Tridiagonal pairs of Krawtchouk type

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Abstract

Let $\mathbb{F}$ denote an algebraically closed field with characteristic 0 and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let A, A^* denote a tridiagonal pair on V with diameter d. We say that A, A^* has *Krawtchouk type* whenever the sequence $\{d - 2i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A and a standard ordering of the eigenvalues of A^*. Assume A, A^* has Krawtchouk type. We show that there exists a nondegenerate symmetric bilinear form $\langle \, , \, \rangle$ on V such that $\langle Au, v \rangle = \langle u, Av \rangle$ and $\langle A^*u, v \rangle = \langle u, A^*v \rangle$ for $u, v \in V$. We show that the following tridiagonal pairs are isomorphic: (i) A, A^*; (ii) $-A, -A^*$; (iii) A^*, A; (iv) $-A^*, -A$. We give a number of related results and conjectures.

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1 Tridiagonal pairs

Throughout this paper $\mathbb{F}$ denotes a field.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. For a subspace $W \subseteq V$ and $A \in \mathrm{End}(V)$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that

$$W = \{v \in V \mid Av = \theta v\}.$$

We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [10] Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair A, A^* of elements in $\mathrm{End}(V)$ that satisfy the following four conditions.

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(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d), \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta), \tag{2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \ne 0$, $W \ne V$.

We say the pair A, A^* is *over* $\mathbb{F}$. We call V the *vector space underlying* A, A^*. We call $\mathrm{End}(V)$ the *ambient algebra* of A, A^*.

Note 1.2 According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

We refer the reader to [1], [2], [10], [11], [12], [13], [19], [26], [37] and the references therein for background on tridiagonal pairs.

In order to motivate our results we recall a few facts about TD pairs. Let A, A^* denote a TD pair on V, as in Definition 1.1. It turns out that the integers d and δ from conditions (ii) and (iii) are equal [10, Lemma 4.5]; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces of A (resp. A^*) will be called *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. Then the ordering $\{V_{d-i}\}_{i=0}^d$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the eigenvalues of A (resp. A^*) will be called *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). By [10, Corollary 5.7], for $0 \le i \le d$ the spaces V_i, V_i^* have the same dimension; we denote this common dimension by ρ_i. By the construction $\rho_i \ne 0$. By [10, Corollary 5.7] and [10, Corollary 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \le i \le d$ and $\rho_{i-1} \le \rho_i$ for $1 \le i \le d/2$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, A^*.

The following special case has received a lot of attention. By a *Leonard pair* we mean a TD pair with shape $(1, 1, \ldots, 1)$ [29, Definition 1.1], [30, Lemma 2.2]. There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials [4], [7] and their relatives [17]. This family coincides with the terminating branch of the Askey scheme [35]. See [18], [20], [21], [22], [23], [24], [25], [26], [30], [31], [32], [33], [34], [36], [38] for more information about Leonard pairs.

We mention some notation for later use. For an indeterminate λ let $\mathbb{F}[\lambda]$ denote the $\mathbb{F}$-algebra of all polynomials in λ that have coefficients in $\mathbb{F}$.

We now summarize our results. For the rest of this section assume $\mathbb{F}$ is algebraically closed with characteristic 0. Let A, A^* denote a TD pair over $\mathbb{F}$ and let d denote the diameter. We say that A, A^* has *Krawtchouk type* whenever the sequence $\{d - 2i\}_{i=0}^d$ is a standard ordering of the eigenvalues of A and a standard ordering of the eigenvalues of A^*. Assume A, A^* has Krawtchouk type and let V denote the underlying vector space. We will prove the following.

- Let $\{\rho_i\}_{i=0}^d$ denote the shape of A, A^*. Then there exists a nonnegative integer N and positive integers $d_1, d_2, \ldots, d_N$ such that

$$\sum_{i=0}^d \rho_i \lambda^i = \prod_{j=1}^N (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

- There exists a nonzero bilinear form $\langle \, , \, \rangle$ on V such that $\langle Au, v \rangle = \langle u, Av \rangle$ and $\langle A^*u, v \rangle = \langle u, A^*v \rangle$ for $u, v \in V$. The form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. The form is nondegenerate and symmetric.

- There exists a unique antiautomorphism $\dagger$ of $\mathrm{End}(V)$ that fixes each of A, A^*. Moreover $X^{\dagger\dagger} = X$ for all $X \in \mathrm{End}(V)$.

- Let B (resp. B^*) denote the image of A (resp. A^*) under the canonical anti-isomorphism $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$, where $\tilde{V}$ is the vector space dual to V. Then the TD pairs A, A^* and B, B^* are isomorphic.

- The following TD pairs are mutually isomorphic:

$$A, A^*; \qquad -A, -A^*; \qquad A^*, A; \qquad -A^*, -A.$$

In addition we prove the following. We associate with A, A^* a polynomial P_{A,A^*} in $\mathbb{F}[\lambda]$ called the Drinfel'd polynomial. We show that the map $A, A^* \mapsto P_{A,A^*}$ induces a bijection between (i) the set of isomorphism classes of TD pairs over $\mathbb{F}$ that have Krawtchouk type; (ii) the set of polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and do not vanish at $\lambda = 1$.

To obtain the above results we use a bijection due to Hartwig [8] involving the TD pairs of Krawtchouk type and the finite-dimensional irreducible modules for the tetrahedron algebra [9].

We finish the paper with a number of conjectures and problems concerning general TD pairs.

The paper is organized as follows. In Sections 2–5 we review some facts about general TD pairs. In Sections 6, 7 we recall the tetrahedron algebra and its relationship to the TD pairs of Krawtchouk type. We use this relationship in Sections 8–13 to prove our results on TD pairs of Krawtchouk type. In Section 14 we give our conjectures and open problems.

We remark that Alnajjar and Curtin [3] obtained some results similar to ours for the TD pairs of q-Serre type.

2   Isomorphisms of TD pairs

In this section we consider the concept of isomorphism for TD pairs from several points of view.

Definition 2.1 Let A, A^* and B, B^* denote TD pairs over $\mathbb{F}$. By an *isomorphism of TD pairs* from A, A^* to B, B^* we mean a vector space isomorphism γ from the vector space underlying A, A^* to the vector space underlying B, B^* such that both

$$\gamma A = B\gamma, \qquad \gamma A^* = B^*\gamma.$$

We have a comment.

Lemma 2.2 *Assume $\mathbb{F}$ is algebraically closed. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V. Then the following (i), (ii) are equivalent for $\gamma \in \mathrm{End}(V)$.*

(i) γ commutes with each of A, A^*;

(ii) there exists $\alpha \in \mathbb{F}$ such that $\gamma = \alpha I$.

Proof: $(i) \Rightarrow (ii)$ Since V has finite positive dimension and since $\mathbb{F}$ is algebraically closed there exists at least one eigenspace $W \subseteq V$ for γ. Note that $AW \subseteq W$ since $\gamma A = A\gamma$ and $A^*W \subseteq W$ since $\gamma A^* = A^*\gamma$. Also $W \neq 0$ by the definition of an eigenspace so $W = V$ in view of Definition 1.1(iv). The result follows.
$(ii) \Rightarrow (i)$ Clear. $\qquad\qquad\square$

Proposition 2.3 *Assume $\mathbb{F}$ is algebraically closed. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V. Then the following (i), (ii) are equivalent for $\gamma \in \mathrm{End}(V)$.*

(i) γ is an isomorphism of TD pairs from A, A^* to A, A^*;

(ii) there exists a nonzero $\alpha \in \mathbb{F}$ such that $\gamma = \alpha I$.

Proof: $(i) \Rightarrow (ii)$ The map γ is nonzero and commutes with each of A, A^* so the result follows in view of Lemma 2.2.
$(ii) \Rightarrow (i)$ Clear. $\qquad\qquad\square$

We have been discussing the concept of isomorphism for TD pairs. In this discussion we now shift the emphasis from maps involving the underlying vector spaces to maps involving the ambient algebras. We do this as follows. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let $\gamma : V \to V'$ denote an isomorphism of vector spaces. Observe that there exists an $\mathbb{F}$-algebra isomorphism $\varrho : \mathrm{End}(V) \to \mathrm{End}(V')$ such that $X^\varrho = \gamma X \gamma^{-1}$ for all $X \in \mathrm{End}(V)$. Conversely let $\varrho : \mathrm{End}(V) \to \mathrm{End}(V')$ denote an $\mathbb{F}$-algebra isomorphism. Then by the Skolem-Noether theorem [28, Corollary 9.122] there exists an isomorphism of vector spaces $\gamma : V \to V'$ such that $X^\varrho = \gamma X \gamma^{-1}$ for all $X \in \mathrm{End}(V)$. Combining these comments with Definition 2.1 we obtain the following result.

Corollary 2.4 *Let A, A^* and B, B^* denote TD pairs over $\mathbb{F}$. Then these TD pairs are isomorphic if and only if there exists an $\mathbb{F}$-algebra isomorphism from the ambient algebra of A, A^* to the the ambient algebra of B, B^* that sends A to B and A^* to B^*.*

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By an *automorphism* of $\operatorname{End}(V)$ we mean an $\mathbb{F}$-algebra isomorphism from $\operatorname{End}(V)$ to $\operatorname{End}(V)$.

Corollary 2.5 *Assume $\mathbb{F}$ is algebraically closed. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V. Then the identity map is the unique automorphism of $\operatorname{End}(V)$ that fixes each of A, A^*.*

Proof: Immediate from Proposition 2.3 and our comments above Corollary 2.4. $\qquad\square$

3 Bilinear forms

Later in the paper we will discuss some bilinear forms related to TD pairs; to prepare for this we recall some relevant linear algebra.

Let V and V' denote vector spaces over $\mathbb{F}$ with the same finite positive dimension. A map $\langle\,,\,\rangle : V \times V' \to \mathbb{F}$ is called a *bilinear form* whenever the following four conditions hold for $u, v \in V$, for $u', v' \in V'$, and for $\alpha \in \mathbb{F}$: (i) $\langle u+v, u'\rangle = \langle u, u'\rangle + \langle v, u'\rangle$; (ii) $\langle \alpha u, u'\rangle = \alpha\langle u, u'\rangle$; (iii) $\langle u, u' + v'\rangle = \langle u, u'\rangle + \langle u, v'\rangle$; (iv) $\langle u, \alpha u'\rangle = \alpha\langle u, u'\rangle$. We observe that a scalar multiple of a bilinear form is a bilinear form. Let $\langle\,,\,\rangle : V \times V' \to \mathbb{F}$ denote a bilinear form. Then the following are equivalent: (i) there exists a nonzero $v \in V$ such that $\langle v, v'\rangle = 0$ for all $v' \in V'$; (ii) there exists a nonzero $v' \in V'$ such that $\langle v, v'\rangle = 0$ for all $v \in V$. The form $\langle\,,\,\rangle$ is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise. Here is an example of a nondegenerate bilinear form. Let $\tilde{V}$ denote the vector space dual to V, consisting of the linear transformations from V to $\mathbb{F}$. Define a map $\langle\,,\,\rangle : V \times \tilde{V} \to \mathbb{F}$ such that $\langle v, f\rangle = f(v)$ for all $v \in V$ and $f \in \tilde{V}$. Then $\langle\,,\,\rangle$ is a nondegenerate bilinear form. We call this form the *canonical* bilinear form between V and $\tilde{V}$. By a *bilinear form on V* we mean a bilinear form $\langle\,,\,\rangle : V \times V \to \mathbb{F}$. This form is said to be *symmetric* whenever $\langle u, v\rangle = \langle v, u\rangle$ for all $u, v \in V$.

Again let V and V' denote vector spaces over $\mathbb{F}$ with the same finite positive dimension. By an *$\mathbb{F}$-algebra anti-isomorphism* from $\operatorname{End}(V)$ to $\operatorname{End}(V')$ we mean an $\mathbb{F}$-linear bijection $\varrho : \operatorname{End}(V) \to \operatorname{End}(V')$ such that $(XY)^\varrho = Y^\varrho X^\varrho$ for all $X, Y \in \operatorname{End}(V)$. Let $\langle\,,\,\rangle : V \times V' \to \mathbb{F}$ denote a nondegenerate bilinear form. Then there exists a unique $\mathbb{F}$-algebra anti-isomorphism $\varrho : \operatorname{End}(V) \to \operatorname{End}(V')$ such that $\langle Xv, v'\rangle = \langle v, X^\varrho v'\rangle$ for all $X \in \operatorname{End}(V)$, $v \in V$, $v' \in V'$. We say that ϱ *corresponds* to $\langle\,,\,\rangle$. By the *canonical* $\mathbb{F}$-algebra anti-isomorphism from $\operatorname{End}(V)$ to $\operatorname{End}(\tilde{V})$ we mean the one that corresponds to the canonical bilinear form. By an *antiautomorphism* of $\operatorname{End}(V)$ we mean an $\mathbb{F}$-algebra anti-isomorphism from $\operatorname{End}(V)$ to $\operatorname{End}(V)$.

4   New TD pairs from old

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V. We can modify this pair in several ways to get another TD pair. For instance the ordered pair A^*, A is a TD pair on V which is potentially nonisomorphic to A, A^*. Let $\alpha, \alpha^*, \beta, \beta^*$ denote scalars in $\mathbb{F}$ such that each of α, α^* is nonzero. Then the pair

$$\alpha A + \beta I, \qquad \alpha^* A^* + \beta^* I$$

is a TD pair on V which is potentially nonisomorphic to A, A^*. Let $\tilde{V}$ denote the vector space dual to V. Let B (resp. B^*) denote the image of A (resp. A^*) under the canonical $\mathbb{F}$-algebra anti-isomorphism $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ from Section 3. By [3, Theorem 1.2] the pair B, B^* is a TD pair on $\tilde{V}$ which is potentially nonisomorphic to A, A^*.

5   The split decomposition of a TD pair

Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues for A (resp. A^*). We recall the corresponding split decomposition and split sequence. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote the ordering of the eigenspaces of A (resp. A^*) associated with $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). For $0 \leq i \leq d$ define

$$U_i = (V_0^* + V_1^* + \cdots + V_i^*) \cap (V_i + V_{i+1} + \cdots + V_d).$$

By [10, Theorem 4.6]

$$V = \sum_{i=0}^d U_i \qquad \text{(direct sum)},$$

and for $0 \leq i \leq d$ both

$$
\begin{aligned}
U_0 + U_1 + \cdots + U_i &= V_0^* + V_1^* + \cdots + V_i^*, \\
U_i + U_{i+1} + \cdots + U_d &= V_i + V_{i+1} + \cdots + V_d.
\end{aligned}
$$

By [10, Corollary 5.7] U_i has dimension ρ_i where $\{\rho_i\}_{i=0}^d$ is the shape of A, A^*. By [10, Theorem 4.6] both

$$
\begin{aligned}
(A - \theta_i I)U_i &\subseteq U_{i+1}, & (3) \\
(A^* - \theta_i^* I)U_i &\subseteq U_{i-1}, & (4)
\end{aligned}
$$

where $U_{-1} = 0$ and $U_{d+1} = 0$. The sequence $\{U_i\}_{i=0}^d$ is called the *split decomposition* of V with respect to $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ [10, Section 4]. Now assume $\rho_0 = 1$, so that U_0 has dimension 1. For $0 \leq i \leq d$ the space U_0 is invariant under

$$(A^* - \theta_1^* I)(A^* - \theta_2^* I) \cdots (A^* - \theta_i^* I)(A - \theta_{i-1} I) \cdots (A - \theta_1 I)(A - \theta_0 I); \qquad (5)$$

let ζ_i denote the corresponding eigenvalue. We call the sequence $\{\zeta_i\}_{i=0}^d$ the *split sequence* of A, A^* with respect to $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$.

Note 5.1 In the literature on Leonard pairs there are two sequences of scalars called the first split sequence and the second split sequence [29, Section 3]. These sequences are related to the above split sequence as follows. Let A, A^* denote a Leonard pair and fix a standard ordering $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) of the eigenvalues of A (resp. A^*). Let $\{\varphi_i\}_{i=1}^d$ (resp. $\{\phi_i\}_{i=1}^d$) denote the corresponding first split sequence (resp. second split sequence) in the sense of [29]. Then the sequence $\{\varphi_1 \varphi_2 \cdots \varphi_i\}_{i=0}^d$ (resp. $\{\phi_1 \phi_2 \cdots \phi_i\}_{i=0}^d$) is the split sequence of A, A^* associated with $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ (resp. $\{\theta_{d-i}\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$).

6 The tetrahedron algebra

From now until the end of Section 13 we assume:

The field $\mathbb{F}$ is algebraically closed with characteristic 0.

We now recall some facts about the tetrahedron algebra that we will use later in the paper. We start with a definition.

Definition 6.1 [9, Definition 1.1] Let $\boxtimes$ denote the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad \mathbb{I} = \{0, 1, 2, 3\} \tag{6}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{k\ell}]]] = 4[x_{ij}, x_{k\ell}]. \tag{7}$$

We call $\boxtimes$ the *tetrahedron algebra* or "tet" for short.

Let V denote a finite-dimensional irreducible $\boxtimes$-module. We recall the diameter of V and the shape of V. By [8, Theorem 3.8] each generator x_{ij} of $\boxtimes$ is diagonalizable on V. Also by [8, Theorem 3.8] there exists an integer $d \geq 0$ such that for each generator x_{ij} the set of distinct eigenvalues on V is $\{d - 2n \mid 0 \leq n \leq d\}$. We call d the *diameter* of V. By [8, Corollary 3.6], for $0 \leq n \leq d$ there exists a positive integer ρ_n such that for each generator x_{ij} the $(2n - d)$-eigenspace in V has dimension ρ_n. We call the sequence $\{\rho_n\}_{n=0}^d$ the *shape* of V.

Theorem 6.2 [16, Theorem 16.5] *Let $\{\rho_n\}_{n=0}^d$ denote the shape of a finite-dimensional irreducible $\boxtimes$-module. Then there exists a nonnegative integer N and positive integers $\{d_j\}_{j=1}^N$ such that*

$$\sum_{n=0}^d \rho_n \lambda^n = \prod_{j=1}^N (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

In particular $\rho_0 = 1$.

We recall how the symmetric group S_4 acts on $\boxtimes$ as a group of automorphisms. We identify S_4 with the group of permutations of $\mathbb{I}$. We use the cycle notation; for example $(1, 2, 3)$ denotes the element of S_4 that sends $1 \mapsto 2 \mapsto 3 \mapsto 1$ and $0 \mapsto 0$. Note that S_4 acts on the set of generators for $\boxtimes$ by permuting the indices:

$$\sigma(x_{ij}) = x_{\sigma(i),\sigma(j)} \qquad \sigma \in S_4, \qquad i, j \in \mathbb{I}, \quad i \neq j.$$

This action leaves invariant the defining relations for $\boxtimes$ and therefore induces an action of S_4 on $\boxtimes$ as a group of automorphisms.

Let G denote the unique normal subgroup of S_4 that has cardinality 4. G consists of

$$(01)(23), \qquad (02)(13), \qquad (03)(12)$$

and the identity element.

Theorem 6.3 [16, Theorem 19.1] *Let V denote a finite-dimensional irreducible $\boxtimes$-module. For a nonidentity $\sigma \in G$ there exists a nonzero bilinear form $\langle \, , \, \rangle_\sigma$ on V such that*

$$\langle \xi.u, v \rangle_\sigma = -\langle u, \sigma(\xi).v \rangle_\sigma \qquad \xi \in \boxtimes, \qquad u, v \in V. \tag{8}$$

This form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. This form is nondegenerate and symmetric.

Let V denote a $\boxtimes$-module. For $\sigma \in S_4$ there exists a $\boxtimes$-module structure on V, called V *twisted via* σ, that behaves as follows: for all $\xi \in \boxtimes$ and $v \in V$ the vector $\xi.v$ computed in V twisted via σ coincides with the vector $\sigma^{-1}(\xi).v$ computed in the original $\boxtimes$-module V. See [16, Section 7] for more information on twisting.

Theorem 6.4 [16, Corollary 15.5] *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then for $\sigma \in G$ the following are isomorphic:*

(i) the $\boxtimes$-module V twisted via σ;

(ii) the $\boxtimes$-module V.

Let V denote a finite-dimensional irreducible $\boxtimes$-module. We recall the corresponding Drinfel'd polynomial. Abbreviate

$$e^+ := \frac{x_{01} + x_{20}}{2}, \qquad e^- := \frac{x_{23} + x_{02}}{2}. \tag{9}$$

Let U denote the eigenspace of x_{20} on V for the eigenvalue $-d$, where d denotes the diameter of V. Note that U has dimension 1 by the last line of Theorem 6.2. For $0 \leq i \leq d$ the space U is invariant under $(e^-)^i(e^+)^i$ [16, Lemma 17.1]; let $\vartheta_i = \vartheta_i(V)$ denote the corresponding eigenvalue. Define a polynomial $P_V \in \mathbb{F}[\lambda]$ by

$$P_V = \sum_{i=0}^{d} \frac{(-1)^i \vartheta_i \lambda^i}{(i!)^2}. \tag{10}$$

We call P_V the *Drinfel'd polynomial* of V [16, Definition 17.3].

Theorem 6.5 [16, Corollary 17.7] *The map $V \mapsto P_V$ induces a bijection between the following two sets:*

(i) the isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules;

(ii) the polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and are nonzero at $\lambda = 1$.

See [5], [6], [8], [9], [14], [15], [16], [27] for more background information on $\boxtimes$ and related topics.

7 TD pairs of Krawtchouk type and the tetrahedron algebra

In [8] Hartwig relates the TD pairs of Krawtchouk type to the finite-dimensional irreducible $\boxtimes$-modules. His results are summarized in the following two theorems and subsequent remark.

Theorem 7.1 [8, Theorem 1.7] *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then the generators x_{01}, x_{23} act on V as a TD pair of Krawtchouk type.*

Theorem 7.2 [8, Theorem 1.8] *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V that has Krawtchouk type. Then there exists a unique $\boxtimes$-module structure on V such that the generators x_{01}, x_{23} act on V as A, A^* respectively. This $\boxtimes$-module is irreducible.*

Remark 7.3 [8, Remark 1.9] Combining the previous two theorems we obtain a bijection between the following two sets:

(i) the isomorphism classes of TD pairs over $\mathbb{F}$ that have Krawtchouk type;

(ii) the isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules.

9

8   TD pairs of Krawtchouk type; the shape

In this section we describe the shape of a TD pair that has Krawtchouk type. We start with two observations.

Lemma 8.1 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then the following coincide:*

   (i) the diameter of the $\boxtimes$-module V from above Theorem 6.2;

   (ii) the diameter of the TD pair x_{01}, x_{23} on V, in the sense of Section 1.

Proof: In the context of either (i), (ii) above the diameter is one less than the number of distinct eigenvalues for x_{01} on V. $\qquad\square$

Lemma 8.2 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then the following coincide:*

   (i) the shape of the $\boxtimes$-module V from above Theorem 6.2;

   (ii) the shape of the TD pair x_{01}, x_{23} on V, in the sense of Section 1.

Proof: With reference to Lemma 8.1 let d denote the diameter. In the context of either (i), (ii) above, for $0 \leq n \leq d$ the nth component of the shape vector is equal to the dimension of the eigenspace for x_{01} on V associated with the eigenvalue $2n - d$. $\qquad\square$

Theorem 8.3 *Let $\{\rho_i\}_{i=0}^d$ denote the shape of a tridiagonal pair over $\mathbb{F}$ that has Krawtchouk type. Then there exists a nonnegative integer N and positive integers $d_1, d_2, \ldots, d_N$ such that*

$$\sum_{i=0}^{d} \rho_i \lambda^i = \prod_{j=1}^{N} (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

In particular $\rho_0 = 1$.

Proof: Let A, A^* denote a TD pair over $\mathbb{F}$ that has shape $\{\rho_i\}_{i=0}^d$ and let V denote the underlying vector space. By Theorem 7.2 there exists an irreducible $\boxtimes$-module structure on V such that the generators x_{01}, x_{23} act as A, A^* respectively. By Lemma 8.2 the shape of the $\boxtimes$-module V is equal to $\{\rho_i\}_{i=0}^d$. The result follows in view of Theorem 6.2. $\qquad\square$

9 TD pairs of Krawtchouk type; the bilinear form

In this section we describe a bilinear form associated with a TD pair of Krawtchouk type.

Theorem 9.1 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V that has Krawtchouk type. Then there exists a nonzero bilinear form $\langle\,,\,\rangle$ on V such that both*

$$\langle Au, v\rangle = \langle u, Av\rangle, \qquad \langle A^*u, v\rangle = \langle u, A^*v\rangle \qquad (11)$$

for $u, v \in V$. This form is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. This form is nondegenerate and symmetric.

Proof: By Theorem 7.2 there exists a $\boxtimes$-module structure on V such that the generators x_{01}, x_{23} act as A, A^* respectively. Consider the element $\sigma = (01)(23)$ in G and let $\langle\,,\,\rangle = \langle\,,\,\rangle_\sigma$ denote the corresponding bilinear form on V from Theorem 6.3. We show $\langle\,,\,\rangle$ satisfies (11). Observe

$$
\begin{aligned}
\langle Au, v\rangle &= \langle x_{01}.u, v\rangle \\
&= -\langle u, \sigma(x_{01}).v\rangle \\
&= -\langle u, x_{10}.v\rangle \\
&= \langle u, x_{01}.v\rangle \\
&= \langle u, Av\rangle
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\langle A^*u, v\rangle &= \langle x_{23}.u, v\rangle \\
&= -\langle u, \sigma(x_{23}).v\rangle \\
&= -\langle u, x_{32}.v\rangle \\
&= \langle u, x_{23}.v\rangle \\
&= \langle u, A^*v\rangle.
\end{aligned}
$$

Therefore $\langle\,,\,\rangle$ satisfies (11). By Theorem 6.3 the form $\langle\,,\,\rangle$ is nondegenerate and symmetric. Concerning the uniqueness of $\langle\,,\,\rangle$ let $\langle\,,\,\rangle'$ denote any bilinear form on V that satisfies (11). We show that $\langle\,,\,\rangle'$ is a scalar multiple of $\langle\,,\,\rangle$. Pick a basis for V, and let A_b (resp. A_b^*) denote the matrix that represents A (resp. A^*) with respect to this basis. Let M (resp. N) denote the matrix that represents $\langle\,,\,\rangle$ (resp. $\langle\,,\,\rangle'$) with respect to the basis. Note that M is invertible since $\langle\,,\,\rangle$ is nondegenerate. By (11) we have $\xi_b^t M = M\xi_b$ and $\xi_b^t N = N\xi_b$ for $\xi \in \{A, A^*\}$. Combining these equations we find that $M^{-1}N$ commutes with each of A_b, A_b^*. Let γ denote the element of $\mathrm{End}(V)$ that is represented by $M^{-1}N$ with respect to the above basis. Then γ commutes with each of A, A^* and is therefore a scalar multiple of the identity by Lemma 2.2. Now $M^{-1}N$ is a scalar multiple of the identity so N is a scalar multiple of M and therefore $\langle\,,\,\rangle'$ is a scalar multiple of $\langle\,,\,\rangle$. It follows that $\langle\,,\,\rangle$ is unique up to multiplication by a nonzero scalar in $\mathbb{F}$. $\square$

10   TD pairs of Krawtchouk type; the map †

In this section we describe an antiautomorphism † associated with a TD pair of Krawtchouk type. We start with a definition.

Definition 10.1 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V that has Krawtchouk type. Let † denote the antiautomorphism of $\mathrm{End}(V)$ that corresponds to the bilinear form in Theorem 9.1. We emphasize*

$$\langle Xu, v \rangle = \langle u, X^\dagger v \rangle \qquad X \in \mathrm{End}(V), \quad u, v \in V. \tag{12}$$

Theorem 10.2 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V that has Krawtchouk type. Then † is the unique antiautomorphism of $\mathrm{End}(V)$ that fixes each of A, A^*. Moreover $X^{\dagger\dagger} = X$ for all $X \in \mathrm{End}(V)$.*

Proof: We first show that $A^\dagger = A$. For $u, v \in V$ we have $\langle Au, v \rangle = \langle u, Av \rangle$ by (11) and $\langle Au, v \rangle = \langle u, A^\dagger v \rangle$ by (12) so $\langle u, (A^\dagger - A)v \rangle = 0$. Now $(A^\dagger - A)v = 0$ since $\langle \, , \, \rangle$ is nondegenerate and therefore $A^\dagger = A$. Similarly $A^{*\dagger} = A^*$. Concerning the uniqueness of †, let ψ denote any antiautomorphism of $\mathrm{End}(V)$ that fixes each of A, A^*. We show $\psi = \dagger$. The composition $\psi^{-1}\dagger$ is an automorphism of $\mathrm{End}(V)$ that fixes each of A, A^*. This composition is the identity by Corollary 2.5 so $\psi = \dagger$. Also since $\dagger^{-1}$ is an antiautomorphism of $\mathrm{End}(V)$ that fixes each of A, A^* we can take $\psi = \dagger^{-1}$ in our above argument to get $\dagger^{-1} = \dagger$. This gives $X^{\dagger\dagger} = X$ for all $X \in \mathrm{End}(V)$. $\qquad\square$

11   TD pairs of Krawtchouk type; the dual space

In this section we show that each TD pair of Krawtchouk type is isomorphic to its dual TD pair in the sense of [3, Theorem 1.2].

Theorem 11.1 *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a TD pair on V that has Krawtchouk type. Let B (resp. B^*) denote the image of A (resp. A^*) under the canonical $\mathbb{F}$-algebra anti-isomorphism $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$, where $\tilde{V}$ is the vector space dual to V. Then the TD pairs A, A^* and B, B^* are isomorphic.*

Proof: By Theorem 10.2 the map † is an antiautomorphism of $\mathrm{End}(V)$ that fixes each of A, A^*. By construction the canonical $\mathbb{F}$-algebra anti-isomorphism $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ sends A to B and A^* to B^*. The composition of these two maps is an $\mathbb{F}$-algebra isomorphism $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ that sends A to B and A^* to B^*. The result follows in view of Corollary 2.4. $\qquad\square$

12 TD pairs of Krawtchouk type; some isomorphisms

In this section we display some isomorphisms between TD pairs of Krawtchouk type.

Theorem 12.1 *Let A, A^* denote a TD pair over $\mathbb{F}$ that has Krawtchouk type. Then the following TD pairs are mutually isomorphic:*

$$A, A^*; \qquad -A, -A^*; \qquad A^*, A; \qquad -A^*, -A.$$

Proof: We first show that the TD pairs A, A^* and $-A, -A^*$ are isomorphic. Let V denote the vector space underlying A, A^*. By Theorem 7.2 there exists a $\boxtimes$-module structure on V such that the generators x_{01}, x_{23} act as A, A^* respectively. Consider the element $\sigma = (01)(23)$ in G. By Theorem 6.4 the $\boxtimes$-module V is isomorphic to the $\boxtimes$-module V twisted via σ. Let $\gamma : V \to V$ denote an isomorphism of $\boxtimes$-modules from V to V twisted via σ. By the definition of twisting and since $\sigma^2 = 1$,

$$\gamma \xi.v = \sigma(\xi).\gamma v \qquad \xi \in \boxtimes, \quad v \in V.$$

We show that γ is an isomorphism of TD pairs from A, A^* to $-A, -A^*$. By construction γ is an isomorphism of vector spaces from V to V. For $v \in V$,

$$
\begin{aligned}
\gamma A v &= \gamma x_{01}.v \\
&= \sigma(x_{01}).\gamma v \\
&= x_{10}.\gamma v \\
&= -x_{01}.\gamma v \\
&= -A \gamma v
\end{aligned}
$$

so $\gamma A = -A\gamma$. Similarly

$$
\begin{aligned}
\gamma A^* v &= \gamma x_{23}.v \\
&= \sigma(x_{23}).\gamma v \\
&= x_{32}.\gamma v \\
&= -x_{23}.\gamma v \\
&= -A^* \gamma v
\end{aligned}
$$

so $\gamma A^* = -A^* \gamma$. By the above comments and Definition 2.1 the map γ is an isomorphism of TD pairs from A, A^* to $-A, -A^*$. Therefore the TD pairs A, A^* and $-A, -A^*$ are isomorphic. The above argument with $\sigma = (02)(13)$ (resp. $\sigma = (03)(12)$) shows that the TD pairs A, A^* and A^*, A (resp. A, A^* and $-A^*, -A$) are isomorphic. $\qquad\square$

13 TD pairs of Krawtchouk type; the Drinfel'd polynomial

In this section we introduce a Drinfel'd polynomial for TD pairs of Krawtchouk type.

Definition 13.1 Let A, A^* denote a TD pair over $\mathbb{F}$ that has Krawtchouk type. We define a polynomial $P = P_{A,A^*}$ in $\mathbb{F}[\lambda]$ by

$$P = \sum_{i=0}^{d} \frac{(-1)^i \zeta_i \lambda^i}{(i!)^2 4^i}$$

where $\{\zeta_i\}_{i=0}^{d}$ is the split sequence for A, A^* associated with the standard ordering $\{d-2i\}_{i=0}^{d}$ (resp. $\{2i - d\}_{i=0}^{d}$) of the eigenvalues for A (resp. A^*). We call P the *Drinfel'd polynomial* of A, A^*.

Lemma 13.2 *Let V denote a finite-dimensional irreducible $\boxtimes$-module. Then the following coincide:*

(i) *the Drinfel'd polynomial P_V from line (10);*

(ii) *the Drinfel'd polynomial for the TD pair x_{01}, x_{23} on V, as in Definition 13.1.*

Proof: Let $A : V \to V$ (resp. $A^* : V \to V$) denote the action of x_{01} (resp. x_{23}) on V. By Theorem 7.1 the pair A, A^* is a TD pair on V that has Krawtchouk type. We show $P_V = P_{A,A^*}$. Let d denote the diameter of A, A^* and abbreviate $\theta_i = d - 2i$ and $\theta_i^* = 2i - d$ for $0 \leq i \leq d$. Let $\{U_i\}_{i=0}^{d}$ denote the split decomposition of V for A, A^* and with respect to $\{\theta_i\}_{i=0}^{d}$, $\{\theta_i^*\}_{i=0}^{d}$. By [16, Section 3] for $0 \leq i \leq d$ the space U_i is the eigenspace for x_{20} on V associated with the eigenvalue $2i - d$. In particular U_0 is the eigenspace for x_{20} on V associated with the eigenvalue $-d$. Therefore U_0 coincides with the space U from below (9). Using (9) we observe that for $0 \leq i \leq d$ the action of $2e^+$ (resp. $2e^-$) on U_i coincides with the restriction of $A - \theta_i I$ (resp. $A^* - \theta_i^* I$) to U_i. By this and (3), (4) we find $e^+ U_i \subseteq U_{i+1}$ and $e^- U_i \subseteq U_{i-1}$. By these comments the action of $4^i (e^-)^i (e^+)^i$ on U_0 coincides with the restriction of the linear transformation (5) to U_0. Therefore $4^i \vartheta_i = \zeta_i$ where ϑ_i is from above (10) and ζ_i is from below (5). Comparing (10) with Definition 13.1 we find $P_V = P_{A,A^*}$ and the result follows. $\qquad \square$

Theorem 13.3 *The map $A, A^* \mapsto P_{A,A^*}$ induces a bijection between the following two sets:*

(i) *the isomorphism classes of TD pairs over $\mathbb{F}$ that have Krawtchouk type;*

(ii) *the polynomials in $\mathbb{F}[\lambda]$ that have constant coefficient 1 and are nonzero at $\lambda = 1$.*

Proof: The composition of the bijection in Remark 7.3 and the bijection in Theorem 6.5 is a bijection from set (i) to set (ii) above. The map $A, A^* \mapsto P_{A,A^*}$ induces this bijection in view of Lemma 13.2. $\qquad \square$

14 Directions for further research

In Sections 8–13 we obtained some results that apply to TD pairs of Krawtchouk type. In this section we consider whether these results apply to more general TD pairs. We also consider some formulae involving the split sequence. We content ourselves with a list of conjectures and problems.

Throughout this section the field $\mathbb{F}$ is arbitrary.

Conjecture 14.1 *Let* $\{\rho_i\}_{i=0}^d$ *denote the shape of a tridiagonal pair over* $\mathbb{F}$*. Then there exists a nonnegative integer* N *and positive integers* $d_1, d_2, \ldots, d_N$ *such that*

$$\sum_{i=0}^d \rho_i \lambda^i = \rho_0 \prod_{j=1}^N (1 + \lambda + \lambda^2 + \cdots + \lambda^{d_j}).$$

Moreover if $\mathbb{F}$ *is algebraically closed then* $\rho_0 = 1$*.*

Conjecture 14.2 *Assume* $\mathbb{F}$ *is algebraically closed. Let* V *denote a vector space over* $\mathbb{F}$ *with finite positive dimension and let* A, A^* *denote a TD pair on* V*. Then there exists a nonzero bilinear form* $\langle \, , \, \rangle$ *on* V *such that both*

$$\langle Au, v \rangle = \langle u, Av \rangle, \qquad \langle A^*u, v \rangle = \langle u, A^*v \rangle$$

for $u, v \in V$*. This form is unique up to multiplication by a nonzero scalar in* $\mathbb{F}$*. This form is nondegenerate and symmetric.*

Conjecture 14.3 *Assume* $\mathbb{F}$ *is algebraically closed. Let* V *denote a vector space over* $\mathbb{F}$ *with finite positive dimension and let* A, A^* *denote a TD pair on* V*. Then there exists a unique antiautomorphism* $\dagger$ *of* $\mathrm{End}(V)$ *that fixes each of* A, A^**. Moreover* $X^{\dagger\dagger} = X$ *for all* $X \in \mathrm{End}(V)$*.*

Conjecture 14.4 *Assume* $\mathbb{F}$ *is algebraically closed. Let* V *denote a vector space over* $\mathbb{F}$ *with finite positive dimension and let* A, A^* *denote a TD pair on* V*. Let* B *(resp.* B^**) denote the image of* A *(resp.* A^**) under the canonical* $\mathbb{F}$*-algebra anti-isomorphism* $\mathrm{End}(V) \to \mathrm{End}(\tilde{V})$*, where* $\tilde{V}$ *is the vector space dual to* V*. Then the TD pairs* A, A^* *and* B, B^* *are isomorphic.*

In order to state the next conjecture we make a definition.

Definition 14.5 Let A, A^* denote a TD pair and assume the shape vector satisfies $\rho_0 = 1$. By a *parameter array* of A, A^* we mean a sequence $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is a standard ordering of the eigenvalues of A (resp. A^*) and $\{\zeta_i\}_{i=0}^d$ is the split sequence of A, A^* with respect to $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$.

Conjecture 14.6 *Assume* $\mathbb{F}$ *is algebraically closed. Let* d *denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \tag{13}$$

denote a sequence of scalars taken from $\mathbb{F}$*. Then there exists a TD pair* A, A^* *over* $\mathbb{F}$ *with parameter array (13) if and only if (i)–(iii) hold below.*

(i) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \zeta_i (\theta_0 - \theta_{i+1}) \cdots (\theta_0 - \theta_d)(\theta_0^* - \theta_{i+1}^*) \cdots (\theta_0^* - \theta_d^*);$$

(ii) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$;

(iii) *the expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d - 1$.

Suppose (i)–(iii) hold. Then A, A^* is unique up to isomorphism of TD pairs.

As a first step in the proof of Conjecture 14.6, try to prove the following conjecture.

Conjecture 14.7 *Assume $\mathbb{F}$ is algebraically closed. Then two TD pairs over $\mathbb{F}$ are isomorphic if and only if they have a parameter array in common.*

Conjecture 14.8 *Let A, A^* denote a TD pair. Assume the shape vector satisfies $\rho_0 = 1$ and let $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ denote a parameter array of A, A^*. Then for $0 \leq i \leq d$ both*

$$\zeta_i = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*) \mathrm{tr}(\tau_i(A) E_0^*),$$
$$\zeta_i = (\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i) \mathrm{tr}(\tau_i^*(A^*) E_0)$$

where E_0 (resp. E_0^*) is the primitive idempotent of A (resp. A^*) for θ_0 (resp. θ_0^*) [10, Section 2] and

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}),$$
$$\tau_i^* = (\lambda - \theta_0^*)(\lambda - \theta_1^*) \cdots (\lambda - \theta_{i-1}^*).$$

Conjecture 14.9 *With the notation and assumptions of Conjecture 14.8 we have $\mathrm{tr}(E_0 E_0^*) \neq 0$. Moreover ζ_i is equal to each of*

$$\frac{\mathrm{tr}(E_0 \tau_i^*(A^*) \tau_i(A) E_0^*)}{\mathrm{tr}(E_0 E_0^*)}, \qquad \frac{\mathrm{tr}(E_0^* \tau_i(A) \tau_i^*(A^*) E_0)}{\mathrm{tr}(E_0^* E_0)}$$

for $0 \leq i \leq d$.

Problem 14.10 *Let A, A^* denote a TD pair. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) denote a standard ordering of the eigenvalues of A (resp. A^*). Assume the shape vector satisfies $\rho_0 = 1$. Find the algebraic relationships between the following eight sequences.*

(i) The split sequences of A, A^* with respect to

$$\{\theta_i\}_{i=0}^d \text{ and } \{\theta_i^*\}_{i=0}^d; \qquad \{\theta_{d-i}\}_{i=0}^d \text{ and } \{\theta_i^*\}_{i=0}^d;$$
$$\{\theta_i\}_{i=0}^d \text{ and } \{\theta_{d-i}^*\}_{i=0}^d; \qquad \{\theta_{d-i}\}_{i=0}^d \text{ and } \{\theta_{d-i}^*\}_{i=0}^d;$$

(ii) The split sequences of A^*, A with respect to

$$\{\theta_i^*\}_{i=0}^d \text{ and } \{\theta_i\}_{i=0}^d; \qquad\qquad \{\theta_{d-i}^*\}_{i=0}^d \text{ and } \{\theta_i\}_{i=0}^d;$$
$$\{\theta_i^*\}_{i=0}^d \text{ and } \{\theta_{d-i}\}_{i=0}^d; \qquad\qquad \{\theta_{d-i}^*\}_{i=0}^d \text{ and } \{\theta_{d-i}\}_{i=0}^d.$$

Conjecture 14.11 *Let* A, A^* *denote a TD pair. Let* $\{\theta_i\}_{i=0}^d$ *(resp.* $\{\theta_i^*\}_{i=0}^d$*) denote a standard ordering of the eigenvalues of* A *(resp.* A^**). Assume the shape vector satisfies* $\rho_0 = 1$. *Then the following coincide:*

(i) *The split sequence of* A, A^* *with respect to* $\{\theta_i\}_{i=0}^d$ *and* $\{\theta_i^*\}_{i=0}^d$;

(ii) *The split sequence of* A^*, A *with respect to* $\{\theta_i^*\}_{i=0}^d$ *and* $\{\theta_i\}_{i=0}^d$.

Problem 14.12 *Referring to Proposition 2.3 and Corollary 2.5, what conclusions can we obtain if we drop the assumption that* $\mathbb{F}$ *is algebraically closed?*

Problem 14.13 *Referring to Theorem 12.1, let* B, B^* *denote one of the TD pairs listed in that theorem, other than* A, A^*. *Let* γ *denote an isomorphism of TD pairs from* A, A^* *to* B, B^*. *Express* γ *explicitly as a polynomial in* A, A^*.

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