The quantum algebra Uq(sl2) and its equitable presentation

The quantum algebra $U_q(\mathfrak{sl}_2)$ and its equitable presentation*

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Abstract

We show that the quantum algebra $U_q(\mathfrak{sl}_2)$ has a presentation with generators $x^{\pm 1}, y, z$ and relations $xx^{-1} = x^{-1}x = 1$,

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We call this the *equitable* presentation. We show that y (resp. z) is not invertible in $U_q(\mathfrak{sl}_2)$ by displaying an infinite dimensional $U_q(\mathfrak{sl}_2)$-module that contains a nonzero null vector for y (resp. z). We consider finite dimensional $U_q(\mathfrak{sl}_2)$-modules under the assumption that q is not a root of 1 and $\mathrm{char}(\mathbb{K}) \neq 2$, where $\mathbb{K}$ is the underlying field. We show that y and z are invertible on each finite dimensional $U_q(\mathfrak{sl}_2)$-module. We display a linear operator Ω that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules, and satisfies

$$\Omega^{-1} x\, \Omega = y, \qquad \Omega^{-1} y\, \Omega = z, \qquad \Omega^{-1} z\, \Omega = x$$

on these modules. We define Ω using the q-exponential function.

1 The algebra $U_q(\mathfrak{sl}_2)$

Let $\mathbb{K}$ denote a field and let q denote a nonzero scalar in $\mathbb{K}$ such that $q^2 \neq 1$. For an integer n we define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and for $n \geq 0$ we define

$$[n]^! = [n][n-1]\cdots[2][1].$$

We interpret $[0]^! = 1$. We now recall the quantum algebra $U_q(\mathfrak{sl}_2)$.

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Definition 1.1 We let $U_q(\mathfrak{sl}_2)$ denote the unital associative $\mathbb{K}$-algebra with generators $k^{\pm 1}$,e, f and the following relations:

$$\begin{aligned} kk^{-1} &= k^{-1}k = 1, \\ ke &= q^2 ek, \\ kf &= q^{-2} fk, \\ ef - fe &= \frac{k - k^{-1}}{q - q^{-1}}. \end{aligned}$$

We call $k^{\pm 1}, e, f$ the *Chevalley generators* for $U_q(\mathfrak{sl}_2)$.

We refer the reader to [20], [21] for background information on $U_q(\mathfrak{sl}_2)$. We will generally follow the notational conventions of [20].

2 The equitable presentation for $U_q(\mathfrak{sl}_2)$

In the presentation for $U_q(\mathfrak{sl}_2)$ given in Definition 1.1 the generators $k^{\pm 1}$ and the generators e, f play a very different role. We now introduce a presentation for $U_q(\mathfrak{sl}_2)$ whose generators are on a more equal footing.

Theorem 2.1 *The algebra $U_q(\mathfrak{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $x^{\pm 1}$, y, z and the following relations:*

$$xx^{-1} = x^{-1}x = 1, \tag{1}$$

$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \tag{2}$$

$$\frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \tag{3}$$

$$\frac{qzx - q^{-1}xz}{q - q^{-1}} = 1. \tag{4}$$

An isomorphism with the presentation in Definition 1.1 is given by:

$$\begin{aligned} x^{\pm 1} &\rightarrow k^{\pm 1}, \\ y &\rightarrow k^{-1} + f(q - q^{-1}), \\ z &\rightarrow k^{-1} - k^{-1}eq(q - q^{-1}). \end{aligned}$$

The inverse of this isomorphism is given by:

$$\begin{aligned} k^{\pm 1} &\rightarrow x^{\pm 1}, \\ f &\rightarrow (y - x^{-1})(q - q^{-1})^{-1}, \\ e &\rightarrow (1 - xz)q^{-1}(q - q^{-1})^{-1}. \end{aligned}$$

Proof: One readily checks that each map is a homomorphism of $\mathbb{K}$-algebras and that the maps are inverses. It follows that each map is an isomorphism of $\mathbb{K}$-algebras. $\square$

The generators $x^{\pm 1}$, y, z from Theorem 2.1 are on an equal footing, more or less. In view of this we make a definition.

Definition 2.2 By the *equitable presentation* for $U_q(\mathfrak{sl}_2)$ we mean the presentation given in Theorem 2.1. We call $x^{\pm 1}, y, z$ the *equitable generators.*

We remark that the isomorphism given in Theorem 2.1 is not unique. This is a consequence of the following lemma.

Lemma 2.3 *For an integer i and nonzero $\alpha \in \mathbb{K}$ there exists a $\mathbb{K}$-algebra automorphism of $U_q(\mathfrak{sl}_2)$ that satisfies*

$$k^{\pm 1} \to k^{\pm 1}, \qquad e \to \alpha e k^i, \qquad f \to \alpha^{-1} k^{-i} f.$$

Proof: Routine. $\square$

3 The elements y and z are not invertible in $U_q(\mathfrak{sl}_2)$

In this section we show that the equitable generators y and z are not invertible in $U_q(\mathfrak{sl}_2)$. In order to show that y (resp. z) is not invertible in $U_q(\mathfrak{sl}_2)$ we display an infinite dimensional $U_q(\mathfrak{sl}_2)$-module that contains a nonzero null vector for y (resp. z).

Lemma 3.1 *There exists a $U_q(\mathfrak{sl}_2)$-module Γ_y with the following property: Γ_y has a basis*

$$u_{ij} \qquad i, j \in \mathbb{Z}, \quad j \geq 0$$

such that

$$\begin{aligned} x u_{ij} &= u_{i+1,j}, \\ x^{-1} u_{ij} &= u_{i-1,j}, \\ y u_{ij} &= q^{2i-j}(q^j - q^{-j}) u_{i,j-1} - q^i (q^i - q^{-i}) u_{i-1,j}, \\ z u_{ij} &= q^{-2i} u_{i,j+1} + q^{-i}(q^i - q^{-i}) u_{i-1,j} \end{aligned}$$

for all $i, j \in \mathbb{Z}$ with $j \geq 0$. In the above equations $u_{r,-1} := 0$ for $r \in \mathbb{Z}$.

Proof: One routinely verifies that the given actions of $x^{\pm 1}, y, z$ satisfy the relations (1)–(4). $\square$

Lemma 3.2 *The following* (i)–(iii) *hold.*

(i) $y u_{00} = 0$, *where the vector u_{00} is from Lemma 3.1.*

(ii) *y is not invertible on Γ_y, where Γ_y is the $U_q(\mathfrak{sl}_2)$-module from Lemma 3.1.*

(iii) *y is not invertible in $U_q(\mathfrak{sl}_2)$.*

Proof: Immediate. $\square$

Remark 3.3 Referring to Lemma 3.1, we have $u_{ij} = x^i z^j u_{00}$ for $i, j \in \mathbb{Z}$, $j \geq 0$.

Lemma 3.4 *There exists a $U_q(\mathfrak{sl}_2)$-module Γ_z with the following property: Γ_z has a basis*

$$v_{ij} \qquad i, j \in \mathbb{Z}, \quad j \geq 0$$

such that

$$\begin{aligned}
xv_{ij} &= v_{i+1,j}, \\
x^{-1}v_{ij} &= v_{i-1,j}, \\
yv_{ij} &= q^{2i}v_{i,j+1} - q^i(q^i - q^{-i})v_{i-1,j}, \\
zv_{ij} &= q^{-i}(q^i - q^{-i})v_{i-1,j} - q^{j-2i}(q^j - q^{-j})v_{i,j-1}
\end{aligned}$$

for all $i, j \in \mathbb{Z}$ with $j \geq 0$. In the above equations $v_{r,-1} := 0$ for $r \in \mathbb{Z}$.

Proof: One routinely verifies that the given actions of $x^{\pm 1}, y, z$ satisfy the relations (1)–(4). $\square$

Lemma 3.5 *The following* (i)–(iii) *hold.*

(i) *$zv_{00} = 0$, where the vector v_{00} is from Lemma 3.4.*

(ii) *z is not invertible on Γ_z, where Γ_z is the $U_q(\mathfrak{sl}_2)$-module from Lemma 3.4.*

(iii) *z is not invertible in $U_q(\mathfrak{sl}_2)$.*

Proof: Immediate. $\square$

Remark 3.6 Referring to Lemma 3.4, we have $v_{ij} = x^i y^j v_{00}$ for $i, j \in \mathbb{Z}$, $j \geq 0$.

4 Finite dimensional $U_q(\mathfrak{sl}_2)$-modules

From now on we restrict our attention to finite dimensional $U_q(\mathfrak{sl}_2)$-modules. In order to simplify things we make the following assumption.

For the rest of this paper, we assume q is not a root of 1, and that $\mathrm{char}(\mathbb{K}) \neq 2$.

In this section we show that the equitable generators y and z are invertible on each finite dimensional $U_q(\mathfrak{sl}_2)$-module.

We begin with some general comments. By [20, Theorems 2.3, 2.9] each finite dimensional $U_q(\mathfrak{sl}_2)$-module M is semi-simple; this means that M is a direct sum of simple $U_q(\mathfrak{sl}_2)$-modules. The finite dimensional simple $U_q(\mathfrak{sl}_2)$-modules are described as follows.

Lemma 4.1 *[20, Theorem 2.6] There exists a family of finite dimensional simple $U_q(\mathfrak{sl}_2)$-modules*

$$L(n,\varepsilon) \qquad \varepsilon \in \{1,-1\}, \qquad n=0,1,2,\ldots \tag{5}$$

with the following properties: $L(n,\varepsilon)$ has a basis $v_0, v_1, \ldots, v_n$ such that $kv_i = \varepsilon q^{n-2i} v_i$ for $0 \le i \le n$, $fv_i = [i+1]v_{i+1}$ for $0 \le i \le n-1$, $fv_n = 0$, $ev_i = \varepsilon[n-i+1]v_{i-1}$ for $1 \le i \le n$, $ev_0 = 0$. Every finite dimensional simple $U_q(\mathfrak{sl}_2)$-module is isomorphic to exactly one of the modules (5).

The equitable generators act on the modules $L(n,\varepsilon)$ as follows.

Lemma 4.2 *For an integer $n \ge 0$ and for $\varepsilon \in \{1,-1\}$, the $U_q(\mathfrak{sl}_2)$-module $L(n,\varepsilon)$ has a basis $u_0, u_1, \ldots, u_n$ such that*

$$\begin{aligned} \varepsilon x u_i &= q^{n-2i} u_i \qquad (0 \le i \le n), & (6)\\ (\varepsilon y - q^{2i-n}) u_i &= (q^{-n} - q^{2i+2-n}) u_{i+1} \qquad (0 \le i \le n-1), \qquad (\varepsilon y - q^n) u_n = 0, & (7)\\ (\varepsilon z - q^{2i-n}) u_i &= (q^{n} - q^{2i-2-n}) u_{i-1} \qquad (1 \le i \le n), \qquad (\varepsilon z - q^{-n}) u_0 = 0. & (8) \end{aligned}$$

Proof: For the purpose of this proof we identify the copy of $U_q(\mathfrak{sl}_2)$ given in Definition 1.1 with the copy given in Theorem 2.1, via the isomorphism in Theorem 2.1. Let the basis $v_0, v_1, \ldots, v_n$ for $L(n,\varepsilon)$ be as in Lemma 4.1. Define $u_i = \gamma_i v_i$ for $0 \le i \le n$, where $\gamma_0 = 1$ and $\gamma_i = -\varepsilon q^{n-i}\gamma_{i-1}$ for $1 \le i \le n$. Using $x = k$, $y = k^{-1} + f(q-q^{-1})$, and $z = k^{-1} - k^{-1}eq(q-q^{-1})$, together with the data in Lemma 4.1, we routinely verify (6)–(8). □

Note 4.3 The basis $u_0, u_1, \ldots, u_n$ in Lemma 4.2 is normalized so that $yu = \varepsilon q^{-n} u$ and $zu = \varepsilon q^n u$ for $u = \sum_{i=0}^n u_i$.

Corollary 4.4 *For an integer $n \ge 0$ and for $\varepsilon \in \{1,-1\}$, the following* (i), (ii) *hold on the $U_q(\mathfrak{sl}_2)$-module $L(n,\varepsilon)$.*

(i) *Each of x, y, z is semi-simple with eigenvalues $\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n}$.*

(ii) *Each of x, y, z is invertible.*

Proof: For x this is clear from (6). We now verify our assertions for y. With respect to the basis $u_0, u_1, \ldots, u_n$ for $L(n,\varepsilon)$ given in Lemma 4.2, by (7) the matrix representing y is lower triangular with (i,i) entry εq^{2i-n} for $0 \le i \le n$. Therefore the action of y on $L(n,\varepsilon)$ has eigenvalues $\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n}$. These eigenvalues are mutually distinct so this action is semi-simple. These eigenvalues are nonzero so this action is invertible. We have now verified our assertions for y. Our assertions for z are similarly verified. □

By Corollary 4.4 and since each finite dimensional $U_q(\mathfrak{sl}_2)$-module is semi-simple we obtain the following result.

Corollary 4.5 *On each finite dimensional $U_q(\mathfrak{sl}_2)$-module the actions of y and z are invertible.*

Motivated by Corollary 4.5 we make the following definition.

Definition 4.6 We let y^{-1} (resp. z^{-1}) denote the linear operator that acts on each finite dimensional $U_q(\mathfrak{sl}_2)$-module as the inverse of y (resp. z).

5 The elements n_x, n_y, n_z

In this section we define some elements n_x, n_y, n_z of $U_q(\mathfrak{sl}_2)$ and show that these are nilpotent on each finite dimensional $U_q(\mathfrak{sl}_2)$-module. We then recall the q-exponential function $\exp_q$ and derive a number of equations involving $\exp_q(n_x), \exp_q(n_y), \exp_q(n_z)$. These equations will show that on finite dimensional $U_q(\mathfrak{sl}_2)$-modules the operators y^{-1}, z^{-1} from Definition 4.6 satisfy

$$y^{-1} = \exp_q(n_z)\, x \exp_q(n_z)^{-1}, \tag{9}$$
$$z^{-1} = \exp_q(n_y)^{-1}\, x \exp_q(n_y). \tag{10}$$

We begin with an observation.

Lemma 5.1 *The equitable generators x, y, z of $U_q(\mathfrak{sl}_2)$ satisfy*

$$\begin{aligned} q(1-yz) &= q^{-1}(1-zy), \\ q(1-zx) &= q^{-1}(1-xz), \\ q(1-xy) &= q^{-1}(1-yx). \end{aligned}$$

Proof: These equations are reformulations of (2)–(4). □

Definition 5.2 We let n_x, n_y, n_z denote the following elements in $U_q(\mathfrak{sl}_2)$:

$$n_x = \frac{q(1-yz)}{q-q^{-1}} = \frac{q^{-1}(1-zy)}{q-q^{-1}}, \tag{11}$$
$$n_y = \frac{q(1-zx)}{q-q^{-1}} = \frac{q^{-1}(1-xz)}{q-q^{-1}}, \tag{12}$$
$$n_z = \frac{q(1-xy)}{q-q^{-1}} = \frac{q^{-1}(1-yx)}{q-q^{-1}}. \tag{13}$$

Note 5.3 Under the isomorphism given in Theorem 2.1 the preimage of n_y (resp. n_z) is e (resp. $-qkf$).

We recall some notation. Let V denote a finite dimensional vector space over $\mathbb{K}$. A linear transformation $T : V \to V$ is called *nilpotent* whenever there exists a positive integer r such that $T^r V = 0$.

We are going to show that each of n_x, n_y, n_z is nilpotent on all finite dimensional $U_q(\mathfrak{sl}_2)$-modules. We will show this using the following lemma.

Lemma 5.4 *The following relations hold in* $U_q(\mathfrak{sl}_2)$*:*

$$xn_y = q^2 n_y x, \qquad xn_z = q^{-2} n_z x, \tag{14}$$
$$yn_z = q^2 n_z y, \qquad yn_x = q^{-2} n_x y, \tag{15}$$
$$zn_x = q^2 n_x z, \qquad zn_y = q^{-2} n_y z. \tag{16}$$

Proof: In order to verify these equations, eliminate n_x, n_y, n_z using Definition 5.2 and simplify the result. □

Lemma 5.5 *Each of* n_x, n_y, n_z *is nilpotent on finite dimensional* $U_q(\mathfrak{sl}_2)$*-modules.*

Proof: We prove the result for n_x; the proof for n_y and n_z is similar. Since each finite dimensional $U_q(\mathfrak{sl}_2)$-module is semi-simple and in view of Lemma 4.1, it suffices to show that n_x is nilpotent on each module $L(n, \varepsilon)$. By Corollary 4.4(i), $L(n, \varepsilon)$ has a basis $w_0, w_1, \ldots, w_n$ such that $yw_i = \varepsilon q^{n-2i} w_i$ for $0 \le i \le n$. Using the equation on the right in (15) we routinely find that $n_x w_i$ is a scalar multiple of w_{i+1} for $0 \le i \le n-1$ and $n_x w_n = 0$. This shows that n_x is nilpotent on $L(n, \varepsilon)$ and the result follows. □

We now recall the q-exponential function.

Definition 5.6 *[31, p. 204]* Let T denote a linear operator that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules in a nilpotent fashion. We define

$$\exp_q(T) = \sum_{i=0}^{\infty} \frac{q^{i(i-1)/2}}{[i]!} T^i. \tag{17}$$

We view $\exp_q(T)$ as a linear operator that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules.

The following result is well known and easily verified.

Lemma 5.7 *[31, p. 204] Let* T *denote a linear operator that acts on finite dimensional* $U_q(\mathfrak{sl}_2)$*-modules in a nilpotent fashion. Then on each of these modules* $\exp_q(T)$ *is invertible; the inverse is*

$$\exp_{q^{-1}}(-T) = \sum_{i=0}^{\infty} \frac{(-1)^i q^{-i(i-1)/2}}{[i]!} T^i.$$

The following lemma is essentially a special case of [28, Eq. (4.8)]; we include a proof for the sake of completeness.

Lemma 5.8 *[28, Eq. (4.8)] The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_y)^{-1}\, x \exp_q(n_y) = z^{-1}$,

(ii) $\exp_q(n_z)^{-1}\, y \exp_q(n_z) = x^{-1}$,

(iii) $\exp_q(n_x)^{-1}\, z \exp_q(n_x) = y^{-1}$.

Proof: (i) We show

$$x \exp_q(n_y)\, z = \exp_q(n_y). \tag{18}$$

The left side of (18) is equal to $x \exp_q(n_y)\, x^{-1}xz$. Observe $x \exp_q(n_y)\, x^{-1} = \exp_q(xn_yx^{-1})$ by (17) and $xn_yx^{-1} = q^2 n_y$ by (14). Also $xz = 1 - q(q - q^{-1})n_y$ by (12). Using these comments and (17) we routinely find that the left side of (18) is equal to the right side of (18). The result follows.
(ii), (iii) Similar to the proof of (i) above. □

For convenience we display a second "version" of Lemma 5.8.

Lemma 5.9 *The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_z)\, x \exp_q(n_z)^{-1} = y^{-1}$,

(ii) $\exp_q(n_x)\, y \exp_q(n_x)^{-1} = z^{-1}$,

(iii) $\exp_q(n_y)\, z \exp_q(n_y)^{-1} = x^{-1}$.

Proof: For each of the equations in Lemma 5.8 take the inverse of each side and simplify the result. □

We note that the equations (9), (10) are just Lemma 5.9(i) and Lemma 5.8(i), respectively.

6 Some formulae involving the q-exponential function

In the next section we will display a linear operator Ω that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules, and satisfies $\Omega^{-1}x\,\Omega = y$, $\Omega^{-1}y\,\Omega = z$, $\Omega^{-1}z\,\Omega = x$ on these modules. In order to prove that Ω has the desired properties we will first establish a few identities. These identities are given in this section.

Lemma 6.1 *The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_z)^{-1}\, x \exp_q(n_z) = xyx$,

(ii) $\exp_q(n_x)^{-1}\, y \exp_q(n_x) = yzy$,

(iii) $\exp_q(n_y)^{-1}\, z \exp_q(n_y) = zxz$.

Proof: (i) The element xy commutes with n_z by (13) so xy commutes with $\exp_q(n_z)$ in view of (17). Therefore $\exp_q(n_z)^{-1}\, xy \exp_q(n_z) = xy$. By Lemma 5.8(ii) we have $y \exp_q(n_z) = \exp_q(n_z)\, x^{-1}$. Combining these last two equations we routinely obtain the result.
(ii), (iii) Similar to the proof of (i) above. □

Lemma 6.2 *The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_y)\, x \exp_q(n_y)^{-1} = xzx$,

(ii) $\exp_q(n_z)\, y \exp_q(n_z)^{-1} = yxy$,

(iii) $\exp_q(n_x)\, z \exp_q(n_x)^{-1} = zyz$.

Proof: (i) By Lemma 6.1(iii) we have $\exp_q(n_y)\, zxz \exp_q(n_y)^{-1} = z$. In this equation we eliminate $\exp_q(n_y)\, z$ and $z \exp_q(n_y)^{-1}$ using Lemma 5.9(iii). The result follows.
(ii), (iii) Similar to the proof of (i) above. □

Lemma 6.3 *The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_x)^{-1}\, x \exp_q(n_x) = x + y - y^{-1}$,

(ii) $\exp_q(n_y)^{-1}\, y \exp_q(n_y) = y + z - z^{-1}$,

(iii) $\exp_q(n_z)^{-1}\, z \exp_q(n_z) = z + x - x^{-1}$.

Proof: (i) Using (2), (4) and (11) we obtain $xn_x - n_x x = y - z$. By this and a routine induction using (15), (16) we find

$$xn_x^i - n_x^i x = q^{1-i}[i](n_x^{i-1}y - zn_x^{i-1}) \tag{19}$$

for each integer $i \geq 0$. Using (17) and (19) we obtain

$$x \exp_q(n_x) - \exp_q(n_x)\, x = \exp_q(n_x)\, y - z \exp_q(n_x). \tag{20}$$

In line (20) we multiply each term on the left by $\exp_q(n_x)^{-1}$ and evaluate the term containing z using Lemma 5.8(iii) to get the result.
(ii), (iii) Similar to the proof of (i) above. □

Lemma 6.4 *The following* (i)–(iii) *hold on each finite dimensional* $U_q(\mathfrak{sl}_2)$*-module.*

(i) $\exp_q(n_x)\, x \exp_q(n_x)^{-1} = x + z - z^{-1}$,

(ii) $\exp_q(n_y)\, y \exp_q(n_y)^{-1} = y + x - x^{-1}$,

(iii) $\exp_q(n_z)\, z \exp_q(n_z)^{-1} = z + y - y^{-1}$.

Proof: (i) By Lemma 6.3(i) we have

$$x = \exp_q(n_x)\,(x + y - y^{-1}) \exp_q(n_x)^{-1}. \tag{21}$$

By Lemma 5.9(ii) we have $\exp_q(n_x)\, y \exp_q(n_x)^{-1} = z^{-1}$ and $\exp_q(n_x)\, y^{-1} \exp_q(n_x)^{-1} = z$. Evaluating (21) using these comments we obtain the result.
(ii), (iii) Similar to the proof of (i) above. □

7 The operator Ω

In this section we display a linear operator Ω that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules, and satisfies $\Omega^{-1}x\,\Omega = y$, $\Omega^{-1}y\,\Omega = z$, $\Omega^{-1}z\,\Omega = x$ on these modules. In order to define Ω we first recall the notion of a *weight space*.

Definition 7.1 Let M denote a finite dimensional $U_q(\mathfrak{sl}_2)$-module. For an integer λ and for $\varepsilon \in \{1, -1\}$ define

$$M(\varepsilon, \lambda) = \{v \in M \mid xv = \varepsilon q^{\lambda} v\}.$$

We call $M(\varepsilon, \lambda)$ the (ε, λ)-*weight space* of M with respect to x. By Corollary 4.4(i) and since M is semi-simple, M is the direct sum of its weight spaces with respect to x.

Definition 7.2 We define a linear operator Ψ that acts on each finite dimensional $U_q(\mathfrak{sl}_2)$-module M. In order to do this we give the action of Ψ on each weight space of M with respect to x. For an integer λ and for $\varepsilon \in \{1, -1\}$, Ψ acts on the weight space $M(\varepsilon, \lambda)$ as $q^{-\lambda^2/2}I$ (if λ is even) and $q^{(1-\lambda^2)/2}I$ (if λ is odd), where I denotes the identity map. We observe that Ψ is invertible on M.

Lemma 7.3 *For the operator Ψ from Definition 7.2 the following* (i)–(iii) *hold on each finite dimensional $U_q(\mathfrak{sl}_2)$-module.*

(i) $\Psi^{-1}x\Psi = x$,

(ii) $\Psi^{-1}n_y\Psi = xn_yx$,

(iii) $\Psi^{-1}n_z\Psi = x^{-1}n_zx^{-1}$.

Proof: Let M denote a finite dimensional $U_q(\mathfrak{sl}_2)$-module. For an integer λ and for $\varepsilon \in \{1, -1\}$ we show that each of (i)–(iii) holds on $M(\varepsilon, \lambda)$.
(i) On $M(\varepsilon, \lambda)$ each of Ψ, x acts as a scalar multiple of the identity.
(ii) For notational convenience define $s = 0$ (if λ is even) and $s = 1$ (if λ is odd). For $v \in M(\varepsilon, \lambda)$ we show $\Psi^{-1}n_y\Psi v = xn_yxv$. Using the equation on the left in (14) we find $n_yv \in M(\varepsilon, \lambda + 2)$. Using this we find

$$\begin{aligned}
\Psi^{-1}n_y\Psi v &= q^{(s-\lambda^2)/2}\Psi^{-1}n_yv \\
&= q^{(s-\lambda^2)/2}q^{((\lambda+2)^2-s)/2}n_yv \\
&= q^{2\lambda+2}n_yv
\end{aligned}$$

and also

$$\begin{aligned}
xn_yxv &= \varepsilon q^{\lambda}xn_yv \\
&= \varepsilon q^{\lambda}\varepsilon q^{\lambda+2}n_yv \\
&= q^{2\lambda+2}n_yv.
\end{aligned}$$

Therefore $\Psi^{-1}n_y\Psi v = xn_yxv$. We have now shown $\Psi^{-1}n_y\Psi$ and xn_yx coincide on $M(\varepsilon, \lambda)$.
(iii) Similar to the proof of (ii) above. □

Definition 7.4 We define

$$\Omega = \exp_q(n_z)\,\Psi\,\exp_q(n_y), \tag{22}$$

where n_y, n_z are from Definition 5.2 and where Ψ is from Definition 7.2. We view Ω as a linear operator that acts on finite dimensional $U_q(\mathfrak{sl}_2)$-modules.

We now present our main result.

Theorem 7.5 *For the operator Ω from Definition 7.4 the following hold on each finite dimensional $U_q(\mathfrak{sl}_2)$-module:*

$$\Omega^{-1}x\,\Omega = y, \qquad \Omega^{-1}y\,\Omega = z, \qquad \Omega^{-1}z\,\Omega = x.$$

Proof: Observe

$$\begin{aligned}
\Omega^{-1}x\,\Omega &= \exp_q(n_y)^{-1}\,\Psi^{-1}\exp_q(n_z)^{-1}\,x\exp_q(n_z)\,\Psi\exp_q(n_y) \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}xyx\Psi\exp_q(n_y) && \text{(by Lemma 6.1(i))} \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}xy\Psi x\exp_q(n_y) && \text{(by Lemma 7.3(i))} \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}(1-q^{-1}(q-q^{-1})n_z)\Psi x\exp_q(n_y) && \text{(by (13))} \\
&= \exp_q(n_y)^{-1}\,(x-q^{-1}(q-q^{-1})x^{-1}n_z)\exp_q(n_y) && \text{(by Lemma 7.3(iii))} \\
&= \exp_q(n_y)^{-1}\,(x-x^{-1}+y)\exp_q(n_y) && \text{(by (13))} \\
&= y && \text{(by Lemma 6.4(ii))}
\end{aligned}$$

and

$$\begin{aligned}
\Omega^{-1}y\,\Omega &= \exp_q(n_y)^{-1}\,\Psi^{-1}\exp_q(n_z)^{-1}\,y\exp_q(n_z)\,\Psi\exp_q(n_y) \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}x^{-1}\Psi\exp_q(n_y) && \text{(by Lemma 5.8(ii))} \\
&= \exp_q(n_y)^{-1}\,x^{-1}\exp_q(n_y) && \text{(by Lemma 7.3(i))} \\
&= z && \text{(by Lemma 5.8(i))}
\end{aligned}$$

and

$$\begin{aligned}
\Omega^{-1}z\,\Omega &= \exp_q(n_y)^{-1}\,\Psi^{-1}\exp_q(n_z)^{-1}\,z\exp_q(n_z)\,\Psi\exp_q(n_y) \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}(z+x-x^{-1})\Psi\exp_q(n_y) && \text{(by Lemma 6.3(iii))} \\
&= \exp_q(n_y)^{-1}\,\Psi^{-1}(x-q^{-1}(q-q^{-1})n_yx^{-1})\Psi\exp_q(n_y) && \text{(by (12))} \\
&= \exp_q(n_y)^{-1}\,(x-q^{-1}(q-q^{-1})\Psi^{-1}n_y\Psi x^{-1})\exp_q(n_y) && \text{(by Lemma 7.3(i))} \\
&= \exp_q(n_y)^{-1}\,(x-q^{-1}(q-q^{-1})xn_y)\exp_q(n_y) && \text{(by Lemma 7.3(ii))} \\
&= \exp_q(n_y)^{-1}\,xzx\exp_q(n_y) && \text{(by (12))} \\
&= x && \text{(by Lemma 6.2(i)).}
\end{aligned}$$

□

We finish this section with a comment.

Corollary 7.6 *On a finite dimensional $U_q(\mathfrak{sl}_2)$-module, Ω^3 commutes with the action of each element of $U_q(\mathfrak{sl}_2)$.*

Proof: Immediate from Theorem 7.5 and since $x^{\pm 1}, y, z$ generate $U_q(\mathfrak{sl}_2)$. □

8 The action of Ω on $L(n, \varepsilon)$

In this section we describe the action of Ω on the module $L(n, \varepsilon)$. We will do this by displaying the action of Ω and Ω^{-1} on the basis for $L(n, \varepsilon)$ given in Lemma 4.2. We begin with a few observations.

Lemma 8.1 *For an integer $n \geq 0$ and for $\varepsilon \in \{1, -1\}$ let $u_0, u_1, \ldots, u_n$ denote the basis for $L(n, \varepsilon)$ given in Lemma 4.2. Then $\Psi u_i = q^{2i(n-i)+(s-n^2)/2} u_i$ for $0 \leq i \leq n$, where $s = 0$ (if n is even) and $s = 1$ (if n is odd).*

Proof: Immediate from Definition 7.2. □

Lemma 8.2 *For an integer $n \geq 0$ and for $\varepsilon \in \{1, -1\}$ let $u_0, u_1, \ldots, u_n$ denote the basis for $L(n, \varepsilon)$ given in Lemma 4.2. Then the following* (i), (ii) *hold.*

(i) $n_y u_i = -q^{n-i}[n-i+1] u_{i-1}$ $(1 \leq i \leq n)$, $\quad n_y u_0 = 0$.

(ii) $n_z u_i = q^{-i}[i+1] u_{i+1}$ $(0 \leq i \leq n-1)$, $\quad n_z u_n = 0$.

Proof: Use Lemma 4.2 and Definition 5.2. □

We recall some notation. For integers $n \geq i \geq 0$ we define

$$\begin{bmatrix} n \\ i \end{bmatrix} = \frac{[n]^!}{[i]^![n-i]^!}.$$

Lemma 8.3 *For an integer $n \geq 0$ and for $\varepsilon \in \{1, -1\}$ let $u_0, u_1, \ldots, u_n$ denote the basis for $L(n, \varepsilon)$ given in Lemma 4.2. Then for $0 \leq j \leq n$ we have*

$$\exp_q(n_y)\, u_j = \sum_{i=0}^{j} (-1)^{i+j} q^{(j-i)(n-i-1)} \begin{bmatrix} n-i \\ j-i \end{bmatrix} u_i, \qquad (23)$$

$$\exp_q(n_y)^{-1}\, u_j = \sum_{i=0}^{j} q^{(j-i)(n-j)} \begin{bmatrix} n-i \\ j-i \end{bmatrix} u_i, \qquad (24)$$

$$\exp_q(n_z)\, u_j = \sum_{i=j}^{n} q^{j(j-i)} \begin{bmatrix} i \\ j \end{bmatrix} u_i, \qquad (25)$$

$$\exp_q(n_z)^{-1}\, u_j = \sum_{i=j}^{n} (-1)^{i+j} q^{(j-i)(i-1)} \begin{bmatrix} i \\ j \end{bmatrix} u_i. \qquad (26)$$

Proof: In order to verify (23) and (25), evaluate the left-hand side using Lemma 8.2 and Definition 5.6. Lines (24) and (26) are similarly verified using Lemma 5.7. □

Theorem 8.4 *For an integer $n \geq 0$ and for $\varepsilon \in \{1, -1\}$ let $u_0, u_1, \ldots, u_n$ denote the basis for $L(n, \varepsilon)$ given in Lemma 4.2. Then for $0 \leq j \leq n$ we have*

$$\Omega u_j = \sum_{i=0}^{n-j} (-1)^j q^{(n-i-1)j+(s-n^2)/2} \begin{bmatrix} n-i \\ j \end{bmatrix} u_i, \qquad (27)$$

$$\Omega^{-1} u_j = \sum_{i=n-j}^{n} (-1)^{n-j} q^{(1-i)(n-j)+(n^2-s)/2} \begin{bmatrix} i \\ n-j \end{bmatrix} u_i, \qquad (28)$$

where $s = 0$ (if n is even) and $s = 1$ (if n is odd).

Proof: In order to verify (27), evaluate the left-hand side using (22), Lemma 8.1, (23), (25), and simplify the result using the q-Vandermonde summation formula [10, p. 11]. Line (28) is similarly verified. □

We finish this section with a comment.

Corollary 8.5 *For an integer $n \geq 0$ and for $\varepsilon \in \{1, -1\}$, Ω^3 acts as a scalar multiple of the identity on $L(n, \varepsilon)$. The scalar is $q^{-n(n+2)/2}$ (if n is even) and $-q^{(1-n)(n+3)/2}$ (if n is odd).*

Proof: Routine calculation using Theorem 8.4. □

9 Remarks

In this section we make some remarks and tie up some loose ends.

Remark 9.1 In [7] Fairlie considers an associative $\mathbb{K}$-algebra with generators X, Y, Z and relations

$$qXY - q^{-1}YX = Z, \qquad (29)$$

$$qYZ - q^{-1}ZY = X, \qquad (30)$$

$$qZX - q^{-1}XZ = Y. \qquad (31)$$

He interprets this algebra as a q-deformation of SU(2) and he works out the irreducible representations. See [5, Remark 8.11], [8, Section 3], [16], [25] for related work. In spite of the superficial resemblance we do not see any connection between (29)–(31) and the equitable presentation of $U_q(\mathfrak{sl}_2)$.

Remark 9.2 In [46] A. S. Zhedanov introduced the Askey-Wilson algebra. He used it to study the Askey-Wilson polynomials and related polynomials in the Askey scheme [22]. The following attractive version of the algebra appears in [27, p. 101], [30], [43, Section 3.3.3].

For a sequence of scalars $g_x, g_y, g_z, h_x, h_y, h_z$ taken from $\mathbb{K}$, the corresponding Askey-Wilson algebra is the unital associative $\mathbb{K}$-algebra with generators X, Y, Z and relations

$$qXY - q^{-1}YX = g_z Z + h_z, \tag{32}$$
$$qYZ - q^{-1}ZY = g_x X + h_x, \tag{33}$$
$$qZX - q^{-1}XZ = g_y Y + h_y. \tag{34}$$

See [11], [12], [13], [14], [15], [41], [47], [48] for work involving the Askey-Wilson algebra. We note that for $g_x = g_y = g_z = 1$ and $h_x = h_y = h_z = 0$ the relations (32)–(34) become (29)–(31). Moreover for $g_x = g_y = g_z = 0$ and $h_x = h_y = h_z = q - q^{-1}$ the relations (32)–(34) become (2)–(4). In this case, and referring to Theorem 2.1, the Askey-Wilson algebra is isomorphic to the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by x, y, z. As far as we know, this case of the Askey-Wilson algebra has not yet been considered by other researchers.

Remark 9.3 In the literature one can find many presentations of algebras that are related in some way to $\mathfrak{sl}_2$. See for example [1], [2], [3], [4], [6], [9], [23], [24], [26], [27, p. 48], [29], [42], [44], [45]. As far as we know, none of these has a direct connection to the equitable presentation of $U_q(\mathfrak{sl}_2)$.

Remark 9.4 The equitable presentation for $U_q(\mathfrak{sl}_2)$ appears implicitly as part of a presentation given in [19, Theorem 2.1] for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$.

Remark 9.5 In light of Theorem 2.1 and Corollary 4.5 it is natural to consider a unital associative $\mathbb{K}$-algebra that has generators $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$ and relations

$$xx^{-1} = x^{-1}x = 1, \qquad yy^{-1} = y^{-1}y = 1, \qquad zz^{-1} = z^{-1}z = 1,$$
$$\frac{qxy - q^{-1}yx}{q - q^{-1}} = 1, \qquad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1, \qquad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1.$$

We denote this algebra by $U_q^{\Delta}(\mathfrak{sl}_2)$ and call it the *equitable* q-deformation of $\mathfrak{sl}_2$. We invite the reader to investigate $U_q^{\Delta}(\mathfrak{sl}_2)$.

Remark 9.6 For a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$ the second author has obtained a presentation for the quantum group $U_q(\mathfrak{g})$ that is analogous to the equitable presentation for $U_q(\mathfrak{sl}_2)$. See [40] for the details.

Remark 9.7 We discovered the equitable presentation for $U_q(\mathfrak{sl}_2)$ during our recent study of tridiagonal pairs [17], [18], [19], and the closely related Leonard pairs [32], [33], [34], [35], [36], [37], [38], [39], [41]. A Leonard pair is a pair of semi-simple linear transformations on a finite-dimensional vector space, each of which acts tridiagonally on an eigenbasis for the other [32, Definition 1.1]. There is a close connection between Leonard pairs and the orthogonal polynomials that make up the terminating branch of the Askey scheme [22], [32, Appendix A], [38]. A tridiagonal pair is a mild generalization of a Leonard pair [17, Definition 1.1].

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