \le ce^{-m|x-y|}$, $|G_{\tilde{M}L}(x,y)| \le c\log(1+m^{-1})e^{-m|x-y|}$, $(x \in \tilde{M},\ y \in L)$ and $F_L(u)^{-1}(x,y) \le ce^{-m|x-y|}$, $x, y \in L$ uniformly in $u \ge 0$. In fact $F_L^{-1}$ is essentially equal to

$(G_{R^0}^{-1})_L$. Then $\hat{t}^{-1/2}$ has the decay property (4.36) except for the coefficient. We multiply $T_{\tilde{M}}$ to the expression of $T_{\tilde{M}}^{-1/2}$ to obtain (4.35).

To estimate $||\delta T_M||_1$, we expand $(G_L - i\frac{\sqrt{N}}{2\psi_L})^{-1}$ into series of $G_L^{-1}$ which converge absolutely since $|\sqrt{N}/\psi_L(x)| \leq N^{-\delta_1}$. Since $||G_L^{-1}|| \leq 8 + m^2$ and $||G_{\tilde{M}L}||_2^2 = \sum_{xy} G_{\tilde{M}L}^2(x,y) \leq c|L|\log^2(1+m^{-1})m^{-2}$, (4.37) follows from the definition (3.41) of $\varepsilon_0$. Q.E.D.

Let

$$\det(1+A_{R^0}) \;=\; \det(1+A_L)\det\left[1-\delta T_{\tilde{M}}\frac{1}{T_{\tilde{M}}-\frac{i\sqrt{N}}{2\psi_{\tilde{M}}}}\right]\det(1+T_{\tilde{M}}\frac{2i}{\sqrt{N}}\psi_{\tilde{M}}). \tag{4.38}$$

Using $\det(1+A) = \exp[\mathrm{Tr}(A+O(A^2))]$ and $|\det(A_L)| \leq |\det(1+A_L)|$, we have estimates

$$|\det{}^{-\frac{N}{2}}(1-\delta T_{\tilde{M}}\frac{1}{T_{\tilde{M}}-\frac{i\sqrt{N}}{2\psi_{\tilde{M}}}})| \;\leq\; \exp[|L|N^{1-\delta_1+\varepsilon_0}], \tag{4.39}$$

$$\begin{aligned} |\det{}^{-\frac{N}{2}}(1+A_L)| \;&\leq\; \left[\prod_{x\in L}\frac{\sqrt{N}}{2|\psi(x)|}\right]^{N/2}\det{}^{-N/2}(G_L) \\ &\leq\; \exp\left[-\frac{1}{2}N\sum_{x\in L}\left\{\log\left(\frac{2|\psi(x)|}{\sqrt{N}}\right)-\log(8+m^2)\right\}\right] \end{aligned} \tag{4.40}$$

Therefore we have (using 2/5 instead of 1/2) :

**Lemma 17** *If $N \geq N(\beta)$ so that $\delta_1 > \varepsilon_0$, then*

$$\left|\left[\det(1+A_L)\det(1-\delta T_{\tilde{M}}\frac{1}{T_{\tilde{M}}-\frac{2i\sqrt{N}}{\psi_{\tilde{M}}}})\right]\right|^{-\frac{N}{2}} < \exp[-\frac{2}{5}N\sum\log(|\frac{\psi(x)}{\sqrt{N}}|)]. \tag{4.41}$$

It remains to estimate the final determinant in the R.H.S. of eq.(4.38) :

$$\begin{aligned} |\det{}^{-N/2}(1+\frac{2i}{\sqrt{N}}T^{1/2}\psi_{\tilde{M}}T^{1/2})| \;&=\; \det{}^{-N/4}(1+\frac{4}{N}[\hat{T}_0+\hat{T}_1]) \\ &=\; \exp[-\Psi_0-\Psi_1] \end{aligned} \tag{4.42}$$

where $T \equiv T_{\tilde{M}}$ and

$$\hat{T}_0 \;=\; T^{1/2}\psi_{\tilde{M}}T^{1/2}\chi_{\tilde{M}\backslash R(L_0/2)}T^{1/2}\psi_{\tilde{M}}T^{1/2}, \tag{4.43}$$

$$\hat{T}_1 \;=\; T^{1/2}\psi_{\tilde{M}}T^{1/2}\chi_{R(L_0/2)}T^{1/2}\psi_{\tilde{M}}T^{1/2}, \tag{4.44}$$

$$\Psi_0 \;=\; \frac{N}{4}\mathrm{Tr}\log\left(1+\frac{4}{N}\hat{T}_0\right), \tag{4.45}$$

$$\Psi_1 \;=\; \frac{N}{4}\mathrm{Tr}\log\left(1+\frac{4}{N}\frac{1}{(1+\frac{4}{N}\hat{T}_0)^{1/2}}\hat{T}_1\frac{1}{(1+\frac{4}{N}\hat{T}_0)^{1/2}}\right). \tag{4.46}$$

Both $\hat{T}_0$ and $\hat{T}_1$ are positive. Put

$$\begin{aligned}
\Phi_0 &= \mathrm{Tr}\hat{T}_0 = \sum_{x,y\in\tilde{M}} \psi(x)\hat{\mathcal{T}}_0(x,y)\psi(y) \equiv < \psi_{\tilde{M}}\hat{\mathcal{T}}_0\psi_{\tilde{M}} >, && (4.47)\\
\Phi_1 &= \mathrm{Tr}\hat{T}_1 = \sum_{x,y\in\tilde{M}} \psi(x)\hat{\mathcal{T}}_1(x,y)\psi(y) \equiv < \psi_{\tilde{M}}\hat{\mathcal{T}}_1\psi_{\tilde{M}} >, && (4.48)\\
\hat{\mathcal{T}}_1 &= (T_{\tilde{M}}^{1/2}\chi_{R(L_0/2)}T_{\tilde{M}}^{1/2})\circ T_{\tilde{M}}, && (4.49)\\
\hat{\mathcal{T}}_0 &= (T_{\tilde{M}}^{1/2}\chi_{\tilde{M}\backslash R(L_0/2)}T_{\tilde{M}}^{1/2})\circ T_{\tilde{M}} \equiv \mathcal{T}+\delta\mathcal{T}, && (4.50)\\
\mathcal{T} &\equiv (G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2})\circ G_{R^0}. && (4.51)
\end{aligned}$$

where $\tilde{M} = R^0\backslash L$ and note that $\tilde{M}\backslash R(L_0/2) = R^0\backslash R(L_0/2)$. Since $G_{R^0}^{1/2}(x,y) \le ce^{-m|x-y|}$ and $G_{\tilde{M}}^{1/2}(x,y) \le ce^{-m|x-y|}$ (Appendix B), we have

$$\begin{aligned}
|(G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2})(x,y)| &\le N^{-1+\varepsilon_0}, \text{ if } x\in R,\, y\in R^0,\\
|(G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2})(x,y)| &\le N^{-2+\varepsilon_0}, \text{ if } x\in R,\, y\in R.
\end{aligned}$$

Since $\psi_{\tilde{M}} = \psi_{\tilde{R}} + \psi_M$, we have

$$\begin{aligned}
\Phi_0 &= < \psi_{\tilde{R}}, \mathcal{T}\psi_{\tilde{R}} > + \delta\Phi_0, && (4.52)\\
|\delta\Phi_0| &\le \text{const.}|L|L_0^2 N^{1/2+2\delta}e^{-mL_0/2} \le |L|N^{-1/2+2\delta+2\varepsilon_0}. && (4.53)
\end{aligned}$$

The argument of the same type shows that $||\hat{T}_0/N|| \le N^{-1+2\delta+\varepsilon_0}$ and $||\hat{T}_1/N|| \le N^{2\delta_1+\varepsilon_0}$.

We remark the following facts: Let $A$ and $B$ be any positive matrices. Then

(i) $\mathrm{Tr}(xA - \frac{1}{2}x^2A^2) \le \mathrm{Tr}\log(1+A) \le \mathrm{Tr}A$ for any $x\in[0,1]$.

(ii) $A\circ B \ge c\,\mathrm{diag}(A)$ if $B \ge c1$, where 1 is the identity.

The fact (i) is trivial and the fact (ii) follows from $A\circ B = A\circ(c1+(B-c1)) \ge cA\circ 1$ where $A\circ 1 = \mathrm{diag}(A)$. Then we have

$$\begin{aligned}
\Phi_0 &\ge \Psi_0 \ge (1-O(N^{-1}))\Phi_0 = \Phi_0 + O(N^{-1+2\varepsilon_0+2\delta}|R|),\\
\Phi_1 &\ge \Psi_1 \ge (1-O(N^{-2\delta_1+2\varepsilon_0}))N^{-3\delta_1}\Phi_1.
\end{aligned}$$

(we used (i) with $x = N^{-3\delta_1}$ in the second.) To obtain the lower bound for $\Phi_1 > 0$, we apply (ii) by setting $A = T^{1/2}\chi_{M(L_0/2)}T^{1/2}$ and $B = T$, where $T = T_{\tilde{M}}$. Therefore we have

$$\Phi_1 \ge \frac{1}{8+m^2}\sum_{x\in\tilde{M}}\left[\sum_{\zeta\in R(L_0/2)} T^{1/2}(x,\zeta)^2\right]\psi(x)^2 \qquad (4.54)$$

since $||T|| \geq (8+m^2)^{-1}$, see Lemma 16. Here again by Lemma 16, we have $\sum_{\zeta\in R(L_0/2)} T^{1/2}(x,\zeta)^2 = G_{\tilde{M}}(x,x) - O(N^{-1/2+\varepsilon_0}) = \beta - O(N^{-1/2+\varepsilon_0}) >> 1$ for $x \in R(L_0/2)\backslash L(L_0/2)$. Thus we find that

$$\Phi_1 \geq \frac{\beta}{9} \sum_{x\in R(L_0/2)\backslash L(L_0/2)} \psi(x)^2 \tag{4.55}$$

Therefore we choose $\delta_1 > 0$ so that

$$\delta_2 \equiv 2\delta - 3\delta_1 > 1.2\times\varepsilon_0, \quad \delta_1 > 1.2\times\varepsilon_0, \tag{4.56}$$

which are satisfied by $\delta = 1/12$ and $\delta_1 = \delta_2 = 1/24$. ($\delta_2 > 1.2\times\varepsilon_0$ is needed later.)

*Proof of Theorem 11.* Putting $\mathcal{T} = \mathcal{T}_i$, $R = R_i$, $L = L_i$ and so on, we have

$$\begin{aligned} |\det{}^{-N/2}(1+A_{R_i^0})| \;\leq\; & \exp[-<\psi_{\tilde{R}_i}, \mathcal{T}_i\psi_{\tilde{R}_i}> + |L_i|N^{-1/2+\delta+\varepsilon_0} + N^{-1+2\delta+2\varepsilon_0}|R_i| \\ & -c_1 N^{-3\delta_1} \sum_{x\in R_i(L_0/2)\backslash L_i(L_0/2)} \psi^2(x) - \frac{2}{5}\sum_{x\in L_i} N\log|\frac{\psi(x)}{\sqrt{N}}|\,] \end{aligned}$$

where $c_1 \geq \beta/9$. We fix $L_i \subset R_i$ and integrate over $\psi(x)$, $x\in R_i$ noticing that $\int_s^\infty e^{-x^2}dx = e^{-s^2}/2s(1+O(s^{-1}))$ and $c_1|L_i(L_0)|N^{\delta_2} \leq (1/15)\delta_1|L_i|N\log N$:

$$\begin{aligned} & \int |\det{}^{-N/2}(1+A_{R_i^0})| \prod_{x\in L_i} d\psi(x) \prod_{x\in R_i\backslash L_i} d\psi(x) \\ & \leq\; e^{-<\psi_{\tilde{R}_i},\mathcal{T}_i\psi_{\tilde{R}_i}>} \exp[-(c_1-o(1))|R_i|N^{\delta_2} - \frac{1}{3}\delta_1|L_i|N\log N] \end{aligned}$$

Take the sum over all $L_i \subset R_i$ and put $c_2 = c_1 - o(1) - O(e^{-N}) \geq \beta/10$. Q.E.D.

## B. Polymer Expansion of the Gaussian Measure

### 1. *Stability of Small Fields*

For any large field region $R$, we integrate the following function:

$$\Xi_R(\psi) \equiv \prod_i \mathcal{D}(A_{R_i^0}) \det{}^{-\frac{N}{2}}(1+\sum_{i\neq j}\delta A_{ij}) \det{}_3^{-\frac{N}{2}}(1+A_{\tilde{K}} - W_{\tilde{K}})e^{-V}\tau(\psi_K)\tau^c(\psi_R), \tag{4.57}$$

where

$$\mathcal{D}(A_{R_i^0}) \;=\; \det{}_2^{-N/2}(1+A_{R_i^0})\exp[<\psi_{\tilde{R}_i}, \mathcal{T}_i\psi_{\tilde{R}_i}>], \tag{4.58}$$

$$\mathcal{T}_i \;=\; (G_{R_i^0}^{1/2}\chi_{R_i^0\backslash R_i(L_0/2)}G_{R_i^0}^{1/2}) \circ G_{R_i^0} \tag{4.59}$$

and

$$V = < \psi_{\tilde{K}}, G^{\circ 2}\psi_{\tilde{K}} > + \delta V_K \equiv V_0 + V_1, \tag{4.60}$$

$$V_0 = < \psi_{\tilde{K}}, G^{\circ 2}\psi_{\tilde{K}} > +2 < \psi_{\tilde{K}}, G^{\circ 2}\psi_{R^0\backslash R} > + \sum_i < \psi_{R_i^0\backslash R_i}, \mathcal{T}_i \psi_{R_i^0\backslash R_i} >, \tag{4.61}$$

$$V_1 = -\frac{N}{2}\mathrm{Tr}\left(W_{\tilde{K}} - A_{\tilde{K},R^0\backslash R}A_{R^0,\tilde{K}}\right) - \frac{N}{2}\mathrm{Tr}(A_{\tilde{K}}W_{\tilde{K}} - \frac{1}{2}W_{\tilde{K}}^2) \tag{4.62}$$

$$\delta V_K = \sum_i < \psi_{\tilde{R}_i}, \mathcal{T}_i \psi_{\tilde{R}_i} > -\frac{N}{2}\mathrm{Tr}(W_{\tilde{K}} + A_{\tilde{K}}W_{\tilde{K}} - \frac{1}{2}W_{\tilde{K}}^2). \tag{4.63}$$

(Remark that $\tilde{R}_i \equiv R_i^0\backslash R_i$). $V_0$ does not depend on $\psi(x)$, $x \in R$, and $V_1$ contains $\psi(x)$, $x \in R$ only through $W_{\tilde{K}}$. $W_{\tilde{K}}$ is bounded uniformly in $\psi(x)$, $x \in R$, because of the small field region surrounding $R$. We would like to stress that $< \psi_{\tilde{K}}, G^{\circ 2}\psi_{R^0\backslash R} >= -\frac{N}{4}\mathrm{Tr}A_{\tilde{K},R^0\backslash R}A_{R^0,\tilde{K}}$ is extracted from $\mathrm{Tr}W_{\tilde{K}}$.

**Lemma 18** *The following bounds (stability bounds) hold uniformly in $|\psi(x)| > N^\delta$, $x \in R$ and $\psi(x) \in [-N^\delta, N^\delta]$, $x \in K = \Lambda\backslash R$:*

$$|V_1| \le \text{const.}\, N^{-1/2+2\delta+\varepsilon_0}|R|, \tag{4.64}$$

$$V_0 \ge -O(|R|N^{-1/2+2\delta+\varepsilon_0}). \tag{4.65}$$

*Proof.* To show the first, we note that

$$W_{\tilde{K}} - A_{\tilde{K},R^0\backslash R}A_{R^0,\tilde{K}} = -A_{\tilde{K},R^0\backslash R}\frac{A_{R^0}}{1+A_{R^0}}A_{R^0,\tilde{K}} + A_{\tilde{K},R}\frac{1}{1+A_{R^0}}A_{R^0,\tilde{K}}.$$

Then the trace of the left hand side is bounded by $\sum_i O(N^{-3/2+\delta+2\varepsilon_0}|R_i|)$.

To show the second, we introduce the positive function

$$\begin{aligned} P(\psi) &= < \psi_K, [(G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2}) \circ G]\psi_K > \\ &= < \psi_{\tilde{K}}, [(G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2}) \circ G]\psi_{\tilde{K}} > +2 < \psi_{\tilde{K}}, [(G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2}) \circ G]\psi_{\tilde{R}} > \\ &\quad + < \psi_{\tilde{R}}, [(G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2}) \circ G]\psi_{\tilde{R}} > \end{aligned} \tag{4.66}$$

which approximates $V_0$ and interpolates $\tilde{K} = \Lambda\backslash R^0$ and $\tilde{R} = R^0\backslash R$. Since

$$G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2} = G_{R^0} - G_{R^0}^{1/2}\chi_{R(L_0/2)}G_{R^0}^{1/2},$$

$$G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2} = G - G^{1/2}\chi_{R(L_0/2)}G^{1/2},$$

and $G_{R^0} = G$ on $\mathbf{C}^{R^0}$, we see that $G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2}$ is equal to $G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2}$ on $\mathbf{C}^{R^0}$ with an error of order $O(m^{-2}e^{-mL_0/4}) = O(N^{-1/2+\varepsilon_0})$. To prove this, we estimate

$$[G_{R^0}^{1/2}\chi_{R^0\backslash R(L_0/2)}G_{R^0}^{1/2}](x,y) = \sum_{\zeta\in R^0\backslash R(L_0/2)} G_{R^0}^{1/2}(x,\zeta)G_{R^0}^{1/2}(\zeta,y)$$

for $x, y \in R^0$. Since $G_{R_0}^{1/2}(x,y) \le ce^{-m|x-y|}$, if $\mathrm{dist}(x,R) > 3L_0/4$, the sum over $\zeta$ is extended to all $\zeta \in R^0$ with a correction bounded by $O(m^{-2}e^{-mL_0/4})$. Thus this is equal to $G_{R^0}(x,y) = G(x,y)$. If $\mathrm{dist}(x,R) < 3L_0/4$, then $\mathrm{dist}(x,(R^0)^c) \ge L_0/4$ and $G_{R^0}^{1/2}(x,y) = G^{1/2}(x,y)$ with a correction bounded by $O(m^{-2}e^{-mL_0/4})$. Thus we have

$$|\sum_i <\psi_{R_i^0\backslash R_i}, \mathcal{T}_i\psi_{R_i^0\backslash R_i}> - <\psi_{\tilde{R}}, [(G^{1/2}\chi_{\Lambda\backslash R(L_0/2)}G^{1/2})\circ G]\psi_{\tilde{R}}>| \le \sum_i |R_i| N^{-1/2+2\delta_0+\varepsilon_0}$$

since $\mathrm{dist}(R_i,R_j) \ge L$. The same relation holds between the first two terms in $V_0$ and $P(\psi)$. Since $P(\psi) \ge 0$, this implies $V_0 \ge O(|R|N^{-1/2+2\delta_0+\varepsilon_0})$ uniformly in $\psi(x) \in [-N^\delta, N^\delta]$, $x \in \Lambda\backslash R$. Q.E.D.

*2. Proof of Theorem 12 (Small Field Contribution)*

Let $d\mu_{\tilde{K}}(\psi)$ be the Gaussian measure of mean 0 and covariance $\frac{1}{2}[\chi_{\tilde{K}}G^{\circ 2}\chi_{\tilde{K}}]^{-1}$:

$$d\mu_{\tilde{K}}(\psi) = \det{}^{1/2}(C_{\tilde{K}}^{-1})\exp[-<\psi, \chi_{\tilde{K}}C^{-1}\chi_{\tilde{K}}\psi>]\prod_{x\in\tilde{K}}\frac{d\psi_x}{\sqrt{\pi}}, \tag{4.67}$$

where $C^{-1} = G^{\circ 2}$, $C_{\tilde{K}}^{-1} = \chi_{\tilde{K}}C^{-1}\chi_{\tilde{K}}$. We define the small field contribution $Z_{\tilde{K}}$ by

$$Z_{\tilde{K}} = \det{}^{-1/2}[C_{\tilde{K}}^{-1}]\int d\mu_{\tilde{K}}\,\eta_K(\psi), \tag{4.68}$$

$$\eta_K(\psi) = \det{}_3^{-\frac{N}{2}}(1 + A_{\tilde{K}} - W_{\tilde{K}})\exp[-\delta V_K]\prod_{x\in K}\tau(\psi(x)), \tag{4.69}$$

where $\delta V_K$ is defined by eq.(4.63). We again use the cluster expansion of the Gaussian measure. But this time, the covariance $\frac{1}{2}[\chi_{\tilde{K}}G^{\circ 2}\chi_{\tilde{K}}]^{-1}$ depends on locations of $R_i^0$.

We introduce interpolation parameters $s_i \in [0,1]$ into (4.67) as follows [16,5]:

$$C = C_\Lambda \to C(s_1) \equiv (1-s_1)(C_{\Lambda\backslash X_1} + C_{X_1}) + s_1 C_\Lambda.$$

The integral is decoupled into $X_1$ and $\Lambda\backslash X_1$ if $s_1 = 0$. Integration by parts yields

$$\frac{\partial}{\partial s_1}\int d\mu(s_1)e^{-V} = \int d\mu(s_1)\sum_{x,y}\frac{1}{4}(\partial_{s_1}A)_{xy}\frac{\partial^2}{\partial\psi(x)\partial\psi(y)}e^{-V} \tag{4.70}$$

where $A \equiv [\chi_{\tilde{K}}H\chi_{\tilde{K}}]^{-1}$ and $H = C^{-1}(s)$. Then we have ( see Appendix C)

$$\begin{aligned}\partial_{s_1}A &= \mathcal{A}(C_{\Lambda\backslash X_1,X_1} + C_{X_1,\Lambda\backslash X_1})\mathcal{A} \\ &= \sum_{X_2\subset\Lambda\backslash X_1}[\delta F(X_1,X_2) + \delta F(X_2,X_1)]\end{aligned}$$

by decomposing $\mathcal{A} \equiv [\chi_{\tilde{K}}H\chi_{\tilde{K}}]^{-1}\chi_{\tilde{K}}H$ into polymers, $\mathcal{A} = I_{\tilde{K}} + \sum_X \delta C(X)$, where $I_{\tilde{K}}$ is the identity operator on $\mathbf{C}^{\tilde{K}}$ and $\delta C(X)$ is the Green's function represented by random walks passing all squares $\Delta$ only in $X$, $X\cap R^0 \neq \emptyset$ and then exhibits tree decay over $\Delta \subset X$.

Next theorem is an extension of Theorem 3. We would like to remind the reader that $\delta F(X_i, X_j)(x,y) = O(e^{-mL})$ unless $x \in X_i$ and $y \in X_j$. See Appendix C for the construction of $\delta F(X_i, X_j)$. The sum over partitions $Y = \cup X_i$ is harmless thanks to Lemma 10.

**Theorem 19** *Let $\tilde{K} = \cup Y_i$ be partitions of $\tilde{K}$ into paved sets $\{Y_i\}$. Then*

$$\int \eta_K(\psi)d\mu_{\tilde{K}} = \left[\sum_n \frac{1}{n!}\sum_{\cup_1^n Y_i=\tilde{K}}\prod S(Y_i)\right]\eta_K(\psi), \tag{4.71}$$

$$\begin{aligned}S(Y) &= \sum_p \sum_{\cup_1^p X_i = Y}\sum_{T'}\int_0^1 ds_1\cdots ds_{p-1}M_{T'}(s)\int d\mu_Y(\{s\},\psi) \\ &\quad\times\prod_{k=1}^{p-1}\left[\sum_{x_k}\sum_{y_{k+1}}\frac{1}{2}\delta F(X_{j_{a(k)}}, X_{j_{k+1}})(x_k,y_{k+1})\frac{\partial^2}{\partial\psi(x_k)\partial\psi(y_{k+1})}\right]\end{aligned} \tag{4.72}$$

*where $Y = \cup_1^p X_i$ are partitions of $Y$ by unions of $\Delta_j \subset Y$ and $D_k \subset Y$. If $i < j$, then*

$$|\delta F(X_i,X_j)(x,y)| \le \min_\ell \exp[-m_1\mathcal{L}(\Delta_\ell \cup (X_j \wedge D), x, y)],\ (\Delta_\ell \subset X_i), \tag{4.73}$$

*where $X \wedge D$ means that $D_\ell \subset X$ are regarded as one sets $D_\ell$, and $\mathcal{L}(X,x,y)$ means the shortest length of walks from $x$ to $y$ passing all centers of $\Delta_i \subset X$, $x,y \notin \Delta_i$.*

Here and hereafter, we use the following notational convention for paved sets $Y$:

$$R_Y = R\cap Y,\ R_Y^0 = R^0 \cap Y,\ \tilde{Y} = Y\backslash R_Y^0. \tag{4.74}$$

By Lemma 14, we expand $(1 + A_{R^0})^{-1}$ and obtain polymer expansions of $W_{\tilde{K}}$ and $\delta V$.

**Lemma 20** *The following cluster expansion holds:*

$$W_{\tilde{K}} = A_{\tilde{K},R^0}\frac{1}{1+A_{R^0}}A_{R^0,\tilde{K}} = \sum_i W_{Y_i} + \sum_{X\neq Y_i}\delta W_X, \tag{4.75}$$

$$W_Y = A_{Y\backslash R^0_Y,R^0_Y}\frac{1}{1+A_{R^0_Y}}A_{R^0_Y,Y\backslash R^0_Y}, \tag{4.76}$$

$$\delta W(X) = \sum_{\substack{Y_i\cup X'\cup Y_\ell = X\\ Y_j\cup Y_k\subset X'}} A_{Y_i\backslash R^0_i,R^0_{Y_j}}\frac{1}{1+A_{R^0_{Y_j}}}F(X')\frac{1}{1+A_{R^0_{Y_k}}}A_{R^0_{Y_k},Y_\ell\backslash R^0_\ell}, \tag{4.77}$$

*where* $\{Y_i\}_1^p$ *are paved sets in eq.(4.71),* $X$ *is a paved set consisting of* $Y_i$ *more than or equal to 2. (*$R^0$ *must be subtracted. )* $F(X)$ *are the non-diagonal terms coming from the random walk expansion of* $(1+A_{R^0})^{-1}$, $R^0=\cup_i R^0_{Y_i}$*. They satisfy the bounds*

$$\begin{aligned} ||F(X)||_1 &\leq (\sum|R^0_{Y_i}|)\exp[-m_2\min_\gamma\sum_{(ij)\in\gamma}\mathrm{dist}(R^0_{Y_i},R^0_{Y_j})], \\ ||\delta W(X)||_1 &\leq (\sum|R^0_{Y_i}|)\exp[-m_2\min_\gamma\sum_{(ij)\in\gamma}\mathrm{dist}(R^0_{Y_i},R^0_{Y_j})] \end{aligned} \tag{4.78}$$

*where* $\gamma$ *are tree graphs over* $Y_i\subset X$*.*

**Lemma 21** *The following expansion holds for* $\delta V_K$ *defined in (4.63):*

$$\delta V_K = \sum_i \delta V_{Y_i} + \sum_X \delta\tilde{V}(X), \tag{4.79}$$

$$\delta V_Y = \sum_{j:D_j\subset Y} <\psi_{\tilde{R}_j},\mathcal{T}_j\psi_{\tilde{R}_j}> -\frac{N}{2}\mathrm{Tr}\,(W_Y + A_Y W_Y - \frac{1}{2}W_Y^2), \tag{4.80}$$

*where* $Y_i$ *are paved sets made by the expansion of the Gaussian measure,* $X$ *is a paved set consisting of* $Y_i$*. Moreover*

$$||\delta\tilde{V}(X)||_1 \leq (\sum|R^0_{Y_i}|)\exp[-m_2\min_\gamma\sum_{(ij)\in\gamma}\mathrm{dist}(R^0_{Y_i},R^0_{Y_j})], \tag{4.81}$$

*where* $\gamma$ *are tree graphs over* $Y_i\subset X$*.*

For each partition $\cup Y_i$ of $\tilde{K}$, we introduce interpolation parameters $s_{ij}$ connecting $Y_i$ and $Y_j$ in the determinant :

$$\begin{aligned} \frac{2i}{\sqrt{N}}\chi_{\tilde{K}}G\psi\chi_{\tilde{K}} &\to \sum_i\frac{2i}{\sqrt{N}}\chi_{Y_i}G\psi\chi_{Y_i} + \sum_{i<j}s_{ij}\frac{2i}{\sqrt{N}}(\chi_{Y_i}G\psi\chi_{Y_j}+\chi_{Y_j}G\psi\chi_{Y_i}) \\ &\equiv \sum_i A_{Y_i} + \sum_{i<j}s_{ij}B_{Y_i,Y_j}, \end{aligned} \tag{4.82}$$

where $Y_i$ should be regarded as $\tilde{Y}_i = Y_i - R^0$ if $R_{Y_i} \neq \emptyset$. We also introduce interpolation parameters $\{t_X\}$ and $\{\tilde{t}_X\}$ into the decompositions (4.75) and (4.79) of $W_{\tilde{K}}$ and $\delta V_K$ :

$$W_{\tilde{K}} \rightarrow \sum_i W_{Y_i} + \sum_{X=\cup Y_i} t_X \delta W(X), \tag{4.83}$$

$$\delta V_K \rightarrow \sum_i \delta V_{Y_i} + \sum_{X=\cup Y_i} \tilde{t}_X \delta \tilde{V}(X), \tag{4.84}$$

where $X = \cup Y_i$ are paved sets consisting of more than or equal to two $Y_i$'s. Thus we have

$$\sum_p \frac{1}{p!} \sum_{\cup Y_i = \tilde{K}} \prod S(Y_i) \eta_K(\psi) = \sum_p \frac{1}{p!} \sum_{\cup X_i = \tilde{K}} \prod \tilde{\rho}_{X_i}, \tag{4.85}$$

$$\tilde{\rho}_X = \sum_p \frac{1}{p!} \sum_{\cup Y_i = X} \prod S(Y_i) [\sum_{\gamma \in \tilde{T}(\{Y_i\})} \int ds_\gamma \partial_\gamma] \eta_X(s), \tag{4.86}$$

where $X = \cup Y_i$ are partitions of $X$ into decoupled paved sets $Y_i$, $\tilde{T}(\{Y_i\})$ is the set of cennected graphs over $\{Y_i\}$ and $\eta_X$ is the $\eta$ function restricted to the paved set $X = \cup Y_i$.

*Proof of Theorem 12.* (*step 1.*) We consider the action of the differential operators in $S(Y)$ on $\eta_Y$. By integration by parts, we start with

$$\int d\mu_Y(\{s\}, \psi) \left[ \prod_{k=1}^{p-1} \delta F(X_{j_{a(k)}}, X_{j_{k+1}})(x_k, y_{k+1}) \frac{\partial^2}{\partial \psi(x_k) \partial \psi(y_{k+1})} \right] \eta_Y$$

$$= \prod_{k=1}^{p-1} \delta F(X_{j_{a(k)}}, X_{j_{k+1}})(x_k, y_{k+1}) \int d\mu_Y(\{s\}, \psi)\, e^{-\delta V_0(Y)}\, \Phi \Psi, \tag{4.87}$$

where putting $H =< \psi_{\tilde{Y}}, C_Y^{-1}(s) \psi_{\tilde{Y}} >$, $\tilde{Y} = Y \backslash R_Y^0$ and $\tilde{R}_Y = R_Y^0 \backslash R_Y$, we have set

$$\Phi = e^H \prod_i \left[ (-1)^{d_i - 1} \prod_{j=1}^{d_i - 1} \frac{\partial}{\partial \psi(x_{i,j})} \right] e^{-H}, \tag{4.88}$$

$$\Psi = e^{\delta V_0(Y)} \prod_{i=1}^{p-1} \frac{\partial}{\partial \psi(x_{i,d_i})} \eta_Y \tag{4.89}$$

$$\eta_Y = \det{}_3^{-N/2}(1 + A_Y - W_Y) \exp[-\delta V_0(Y) - V_1(Y)] \tau(\psi_Y), \tag{4.90}$$

$$\delta V_0(Y) = 2 < \psi_{\tilde{Y}}, C^{-1} \psi_{\tilde{R}_Y} > + \sum_{i: R_i \subset Y} < \psi_{\tilde{R}_i}, \mathcal{T}_i \psi_{\tilde{R}_i} >, \tag{4.91}$$

$$V_1(Y) = -\frac{N}{2} \mathrm{Tr} \left[ W_Y - A_{\tilde{Y}, R_Y^0 \backslash R_Y} A_{R_Y^0 \backslash R_Y, \tilde{Y}} + A_Y W_Y - \frac{1}{2} W_Y^2 \right], \tag{4.92}$$

and $d_i$ is the number of $\{x_k, y_{k+1}\}_1^{p-1}$ such that $x_k \in X_{j_i}$ or $y_{k+1} \in X_{j_i}$. If $x_k \in X_{j_{a(k)}}$ and $y_{k+1} \in X_{j_{k+1}}$, $d_i$ is the incidence number of the vertex $X_{j_i}$. By Theorem 19

$$|\prod_k \delta F(X_{j_{a(k)}}, X_{j_{k+1}})(x, y)| \leq \exp[-\frac{4}{5} \sum_k m_1 \mathcal{L}(\Delta_{j_{a(k)}} \cup X_{j_{k+1}}, x_k, y_{k+1}) - \sum_i \frac{m_1}{10} L[\frac{d_i}{9}]^{\frac{3}{2}}]$$

( $\wedge D$ is omitted for simplicity.) Then (See Appendix C)

(i) we can extract tree decay factors $\prod \exp[-cm_1 \mathcal{L}(X_k')]$, $X_k' = \Delta_{j_{a(k)}} \cup X_{j_{k+1}}$,

(ii) if $X_{j_{k+1}}$ consists of more than or equal to two $\Delta_k$ or $D_\ell$, $\delta F(X_{j_{a(k)}}, X_{j_{k+1}})$ contains $\exp[-m\mathrm{dist}(X_{j_{a(k)}}, R^0 \cap X_{j_{k+1}})]$,

(iii) $x_k \notin X_{j_{a(k)}}$ takes place if and only if $\delta F$ consists of walks passing through $^\exists R_\ell^0 \subset X_{j_{a(k)}} \cup X_{j_{k+1}}$. So $|\delta F|$ is bounded by $e^{-(L+L_0)m}$.

The fact (i) means that it is enough to show that the derivatives and the summations over $\{x_k, y_{k+1}\}$ do not yield very large terms.

(*step 2.*) We show the stability of $e^{-\delta V_0(Y)} d\mu_Y$. It suffices to consider a paved set $Y$ such that $R_Y \neq \emptyset$. Then $V_0(Y) \equiv < \psi_{\tilde{Y}}, C_Y^{-1}(s)\psi_{\tilde{Y}} > + \delta V_0(Y)$ is given by

$$< (\psi_{\tilde{Y}} + D\psi_{\tilde{R}_Y}), C_Y^{-1}(s)(\psi_{\tilde{Y}} + D\psi_{\tilde{R}_Y}) > + < \psi_{\tilde{R}_Y}, E\psi_{\tilde{R}_Y} > + O(|R_Y| e^{-mL_0})$$

where $D = C_Y(s)(C^{-1})_{\tilde{Y},\tilde{R}_Y}$, $E = \mathcal{T}_{\tilde{Y}} - (C^{-1})^+_{\tilde{R}_Y,\tilde{Y}} C_Y(s)(C^{-1})_{\tilde{Y},\tilde{R}_Y}$ and $\mathcal{T}_{\tilde{Y}} = G_Y^{1/2} \chi_{Y \setminus R_Y(L_0/2)} G_Y^{1/2} \circ G$. Then $E \geq -\mathrm{const.} e^{-mL_0/2}$ on $R_Y^0$ by Lemma 18. (Accurately speaking, $d\mu_Y$ and $C_Y^{-1}$ should be written $d\mu_{\tilde{Y}}$ and $C_{\tilde{Y}}^{-1}$.) Let us define

$$d\tilde{\mu}_Y \equiv \det{}^{1/2}[C_Y^{-1}(s)] \exp[-V_0(Y)] \tau(\psi_{R_Y^0 \setminus R_Y}) \prod_{x \in Y \setminus R_Y} \frac{d\psi(x)}{\sqrt{\pi}}. \tag{4.93}$$

Then $d\tilde{\mu}_Y$ is Gaussian with respect to $\psi_{\tilde{Y}}$ if $\psi_{\tilde{R}_Y}$ are fixed. Since $|\psi(x)| \leq N^\delta$ for $x \in \tilde{R}_Y$, we have $\int d\tilde{\mu}_Y \leq \exp[\pi L_0^2 |R| \, \delta \log N]$. Thus we can regard $d\tilde{\mu}_Y$ as the probability measure with an additional factor bounded by $\exp[\pi L_0^2 |R| \delta \log N]$.

(*step 3.*) The application of $\partial/\partial\psi(\xi)$ on $H$ yields $-\sum_\zeta C_Y^{-1}(\xi,\zeta)\psi(\zeta)$. Then using Schwarz's inequality, we find it enough to estimate

$$\sum_{\{\zeta_i\}} \prod_i |C^{-1}(\xi_i,\zeta_i)| \left[ \int d\mu_Y e^{-\delta V_0(Y)} \prod \psi(\zeta_i)^2 \right]^{1/2} \left[ \int d\mu_Y e^{-\delta V_0(Y)} |\Psi|^2 \right]^{1/2}.$$

where $\{\xi\}$ are $\{x_{i,1}, \cdots, x_{i,d_{i-1}}\}$, see Lemma 8.

Consider $\Psi$. As for the derivatives of $\eta_Y$, we first see that the derivatives of $W_Y$ with respect to $\psi(y)$, $y \in \tilde{Y}$ yield the factor $N^{-1+3\delta}$ thanks to the small fields enclosing the large fields. Thus derivatives of $\det{}_3^{-N/2}(\cdots)$ yield factors bounded by $N^{-1+3\delta}$. We estimate the derivatives of $\delta V_Y = \delta V_0(Y) + V_1(Y)$. The derivatives of $V_1$ yield factors bounded by $N^{-1+2\delta}$. The derivatives of $\delta V_0(Y)$ yield $2\sum_{\zeta\in R^0_Y\setminus R_Y} C^{-1}(y,\zeta)\psi(\zeta)$, $|\psi(\zeta)| < N^{\delta}$. But they come with $\delta F(X_{j_{a(k)}}, X_{j_{k+1}})(x_k, y_{k+1})$ ( $y = y_{k+1}$ or $y = x_k$ ). Then $\mathcal{L}(\Delta_{j_{a(k)}} \cup X_{j_{k+1}}, x_k, y_{k+1}) + |y_{k+1} - \zeta| > L$. Thus we can bound $|\Psi|$ by $N^{-n_Y\delta_0}||e^{\delta V_0}\eta_Y||$ uniformly in $\psi_Y$ by a fraction of $\prod \delta F$. Differentiations of $\tau$ can be treated as before.

Let us consider $\prod \psi(\zeta_i)^2$. We first shift $\psi(x)$, $x \in \tilde{Y}$ by $-(D\psi_{\tilde{R}_Y})(x)$ which is bounded by $e^{-m\mathrm{dist}(x,R^0)}$. Then $d\tilde{\mu}_Y$ decomposes into $d\mu_Y$ and the integration over $\psi(x)$, $x \in \tilde{R}_Y$. Then we can regard $d\tilde{\mu}_Y$ as $d\mu_Y$. Therefore the proof of Lemma 8 can be applied and we obtain the same results by replacing $\mathrm{dist}(\Delta_i, \Delta_j)$ by $\mathcal{L}(\Delta_i \cup X_j)$ and so on. In fact we define

$$\sum_{\xi,\xi'} \delta F(X_{j_{a(k)}}, X_{j_{k+1}})(\xi,\xi')|C^{-1}(\xi, x_k)||C^{-1}(\xi', y_{k+1})| \equiv m^{-4}\delta f(X_{j_{a(k)}}, X_{j_{k+1}})(x_k, y_{k+1}).$$

Then $\delta f(X_{j_{a(k)}}, X_{j_{k+1}})(x_k, y_{k+1})$ again has the property (4.73) except for a multiplicative constant $\log^4(1 + m^{-1})$ which comes from $C^{-1} = G^{\circ 2}$. Then we repeat the arguments in Lemma 8 by replacing $\Delta_i$ by $X_i$ and $\mathrm{dist}(\Delta, x)$ by $\mathcal{L}(X, x, y)$ and so on. We remark that the volume $|X_{j_{k+1}}|$ is compensated by a fraction of $\exp[-m_1\mathcal{L}(X_{j_{k+1}})]$.

(*step 4.*) Finally take the sum over partitions $Y = \cup X_i$. Since we already have tree decay factors of $X'_k$, the proofs of Lemma 8 and Theorem 5 apply to the rest. Q.E.D.

To expand $\det^{\frac{1}{2}}(C_{\tilde{K}})/Z_\infty$, $C_{\tilde{K}} = [\chi_{\tilde{K}} G^{\circ 2}\chi_{\tilde{K}}]^{-1}$, we put $H = G^{\circ 2}$ and observe that

$$\begin{aligned}\det\begin{pmatrix} H_0 & H_{01} \\ H_{10} & H_1 \end{pmatrix} &= \det(H_0)\det(H_1)\det(1 - H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2}) \\ &= \det(H_0)\prod \det(H_{R^0_i})\det(1 + \sum \delta H^1_{ij})\det(1 - H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2}),\end{aligned}$$

where $H_0 = \chi_{\tilde{K}} H \chi_{\tilde{K}}$, $H_1 = \chi_{R^0} H \chi_{R^0}$ and

$$\delta H^1_{ij} = (H_{R^0_i})^{-1}\chi_{R^0_i} H \chi_{R^0_j}, \tag{4.94}$$

and we have used the notational convention $H_X \equiv \chi_X H \chi_X$ and $H_{XY} \equiv \chi_X H \chi_Y$. Thus $H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2}$ is the matrix of size $|R^0| \times |R^0|$.

$H_i(x, y)$ and $(\chi_X H_i \chi_X)^{-1}(x, y)$ decay exponentially fast (see Appendix B). We expand $H_0^{-1}$ and $H_1^{-1/2}$ by introducing interpolation parameters like $[(1-s)(H_{X\setminus\Delta} + H_\Delta) + sH_X]^{-1}$

and repeating the method used in the proof of Lemma 14. (We use $H^{-1/2} = 2\int(H + u^2)^{-1}du/\pi$ to expand $H_1^{-1/2}$).

**Lemma 22** *The matrix $H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2}$ has the following expansion:*

$$H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2} = \sum_i \delta H(Y_i) + \sum_{X=\cup Y_i} \delta H(X),$$

*where $X$ are paved sets consisting of more than or equal to two $Y_i$'s and include at least one $R_i^0 \subset D_i$. The functions $\delta H(Y_i)$ and $\delta H(X)$ depend on variables located on $Y_i$ and $X$ only. The diagonal terms $\delta H(Y)$ are given by*

$$\delta H(Y) = H_{R_Y^0}^{-1/2} H_{R_Y^0, Y\backslash R_Y^0} H_{Y\backslash R_Y^0} H_{Y\backslash R_Y^0, R_Y^0} H_{R_Y^0}^{-1/2}, \; R_Y^0 = R^0 \cap Y.$$

*The non-diagonal terms $\delta H(X)$ $(X = \cup Y_i)$ satisfy the bound*

$$|\delta H(X)(x,y)| \le \exp[-m_1 \mathcal{L}(X,x,y)].$$

The proof of Lemma 16 (1) means that $0 < O(1)m^4 \le 1 - H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2} \le 1$. Then the diagonal terms satisfy the bounds

$$\exp[-\text{const.}\, L_0^2|R|\log m^{-1}] \le \det{}^{\frac{1}{2}}(1 - \delta H(Y)) \le 1. \tag{4.95}$$

Since $L_0 \sim 2m^{-1}\log N \sim \beta e^{2\pi\beta}$, if the condition (4.56) is satisfied, the factors $\exp[\pi\delta L_0^2|R|\log N]$ from $\int d\tilde{\mu}_Y$ and $\exp[O(1)L_0^2|R|\log m^{-1}]$ from $\det^{1/2}(H_{R_i^0})$ are all compensated by $\exp[-(\beta/12)N^{\delta_2}|R|]$ given in Theorem 11 (the large field stability). In fact for $\delta = 1/12$ and $\delta_1 = 1/24$, we have $\delta_2 = 1/24$. If $N \sim e^{400\pi\beta}$, we have $N^{\delta_2} > e^{16\pi\beta} > m^{-8} \sim L^8$.

## V. ANALYTICITY OF THE FREE ENERGY

### A. Proof of Theorem 1 (Former Half)

To carry out the integration over $\{\psi(x); x \in \Lambda\}$, we introduce a series of interpolation parameters $\{s_i, s_{ij}, t_X, \tilde{t}_X, u_Y, v_{ij}, \tilde{v}_{ij}\}$ to decouple $R_i^0 \subset D_i$, $R_j^0 \subset D_j$, $i \neq j$, $Y_k \subset K^0$

$(K^0 \equiv \Lambda\backslash D)$ and $D_i\backslash R_i^0 \subset D$. From now on, let $Y_i$ stand for either $Y_i \subset K^0$ or $D_i\backslash R_i^0 \subset D$ or for their unions. We summarize the interpolation parameters:

(1) Given configuration of $R$, we decompose $\tilde{K} = \Lambda - R^0$ into squares $\Delta_i \subset K^0$ and paved set $D_i$ with $R_i^0$ subtracted. Introduce interpolation parameters $s_i$ to the measure $d\mu_{\tilde{K}}(\psi)$ .

$$Z(R) = \frac{\det^{1/2}(C_{\tilde{K}})}{Z_\infty}\left[\sum_p \frac{1}{p!}\sum_{\cup Y_i=\tilde{K}}\prod_i S_{Y_i}\right]\eta_\Lambda.$$

(2) To each decomposition $\tilde{K} = \cup Y_i$, introduce real interpolation parameters $s_{ij} \in [0,1]$ for $B_{Y_iY_j}$ like eq.(4.82).

(3) Introduce $t_X \in [0,1]$ and $\tilde{t}_X \in [0,1]$ following eq.(4.83) and eq.(4.84).

(4) Introduce $u_Y \in [0,1]$ in such a way that

$$H_1^{-1/2}H_{10}H_0^{-1}H_{01}H_1^{-1/2} \ \to \sum_i \delta H(Y_i) + \sum_{X=\cup Y_i} u_X \delta H(X). \tag{5.1}$$

The diagonal terms $\delta H(Y_i)$ such that $H_0 = H_{Y_i\backslash R^0_{Y_i}}$ and $H_1 = H_{R^0_{Y_i}}$ are untouched and coupled with $\mathcal{D}(A_{R_j^0})$, $D_j \subset Y_i$.

(5) Redefine $A_{ij}$ and $H_{ij}$ by

$$\delta A_{ij} \equiv \sum_{k:R_k\subset Y_i}\sum_{\ell:R_\ell\subset Y_j} A_{R_k^0,R_\ell^0}\frac{1}{1+A_{R_\ell^0}}, \quad \delta H_{ij} \equiv \frac{1}{H_{R^0_{Y_i}}}H_{R^0_{Y_i},R^0_{Y_j}}. \tag{5.2}$$

and introduce $v_{ij} \in [0,1]$ and $\tilde{v}_{ij} \in [0,1]$ in such a way that

$$\sum_{i,j}\delta A_{ij} \to \sum_{ij} v_{ij}\delta A_{ij}, \quad \sum_{i,j}\delta H_{ij} \to \sum_{ij}\tilde{v}_{ij}\delta H_{ij}, \tag{5.3}$$

Thus both of $||\delta A_{ij}||$ and $||\delta H_{ij}||$ are bounded by $m^{-2}\exp[-m\,\mathrm{dist}(R^0_{Y_i}, R^0_{Y_j})]$, and both of $||\delta A_{ij}||_1$ and $||\delta H_{ij}||_1$ are bounded by

$$\min\{|R^0_{Y_i}|, |R^0_{Y_j}|\}\exp[-\frac{4m}{5}\,\mathrm{dist}(R^0_{Y_i}, R^0_{Y_j})],$$

Substituting these into the integrand $\Xi_R(\psi)$ defined by (4.57), we have our final expression of $Z_\Lambda = Z_\infty \sum_R Z(R)$, where

$$Z(R) = \int\prod_{\zeta\in R^0}\frac{d\psi(\zeta)}{\sqrt{\pi}}\left[\sum\frac{1}{p!}\sum_{\cup Y_i=\tilde{K}}\prod S_{Y_i}\cdot \Xi(\{Y_i\}, R; s,\cdots,\tilde{v})|_{s=\cdots=\tilde{v}=1}\right], \tag{5.4}$$

and $\Xi(\{Y_i\}, R; s, \cdots, \tilde{v})$ is the $\Xi-$function with the interpolation parameters introduced through $Y_i$ and explicitly given by

$$\det{}_3^{-N/2}\left[1+\sum_i A_i - \sum_{Y_i\cap R\neq\emptyset} W(Y_i) + \sum_{i\neq j} s_{ij}B_{ij} - \sum_X t_X \delta W_{\tilde{K}}(X)\right]$$
$$\times \det{}^{-N/2}\left[1+\sum_{ij} v_{ij}\delta A_{ij}\right] \det{}^{\frac{1}{2}}\left[1-\sum_i \delta H(Y_i) - \sum_X u_X \delta H(X)\right]$$
$$\times \det{}^{\frac{1}{2}}\left[1+\sum_{ij}\tilde{v}_{ij}\delta H_{ij}\right]\left[\prod_i \mathcal{D}(R^0_{Y_i})\right] \exp\left[-\sum \delta V_{Y_i} - \sum \tilde{t}_X \delta\tilde{V}(X)\right] \tau(\psi_K)\tau^c(\psi_R). \quad (5.5)$$

Here $\mathcal{D}(R^0_Y)$ is the contribution from $R^0_Y = R^0 \cap Y$ with the small fields subtracted :

$$\mathcal{D}(R^0_Y) \equiv \prod_{i:R_i\subset Y}\left[\det{}_2^{-N/2}(1+A_{R^0_i}) \det{}^{\frac{1}{2}}(\chi_{R^0_i} H \chi_{R^0_i}) \exp[< \psi_{R^0_i\setminus R_i}, \mathcal{T}_i \psi_{R^0_i\setminus R_i} >]\right] \quad (5.6)$$

where by Theorem 11

$$\sup_{\psi_{R^0_Y\setminus R_Y}} \int |\mathcal{D}(R^0_Y)|\tau^c(\psi_{R_Y}) \prod_{x\in R_Y} \frac{d\psi(x)}{\sqrt{\pi}} \leq \exp[-\frac{\beta}{11}|R_Y|N^{\delta_2}]. \quad (5.7)$$

If all parameters are set 0, we have the completely decoupled result:

$$\begin{aligned} Z(R) &\rightarrow \sum \frac{1}{p!} \sum_{\cup Y_i = \Lambda} \prod_i \eta(Y_i; R_{Y_i}), \\ \eta(Y; R_Y) &= \int S_Y \Xi(Y; R_Y) \tau(\psi_{R^0_Y\setminus R_Y}) \tau^c(\psi_{R_Y}) \prod_{\zeta\in R^0_Y} \frac{d\psi(\zeta)}{\sqrt{\pi}}, \\ \Xi(Y; R_Y) &= \det{}_3^{-N/2}[1 + A_Y - W(Y)] \exp[-\delta V_Y] \\ &\quad \times \det{}^{1/2}[1 - \delta H(Y)]\, \mathcal{D}(R^0_Y) \tau(\psi_{Y\setminus R^0_Y}). \end{aligned}$$

Here and hereafter, $\eta$ means integrated activities which may contain contributions from $\psi_R$.

If $Y = \Delta$, $R_Y = \emptyset$ and $S_\Delta = d\mu_\Delta(\psi)$ (with $|\psi(x)| < N^\delta$, $x \in \Delta$), and $\eta(\Delta) = \rho_\Delta$. If $Y = D_i$, then $S_Y = d\mu_{Y\setminus R^0_Y}(\psi)$ and by Theorem 11, we have

$$|\eta(D_i; R_i)| \leq \exp[-\frac{\beta}{12}|R_i|N^{\delta_2} + N^{-\hat{\delta}}|D_i| + \pi L_0^2 |R_i| \delta \log N], \quad (5.8)$$

where $|D_i| < 9|R_i|L^2$ and $L_0 < L \sim m^{-1}\log N$. Then

**Lemma 23** *Take the sum over all $R \subset D_i$ which are consistent with $D_i$. Then*

$$|\sum_R \eta(D_i;R)| \ \leq \ \exp[-|D_i|N^{\delta_3}], \tag{5.9}$$

$$\delta_3 \ = \ \delta_2 - O(N^{-\delta_2}). \tag{5.10}$$

*Proof.* Take a square $\Delta \subset D_i$ of size $L \times L$ such that $R \cap \Delta \neq \emptyset$, and take the sum over $R \cap \Delta$ ($|R \cap \Delta| = 1, \cdots, L^2$). Since $L^2 \sim 400m^{-2} \log^2 N \sim N^{\varepsilon_0}$, we have estimates

$$\begin{aligned}\sum_{R\subset\Delta} \exp[-\frac{\beta}{12}|R|N^{\delta_2}] \ &\leq \ (1+\exp[-\frac{\beta}{12}N^{\delta_2}])^{L^2} - 1 \\ &\leq \ \exp[L^2 \exp[-\frac{\beta}{12}N^{\delta_2}]] - 1 \leq \exp[-|\Delta|N^{\delta_3}].\end{aligned}$$

Since $D_i$ is the connected set of $\{\Delta \subset D_i\}$, the conclusion follows [9]. Q.E.D.

We iteratively use the identity $f(1) = \int_0^1 dw \partial_w f(w) + f(0)$ with respect to all interpolation parameters except for $s_i$ already used to expand the Gausiian measure. We thus obtain $Z(R) = \sum_{\mathcal{U}(R)} \prod_{X\in\mathcal{U}(R)} \eta(X;R)$, where $\mathcal{U}(R)$ are partitions of $\Lambda$ into paved sets which consists of $\Delta_i \subset K^0$ and $D_i \subset D$, and $\eta(X;R)$ is the quantity given by

$$\sum_p \frac{1}{p!} \sum_{\cup Y_i = X} \prod_{i=1}^p S_{Y_i}\mathcal{I}(\{Y_i\}) \int \Xi(X,\{Y_i\},R_X)\tau(\psi_{X\cap\tilde{K}})\tau^c(\psi_{R_X}) \prod_{x\in X\cap R^0} \frac{d\psi(x)}{\sqrt{\pi}}. \tag{5.11}$$

Namely if $\mathcal{U} = \{X_1, \cdots, X_n\}$ is a partition, $X_i$ are unions of $\Delta_i$ and $D_j$ and $\cup X_i = \Lambda$. Moreover $\Xi(X,\{Y_i\},R_X)$ is the restriction of $\Xi(\Lambda, R)$ to the region $X$ equipped with $R_X = R \cap X$, together with the interpolation parameters following the decomposition $X = \cup Y_i$. $\mathcal{I}(\{Y_i\})$ is the interpolation operator over $\{Y_i\}$, $R_i^0$ and so on defined by

$$\mathcal{I} = \sum_{\cup X_i^j = X} \prod_{i,j} \mathcal{I}(X_i^j), \tag{5.12}$$

where $X_i^j$ is a paved set consisting of $Y_i \subset X$ connected by the interpolation parameters ($s_{ij}$ for $i=1$, $t_X$ for $i=2$, $\tilde{t}_X$ for $i=3$, $u_X$ for $i=4$, $v_{ij}$ for $i=5$ and $\tilde{v}_{ij}$ for $i=6$). The paved set $X$ cannot be decomposed into two disconnected pieces without bisecting some $X_i^j$ and

$$\mathcal{I}_i(X_i^j) = \sum_{\gamma\in\tilde{T}(X_i^j)} \int_0^1 dw_\gamma \partial_{w_\gamma}, \tag{5.13}$$

where $\tilde{T}(X_i^j)$ is the set of connected graphs over the constituents $Y_k \subset X_i^j$ or $R_i^0 \subset X_i^j$ made by $w$ $(= s_{ij}, t_X, \tilde{t}_X, u_Y, v_{ij}$ and $\tilde{v}_{ij}$ ). (Multi-indices are used for $w_\gamma$.) Then we have

$$Z_\Lambda = Z_\infty \left[ \sum_p \frac{1}{p!} \sum_{\cup_1^p X_i = \Lambda} \prod_i \rho_{X_i} \right], \tag{5.14}$$

$$\rho_X = \sum_{R \subset X} \eta(X; R), \tag{5.15}$$

where the sum over $R \subset X$ is chosen so that the locations of $R$ are consistent with the polymer expansion, i.e., $R^0 \cap \Delta = \emptyset$ for $\Delta \subset \partial X$. We can now prove Theorem 1:

*Proof of Theorem 1 ( former half ).* Put $X = \cup_{i=1}^6 X_i$ and $X_i = \cup_j X_i^j$ where $X_i^j$ is a collection of paved sets $\{Y_k \subset X\}$ such that $X_i = \cup X_i^j$, and is constructed by the action of $\mathcal{I}_i(X_i^j)$ on $\Xi$. $X$ cannot be divided into two disconnected sets without bisecting some $X_i^j$ and $X_i$.

$S_{Y_i}$ yields the tree decay factor $\exp[-\delta_0 n_{Y_i} \log N - m_0 \mathcal{L}(Y_i)]$, $n_{Y_i} \geq 2$ over the squares $\Delta_k \subset Y_i$. Moreover as is seen from Lemmas 9 and 10, the action of $\mathcal{I}_i(X_i^j)$ on $\Xi$ yields the factor $\sigma_i(X_i^j)$ bounded by the tree decay factor:

$$|\sigma_i(X_i^j)| \leq \exp[-\delta_0 \tilde{n}_{X_j^i} \log N - m_0 \mathcal{L}_Y(X_i^j)],$$

where $\tilde{n}_X$ is the number of $Y_i$ contained in $X$ ($\tilde{n}_X \geq 2$) and $\mathcal{L}_Y(X)$ denotes the length of the shortest tree graphs over $Y_i \subset X$ (from center of $\Delta_i \subset Y_i$ to center of $\Delta_j \subset Y_j$).

The factor $\mathcal{D}(R_Y^0)$ is combined with $\det^{1/2}(1 - \delta H(Y)) \leq 1$. By Lemma 23, we see that it yields the factors bounded by $\exp[-\sum_{D_k \subset Y} |D_k| N^{\delta_3}]$. Since $\sigma_s$, $\cdots$, and $\sigma_w$ contain the tree decay factors over $Y_i$ and $D_j$, and since $S_{Y_i}$ contains the tree decay factors over $\Delta_k \subset Y_i$, we can extract a part (e.g. $7/8$ ) of the tree decay factors over $\Delta_i \subset X \backslash D$ and $D_k \subset D \cap X$ in advance from $\sigma_{s,\cdots,w}(X)$ (we denote the remainders again by $\sigma_{s,\cdots,w}(X)$ for simplicity). Thus we have

$$\begin{aligned}|\rho(X)| \leq & \sum_{D_k \subset X^0} \exp\left[ -\frac{7}{8}\delta_0 n_{X \backslash \cup D_k} \log N - c_1 \sum |D_k| N^{\delta_3} - \frac{7}{8} m_0 \mathcal{L}(\{\Delta_i \subset X \backslash \cup D_k\}, \{D_k\}) \right] \\ & \times \left[ \sum_{\cup X_i = X} \left( \sum_{p_1} \sum_{\cup X_1^i = X_1} \frac{1}{p_1!} \prod_{i=1}^{p_1} \sigma_s(X_1^i) \right) \cdots \left( \sum_{p_6} \sum_{\cup X_6^i = X_6} \frac{1}{p_6!} \prod_{i=1}^{p_6} \sigma_w(X_6^i) \right) \right]\end{aligned}$$

where $X^0 = X - \partial X$, $c_1 = O(1) > 0$ (in fact $c_1 \sim 1$), $\{D_k\}$ are the large filed regions consistent with $X_i^j$ and $\mathcal{L}(\{\Delta_i \subset X \backslash \cup D_k\}, \{D_k\})$ is the length of the shortest tree graph over $\{\Delta_i\}$ and $\{D_k\}$. Then we can assume that $X_i$ cannot be bisected without bisecting some $X_i^j$ by adding 1/8 of the dacay factor to each of $\prod \sigma_i(X_i^j)$, $i = 1, \cdots, 6$. Thus the sum over $\{X_i^j\}_j$ is convergent for $i = 1, \cdots, 6$. Since $X$ cannot be devided into two pieces without bisecting some $X_i$, the sum over $X_i$ is again convergent. The result is bounded by $\exp[-\delta_c n_X \log N - m_c \mathcal{L}(X)]$ if $N$ is large, where $\delta_c > \delta_0/8$ and $m_c > m_0/8$. Q.E.D.

**Remark 7** *It is obvious that $m_c$ and $\delta_c$ converge to $m_0$ and $\delta_0$, respectively for large $N$ since the contributions from large fields are exponentially small.*

### B. Proof of Theorem 1 (Latter Half)

We now resum eq.(5.14) in the following form:

$$\begin{aligned} Z_\Lambda &= Z_\infty \exp[-\sum W_\Delta][\sum_p \frac{1}{p!} \sum_{\cup X_i \subset \Lambda} \prod_i \hat{\rho}_{X_i}] \\ &= Z_\infty \exp[-\sum W_\Delta - \sum_Y \hat{W}_Y], \end{aligned} \tag{5.16}$$

where $\hat{\rho}_X \equiv \exp[\sum_{\Delta \subset X} W_\Delta] \rho_X$ is the polymer activity with the single square contributions subtracted. Thus $\hat{\rho}_\Delta = 1$. Moreover $n_Y \geq 2$ ($n_Y$=number of squares in $Y$ ) and

$$\hat{W}_Y = -\sum_k \frac{1}{k!} \sum_{\{X_i; i=1,\cdots,k\}; \cup X_i = Y} \sum_{\gamma_c} \prod_{\ell \in \gamma_c} \epsilon(\ell) \prod_\zeta \hat{\rho}_{X_\zeta}. \tag{5.17}$$

In this equation, $k$ is the number of $\{X_i\}$ and $\gamma_c$ runs over connected graphs of lines $\{\ell\}$ joining vertices $\{1, 2, \cdots, k\}$, $\epsilon(\ell) = -1$ if $X_{\ell_+} \cap X_{\ell_-} \neq \emptyset$ where $\ell = (\ell_+, \ell_-)$ and zero otherwise. Then it follows [5,13,16] from (3.5) that

**Theorem 24** *For given $\beta > 0$, if $N$ is chosen large ($N \geq \exp[400\pi\beta]$), then*

$$\alpha \equiv \frac{W_\Delta}{L^2} + \sum_{Y \ni 0} \frac{1}{|Y|} \hat{W}_Y \tag{5.18}$$

*converges absolutely as $\Lambda \to Z^2$. The free energy $\alpha_F = \alpha_0 + \alpha$ is analytic in $\beta$, where*

$$\alpha_0 \equiv \lim \frac{1}{|\Lambda|} \left[ \frac{N}{2} \log(\det(m^2 - \Delta)) - \frac{1}{2} \log(\det(C_\Lambda)) \right]. \tag{5.19}$$

## VI. CONCLUSION AND SOME REMARKS

We have shown that the free energy is represented by the convergent polymer expansion, which establishes the analyticity of the free energy. Exponential decay of the correlation functions will be proved in the same way, but with some additional tricks. The mass parameter $m \sim e^{-2\pi\beta}$ is almost zero for large $\beta$, and our result is weak in the sense that $\beta_c(N)/N$ increases just logarithmically. Note that we used blocks of single scale only. Our longstanding problem will be solved by iterative usages of block-spin-type calculations.

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## APPENDIX A: PROPERTIES OF $G$, $C$, $\tilde{G}$ AND THEIR INVERSES

We first consider $G(x) = (2\pi)^{-2} \int e^{ipx} g(p) \prod dp$. Since $g(p)$ is ananlytic and periodic in $p$, the integral is invariant by the shift of $p_k$ by $i\varepsilon_k$ where $\varepsilon_k = \varepsilon x_k (x_1^2 + x_2^2)^{-1/2}$, $\varepsilon > 0$. Then $ipx \to ipx - \varepsilon|x|$ and $g(p + i\varepsilon)^{-1}$ is equal to

$$\begin{aligned}
&m^2 + 2\sum_k [1 - \cos(p_k)\cosh(\varepsilon_k)] + 2i\sum_k \sin(p_k)\sinh(\varepsilon_k) \\
&= m^2 + 2\sum_k [1 - \cosh(\varepsilon_k)] + 2\sum (1 - \cos(p_k))\cosh(\varepsilon_k) + 2i\sum_k \sin(p_k)\sinh(\varepsilon_k)
\end{aligned}$$

Here we can set $\varepsilon = m_*$ by $m^2 + 2(1 - \cosh(m_*)) = 0$ since $\sum (1 - \cosh \varepsilon_k) \geq 1 - \cosh \varepsilon$. Then $\varepsilon = O(m)$ and it is immediate to see that $\int |g(p + i\varepsilon)| \prod dp < \text{const.} \log(1 + m^{-1})$

In eq.(3.8), we consider the complex displacement of $p_i$ by $i\varepsilon_i$. We again shift $k_i$ by $i\varepsilon_i/2$ since $g(p - k)$ is periodic. Then $\tilde{g}^2(p + i\varepsilon)$ is equal to

$$\int g(p - k + i\frac{\varepsilon}{2}) g(k + i\frac{\varepsilon}{2}) \prod \frac{dk_i}{2\pi} = \int \frac{D - i(A_1 B_2 + A_2 B_1)}{(A_1^2 + B_1^2)(A_2^2 + B_2^2)} \prod \frac{dk_i}{2\pi}$$

where $A_1 = m^2 + 2\sum[1 - \cos(p_i - k_i)\cosh(\varepsilon_i/2)]$, $B_1 = 2\sum \sin(p_i - k_i)\sinh(\varepsilon_i/2)$, $A_2 = A_1(p \equiv 0)$, $B_2 = -B_1(p \equiv 0)$ and $D \equiv A_1A_2 - B_1B_2$. Note that $2D = (A_1 + B_1)(A_2 - B_2) + (A_1 - B_1)(A_2 + B_2)$, where $A_1 \pm B_1 = m^2 + 4 - 2\sum_i \sqrt{\cosh(\varepsilon_i)}\cos(p_i - k_i \pm \delta_i)$ and so on, where $\tan\delta_i = \tanh(\varepsilon_i/2)$. Then $D > 0$ if $m^2 + 4 - 2\sum\sqrt{\cosh(\varepsilon_i)} > 0$. Since $\varepsilon^2 = \sum \varepsilon_i^2$ and $\sum\sqrt{\cosh(\varepsilon_i)} = 2 + \frac{1}{4}\varepsilon^2 - O(\varepsilon^4)$, $D > 0$ if $|\varepsilon| \le \sqrt{2}m$. Then $C(x,y)$, $\tilde{G}(x,y)$ and $\tilde{G}^{-1}(x,y)$ have uniform exponential decay faster than $\exp[-\sqrt{2}m|x-y|]$.

By Schwarz's inequality, $\int|\tilde{g}(p + i\varepsilon)|dp <$ const. $\log(1 + m^{-1})$ if $|\varepsilon| < \sqrt{2}m$. Thus the bound for $\tilde{G}$ follows. Maximize $A_i^2 + B_i^2$ and integrate $D$ over $k$ to obtain $\mathrm{Re}\,\tilde{g}(p + i\varepsilon)^2 \ge c_0(8 + m^2)^{-2}$, $c_0 = O(1) > 0$. Thus the bounds for $C = [G^{\circ 2}]^{-1}$ and $\tilde{G}^{-1}$ follow.

The function $\tilde{g}(p)$ is exactly obtained in the continuum limit, and is analytic in $|\mathrm{Im} p| < 2m$. Thus our estimate will be improved.

## APPENDIX B: POLYMER EXPANSIONS OF KERNEL FUNCTIONS

Let $H(x)$ be a positive type function defined on $\mathbf{Z}^2$ whose Fourier transform $\tilde{H}(p)$ satisfies the following:

(1) $0 < c_1 \le \tilde{H}(p) \le c_2$.

(2) $\tilde{H}(p)$ is periodic in $p_i$, $i = 1, 2$.

(3) $\tilde{H}(p)$ is analytic in $p \in \Omega_\varepsilon$ where $\Omega_\varepsilon = \{(p_1, p_2); |\mathrm{Im}\, p_i| < \varepsilon_i\}$, $\sum \varepsilon_i^2 < m^2$. $|\tilde{H}(p)|$ and $|\tilde{H}(p)|^{-1}$ are bounded on the boundary.

(4) $0 < c_1' \le \mathrm{Re}\,\tilde{H}(p) \le c_2'$ and $|\mathrm{Im}\tilde{H}(p)| \le c_3'$ for $p \in \Omega_\varepsilon$.

Then we have shown that both $H(x)$ and $H^{-1}(x)$ decrease exponentially fast in $|x|$. Put

$$H(x,y) = \int \exp[ip(x-y)]\tilde{H}(p)\prod\frac{dp_i}{2\pi}.$$

Let $X \subset \Lambda$ and we define the matrix $H_X$ of size $|X|\times|X|$ by $H_X(x,y) \equiv \chi_X(x)H(x-y)\chi_X(y)$. Then $c_1 \le H_X \le c_2$ and we have:

**Theorem B 1** $H_X^{-1}(x,y)$, $H_X^{1/2}(x,y)$ *and* $H_X^{-1/2}(x,y)$ *again decay exponentially fast :*

$$|H_X^{-1}(x,y)| \;<\; \text{const.}\exp[-m|x-y|],$$
$$|H_X^{\pm 1/2}(x,y)| \;<\; \text{const.}\exp[-m|x-y|].$$

*Proof.* First suppose that $X$ is a rectangle of side lengths $X_1$ and $X_2$ with the center at the origin. The operator $H_X(x,y)$ is strictly positive. Let $\tilde{H}_X(p,q)$ be its Fourier kernel:

$$\tilde{H}_X(p,q) = \sum_{x,y\in X} \int e^{i(p+k)x-i(q+k)y} \tilde{H}(k) \prod \frac{dk_i}{2\pi}. \tag{B1}$$

This is strictly positive and hence invertible. The properties (2) and (3) mean that $\tilde{H}_X$ can be analytically continued by

$$\tilde{H}_X(i\varepsilon)(p,q) \equiv \tilde{H}_X(p+i\varepsilon, q+i\varepsilon) = \sum_{x,y\in X} \int e^{i(p+k)x-i(q+k)y} \tilde{H}(k-i\varepsilon) \prod \frac{dk_i}{2\pi}, \tag{B2}$$

and we see that

(i) $\tilde{H}_X$ is strictly positive as an oprator on $\ell^2(X^*)$, where $X^*$ is the dual of $X$: $X^* = \{(2\pi n_1/X_1,\, 2\pi n_2/X_2);\, n_i = 0,1,\cdots,X_i-1\}$

(ii) The self-adjoint part of $\tilde{H}_X(i\varepsilon)$ is strictly positive for $|\varepsilon| < m$.

Since

$$\begin{aligned} H_X(x,y) &= \frac{1}{|X|^2} \sum_{p,q\in X^*} \int e^{-ipx+iqy} \tilde{H}_X(0)(p,q) \\ &= \frac{1}{|X|^2} \sum_{p,q\in X^*} \int e^{-i(p+i\varepsilon)x+i(q+i\varepsilon)y} \tilde{H}_X(i\varepsilon)(p,q), \end{aligned} \tag{B3}$$

we have

$$H_X^{-1}(x,y) = \frac{1}{|X|^2} \sum_{p,q\in X^*} e^{-i(p+i\varepsilon)x+i(q+i\varepsilon)y} \tilde{H}_X(i\varepsilon)^{-1}(p,q) \tag{B4}$$

where $|X| = X_1X_2$. Then take $\varepsilon_k = -m\zeta_k/|\zeta|$, $\zeta = x-y$.

If $X$ is not a rectangle, choose the smallest rectangular set $\hat{X}$ containing $X$. Define $\hat{H}_{\hat{X}} = \chi_X H \chi_X + 1_{\hat{X}\backslash X}$, where $1_{\hat{X}\backslash X}$ is the identity operator on $\hat{X}\backslash X$. Then $\hat{H}_{\hat{X}}$ is strictly positive on $\ell^2(\hat{X})$ and the previous discussion applies. The proof is same for $H_X^{1/2}(x,y)$ and $H_X^{-1/2}(x,y)$. Q.E.D.

For $(G_R)^{-1/2}$, we have an alternative : we can apply polymer expansion or random walk expansion to the right hand side of the integral representation $G_R^{-1/2} = 2\int (G_R+u^2)^{-1} du/\pi$. (This is left to the reader.)

Let $X = X_1 \cup X_2$ where $X_1 \cap X_2 = \emptyset$ and we assume that $X_1$, $X_2$ and $X = X_1 \cup X_2$ are rectangles. Let $H_X(s) \equiv (1-s)(H_{X_1} + H_{X_2}) + sH_X$. Then $H(s)$ is strictly positive uniformly in $s \in [0,1]$. What is important is that the Fourier transform of $H(s)(x,y)$ is

$$\tilde{H}_s(p,q) \equiv (1-s)(\tilde{H}_{X_1}(p,q) + \tilde{H}_{X_2}(p,q)) + s\tilde{H}_{X_2}(p,q)$$

which satisfies the conditions (i) and (ii) uniformly in $s \in [0,1]$. This implies that

**Theorem B 2** *Let $H_X(s)$ be a convex linear combination of $\{H_{X_1} \oplus \cdots \oplus H_{X_n}; X = \cup X_i, X_i \cap X_j = \emptyset, (i \neq j)\}$ . Then the following bound holds uniformly in $s_i \in [0,1]$:*

$$|H_X^{-1}(s)(x,y)| \; < \; \text{const.} \exp[-m|x-y|].$$

For $H_X(s)$ with $X = X_1 \cup X_2$, we have :

$$H_X^{-1} = H_{X_1}^{-1} \oplus H_{X_2}^{-1} - \int_0^1 H_X(s)^{-1}(H_{X_1X_2} + H_{X_2X_1})H_X(s)^{-1} ds$$

This is the first step of the polymer expansion of $H_X^{-1}$ in the form of Lemma 14, but here we have introduced the interpolation parameter $s = s_1$ into the denominator (not in $G^{-1}$ like in Lemma 14). All these mean that we can apply the Brydges-Federbush method to cluster-expand some Green's functions.

## APPENDIX C: POLYMER EXPANSION OF GAUSSIAN MEASURES

We here discuss a cluster expansion of Gaussian measures with an interaction $V$:

$$Z_\Lambda \; = \; \int \exp[-V(\psi)] d\mu \tag{C1}$$

$$d\mu \; = \; \det{}^{-1/2}(C) \exp[- < \psi, C^{-1}\psi >] \prod \frac{d\psi(x)}{\sqrt{\pi}}. \tag{C2}$$

Since $C$ is strictly positive, we use the cluster expansion of Brydges-Federbush type which keeps positivity of the operator. To do so, we first choose $\Delta_1 \subset \Lambda$ and define

$$C(s_1) \; = \; [(1-s_1)\mathcal{P}_1 + 1]C_\Lambda = (1-s_1)(C_{\Lambda\setminus\Delta_1} + C_{\Delta_1}) + s_1 C_\Lambda, \tag{C3}$$

$$\mathcal{P}_1 C_X \; \equiv \; C_{X\setminus\Delta_1} + C_{X\cap\Delta_1}, \tag{C4}$$

where we have used the notational convention $C_X = \chi_X C \chi_X$, $C_{X,Y} = \chi_X C \chi_Y$ and $X^c = \Lambda \backslash X$ as usual. Thus we have ($C$ in [5,16] is written $\frac{1}{2}C$ here)

$$
\begin{aligned}
Z_\Lambda &= \int \exp[-V(\psi)] d\mu(s_1 = 1) \\
&= Z_{\Lambda\backslash\Delta_1} Z_{\Delta_1} + \sum_{\Delta_2 \subset \Lambda\backslash\Delta_1} \int_0^1 ds_1 \int d\mu(s_1) \sum_{x\in\Delta_1} \sum_{y\in\Delta_2} \frac{1}{2} C(x,y) \frac{\partial^2}{\partial\psi(x)\partial\psi(y)} e^{-V}, \quad \text{(C5)}
\end{aligned}
$$

where

$$
d\mu(s_1) = \det{}^{-1/2}[C(s)] \exp[- < \psi, C(s_1)^{-1}\psi >] \prod \frac{d\psi(x)}{\sqrt{\pi}}. \quad \text{(C6)}
$$

In fact, this follows from the observations of

$$
\begin{aligned}
\int d\mu(s_1) e^{i\psi(f)} &= \exp[-\frac{1}{4} < f, C(s_1) f >], \quad \text{(C7)} \\
\frac{\partial}{\partial s_1}(\text{r.h.s.}) &= \frac{1}{4} \sum_{x,y} [\frac{\partial}{\partial s_1} C(s_1)]_{xy} \int d\mu(s_1) \frac{\partial^2}{\partial\psi(x)\partial\psi(y)} e^{i\psi(f)}, \quad \text{(C8)} \\
\frac{\partial}{\partial s_1} C(s_1) &= \sum_{\Delta_2 \subset \Lambda\backslash\Delta_1} (C_{\Delta_1,\Delta_2} + C_{\Delta_2,\Delta_1}). \quad \text{(C9)}
\end{aligned}
$$

This establishes the claim for the decomposition into $\Delta_1$ and $\Lambda\backslash\Delta_1$.

We next apply the same steps to each term of eq.(C5): we introduce an interpolation parameter $s_1$ to $Z_{\Lambda\backslash\Delta_1}$ to decouple $\Delta_2$ from $\Lambda\backslash\Delta_1$ and introduce next interpolation parameter $s_2$ to the rest to decouple $Y \equiv \Delta_1 \cup \Delta_2$ from $\Delta_3 \subset \Lambda\backslash Y$. See [5,16] for the detail.

Tree graphs $T'$ over $\{\Delta_1, \cdots, \Delta_p\}$ with the root $\Delta_1$ are graphs defined by permutations $\{j_1, \cdots, j_p\}$ of $\{1, 2, \cdots, p\}$ with $j_1 = 1$ and a map $a_{T'} : \{1, 2, \cdots, p-1\} \to \{1, 2, \cdots, p-1\}$ such that $a_{T'}(k) \le k$. They define a set of ordered links (tree graph $T'$) $\ell_k = (\Delta_{j_{a(k)}}, \Delta_{j_{k+1}})$, $k = 1, 2, \cdots, p-1$. Set

$$
M_{T'}(s) = \prod_{i=1}^{p-1} \prod_{j=a_{T'}(i)}^{i-1} s_j. \quad \text{(C10)}
$$

**Theorem C 1** *[16] $Z_\Lambda$ have the cluster expansion*

$$
\sum_p \frac{1}{p!} \sum_{Y_1,\cdots,Y_p} \prod_i Z^c_{Y_i} \prod_{\Delta \subset \Lambda\cup Y_i} Z_\Delta, \quad \text{(C11)}
$$

*where $Y_i$ are paved sets which are disjoint each other and consist of more than two $\Delta_i \subset \Lambda$. Let $Y = \cup_{i=1}^p \Delta_i$ be one of $Y_i$. Then $Z_Y^c$ has the following expression:*

$$\sum_{T'} \int_0^1 ds_1 \cdots ds_{p-1} M_{T'}(s) \int d\mu(\{s\}) \times \prod_{k=1}^{p-1} \left[ \sum_{x_k \in \Delta_{j_{a(k)}}} \sum_{y_{k+1} \in \Delta_{j_{k+1}}} \frac{1}{2} C(x_k, y_{k+1}) \frac{\partial^2}{\partial\psi(x_k)\partial\psi(y_{k+1})} \right] \exp[-V(\psi)], \tag{C12}$$

*where $T' = T_a = \{(j_{a(k)}, j_{k+1})\}_k$,*

$$d\mu(\{s\}) = \det{}^{-1/2}[C(s)] \exp[- < \psi, C^{-1}(\{s\})\psi >] \prod \frac{d\psi(x)}{\sqrt{\pi}}, \tag{C13}$$

$$C(\{s\}) = [\prod_{i=1}^{p-1}((1-s_i)\mathcal{P}_i + s_i)]C_\Lambda, \tag{C14}$$

$$\mathcal{P}_i C_X = C_{X\setminus X_i} + C_{X\cap X_i}, \; (X_i = \cup_{k=1}^i \Delta_{j_k}). \tag{C15}$$

There are many tree graphs $T'$ with root $\Delta_1$ which have the same links and vertices with $T$. They differ each other by $M_{T'}(s)$ and $C^{-1}(s)$ [5,16]:

**Theorem C 2** *$M_T \prod ds_i$ is a probability measure in the following sense:*

$$\sum_{T':T(T')=T} \int_0^1 M_{T'} \prod ds_i = 1, \tag{C16}$$

*where $\sum_{T':T(T')=T}$ means the sum over tree graphs $T'$ which have same links with $T$.*

For the Gaussian measure $d\mu_{\tilde{K}}$ restricted to the region $\tilde{K}$, we have :

$$d\mu_{\tilde{K}}(s) = \det{}^{1/2}(\chi_{\tilde{K}} H(s)\chi_{\tilde{K}}) \exp[- < \psi, \chi_{\tilde{K}} H(s)\chi_{\tilde{K}}\psi >] \prod \frac{d\psi(x)}{\sqrt{\pi}}, \tag{C17}$$

where $H(s)^{-1} = C(s) = (1-s)(C_{\Lambda\setminus X_1} + C_{X_1}) + sC_\Lambda$ (we used $X_1$ for $\Delta_1$) and

$$\frac{d}{ds}\int e^{-V} d\mu_{\tilde{K}} = \int d\mu_{\tilde{K}}(s) \sum_{x,y} \frac{1}{4}(ABA)_{xy} \frac{\partial^2}{\partial\psi(x)\partial\psi(y)} e^{-V}, \tag{C18}$$

$$A = [\chi_{\tilde{K}} H(s)\chi_{\tilde{K}}]^{-1}, \tag{C19}$$

$$B = -\frac{\partial}{\partial s}A^{-1} = \chi_{\tilde{K}} H(s)[\frac{\partial}{\partial s}C(s)]H(s)\chi_{\tilde{K}}. \tag{C20}$$

Since $(ABA)_{xy}$ depends on locations of $R_i^0$, we expand $ABA$ into polymers. In fact using the method of Lemma 14 to expand $[\chi_{\tilde{K}} H(s)\chi_{\tilde{K}}]^{-1}$ in terms of $H_{\Delta_i}$ and $H_{D_i\setminus R_i^0}$, we have

$$\begin{aligned}[\chi_{\tilde{K}} H(s)\chi_{\tilde{K}}]^{-1}\chi_{\tilde{K}} H(s) &= I_{\tilde{K}} + [\chi_{\tilde{K}} H(s)\chi_{\tilde{K}}]^{-1}\chi_{\tilde{K}} H(s)\chi_{R^0} \\ &= I_{\tilde{K}} + \sum_{X\cap R^0\neq\emptyset} \delta C(X)\end{aligned}$$

where $I_{\tilde{K}}$ is the identity operator on $\mathbf{C}^{\tilde{K}}$ and $\delta C(X)$ are the polymers expressed by random walks passing all squares $\Delta_i$ only in $X$ and at least one of $\{R_i^0, D_i\backslash R_i^0\}$ if $D_i \subset X$.

We proceed inductively. After $j$ steps, $ABA$ is the sum over $i$ of the following terms:

$$1_{\tilde{K}}\left[C_{X_i,\Lambda\backslash\cup_1^j X_k} + C_{\Lambda\backslash\cup_1^j X_k,X_i}\right]1_{\tilde{K}} + \sum_{X_1'}\delta C(X_1')\,[(\text{ same })]\,1_{\tilde{K}}$$
$$+\sum_{X_2'}1_{\tilde{K}}\,[(\text{ same })]\,\delta C^+(X_2') + \sum_{X_1'}\sum_{X_2'}\delta C(X_1')\,[(\text{ same })]\,\delta C^+(X_2')$$

where $X_\ell \cap X_k = \emptyset$ $(k \neq \ell)$, $1 \le i \le j$ and $\{s_i\}_1^j$ are ommitted. Next step is :

(i) In $1_{\tilde{K}}[\cdots]1_{\tilde{K}}$, choose any $X_{j+1} = \Delta_\ell \subset \Lambda\backslash \cup_1^j X_k$ or $X_{j+1} = D_\ell \subset \Lambda\backslash\cup_1^j X_k$. Define $\delta F_1(X_i, X_{j+1}) \equiv C_{X_i,X_{j+1}}, \quad \delta_1(X_{j+1}, X_i) \equiv C_{X_{j+1},X_i}$.

(ii) In $\delta C(X_1')[\cdots]1_{\tilde{K}}$, choose any $X_{j+1} \subset \Lambda\backslash\cup_1^j X_k$. Define

$\delta F_2(X_i, X_{j+1}) \equiv \sum_{X_1'}\delta C(X_1')C_{X_i,X_{j+1}}, \quad \delta F_2(X_{j+1}, X_i) \equiv \sum_{X_1'}\delta C(X_1')C_{X_{j+1},X_i}$

where $X_1' \subset \cup_1^{j+1}X_k$, and $X_1' \cap X_{j+1}$ must contain $X_{j+1}\cap K^0$ and at least one of $\{R_k^0, D_k\backslash R_k^0\}$ if $D_k \subset X_{j+1}$. This is same for $1_{\tilde{K}}[\cdots]\delta C^+(X_2')$.

(iii) In $\delta C(X_1')1_{\tilde{K}}[\cdots]\delta C(X_2')$, choose any $X_{j+1} \subset \Lambda\backslash\cup_1^j X_k$. Define

$\delta F_4(X_i, X_{j+1}) \equiv \sum_{X_1',X_2'}\delta C(X_1')C_{X_i,\Lambda\backslash\cup_1^j X_k}\delta C^+(X_2')$,

$\delta F_4(X_{j+1}, X_i) \equiv \sum_{X_1',X_2'}\delta C(X_1')C_{\Lambda\backslash\cup_1^j X_k,X_i}\delta C^+(X_2')$

where $X_1'\cup X_2' \subset \cup_1^{j+1}X_k$, and $(X_1'\cup X_2')\cap X_{j+1}$ must contain $X_{j+1}\cap K^0$ and at least one of $\{R_k^0, D_k\backslash R_k^0\}$ if $D_k \subset X_{j+1}$.

Then we define $\delta F(X_i, X_{j+1}) \equiv \sum_{k=1}^4 \delta F_k(X_i, X_{j+1})$ . ( Same for $\delta F(X_{j+1}, X_i)$ ). The following facts are immediate from the construction:

(1) Thanks to the random walk expansion, the sum in the right hand sides converge and exhibits tree decay property with respect to blocks $\Delta_k \subset X_{j+1}$ and $D_\ell \subset X_{j+1}$. The factor $\delta F(X_i, X_{j+1})$, with $i < j+1$ includes the tree decay factor $\exp[-m\mathcal{L}(X_{j+1}\wedge D)]$ and $\exp[-m\mathrm{dist}(X_i, X_{j+1})]$, where $X_{j+1}\wedge D$ implies that $D_\ell \subset X_{j+1}$ must be regarded as one sets and must not be decomposed into $\Delta_k \subset D_\ell$.

(2) If $X_{j+1}$ consists of more than or equal to two $\Delta_k$ or $D_\ell$, then the factor $\delta F(X_i, X_{j+1})$ must contain $\exp[-m\mathrm{dist}(R^0\cap X_{j+1}, X_i)] < \exp[-3mL]$.

(3) The matrix element $\delta F(X_i, X_j)(x,y)$ is less than $\min_\ell \exp[-m\mathcal{L}(\Delta_\ell \cup (X_j \wedge D), x, y)]$, where $\Delta_\ell \subset X_i$.

(4) The matrix element $\delta F(X_i, X_j)(x,y) \neq 0$ even if $x \notin X_i$ or $y \notin X_j$. But it is less than the value given above, and bounded by $\exp[-m(L + L_0)]$ since it contains $R^0$.

We then introduce $s_{j+1}$ to $C(s_1, \cdots, s_j)$ to separate $\cup_1^{j+1} X_k$ from its complement. We repeat the argument and obtain Theorem 19.

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