A classification of sharp tridiagonal pairs

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Abstract

Let $\mathbb{F}$ denote a field and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following conditions: (i) each of A, A^* is diagonalizable; (ii) there exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that $A^* V_i \subseteq V_{i-1} + V_i + V_{i+1}$ for $0 \leq i \leq d$, where $V_{-1} = 0$ and $V_{d+1} = 0$; (iii) there exists an ordering $\{V_i^*\}_{i=0}^{\delta}$ of the eigenspaces of A^* such that $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ for $0 \leq i \leq \delta$, where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$; (iv) there is no subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We call such a pair a *tridiagonal pair* on V. It is known that $d = \delta$ and for $0 \leq i \leq d$ the dimensions of $V_i, V_{d-i}, V_i^*, V_{d-i}^*$ coincide. The pair A, A^* is called *sharp* whenever $\dim V_0 = 1$. It is known that if $\mathbb{F}$ is algebraically closed then A, A^* is sharp. In this paper we classify up to isomorphism the sharp tridiagonal pairs. As a corollary, we classify up to isomorphism the tridiagonal pairs over an algebraically closed field. We obtain these classifications by proving the μ-conjecture.

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1 Tridiagonal pairs

Throughout this paper $\mathbb{F}$ denotes a field and $\overline{\mathbb{F}}$ denotes the algebraic closure of $\mathbb{F}$. An algebra is meant to be associative and have a 1.

We begin by recalling the notion of a tridiagonal pair. We will use the following terms. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. For a linear transformation $A : V \to V$ and a subspace $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{F}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case θ is the *eigenvalue* of A associated with W. We say that A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 1.1 [27, Definition 1.1] Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal pair* (or *TD pair*) on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions.

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(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad\qquad 0 \le i \le d, \tag{1}$$

where $V_{-1} = 0$ and $V_{d+1} = 0$.

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad\qquad 0 \le i \le \delta, \tag{2}$$

where $V_{-1}^* = 0$ and $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

We say the pair A, A^* is *over* $\mathbb{F}$. We call V the *underlying vector space.*

Note 1.2 According to a common notational convention A^* denotes the conjugate-transpose of A. We are not using this convention. In a TD pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

We now give some background on TD pairs; for more information we refer the reader to the survey [75]. The concept of a TD pair originated in algebraic graph theory, or more precisely, the theory of Q-polynomial distance-regular graphs. The concept is implicit in [7, p. 263], [41] and more explicit in [67, Theorem 2.1]. A systematic study began in [27]. Some notable papers on the topic are [6, 13, 28–31, 35–37, 69]. There are connections to representation theory [2,9,21,26,29,31,39,62,63,73], partially ordered sets [71], the bispectral problem [5, 6, 22–24, 80], statistical mechanical models [8–14, 17–20, 60], and other areas of physics [59, 79, 81].

Let A, A^* denote a TD pair on V, as in Definition 1.1. By [27, Lemma 4.5] the integers d and δ from (ii), (iii) are equal; we call this common value the *diameter* of the pair. By [27, Theorem 10.1] the pair A, A^* satisfy two polynomial equations called the tridiagonal relations; these generalize the q-Serre relations [70, Example 3.6] and the Dolan-Grady relations [70, Example 3.2]. See [13, 24, 35, 37, 43, 68, 70, 73, 77, 78] for results on the tridiagonal relations. An ordering of the eigenspaces of A (resp. A^*) is said to be *standard* whenever it satisfies (1) (resp. (2)). We comment on the uniqueness of the standard ordering. Let $\{V_i\}_{i=0}^d$ denote a standard ordering of the eigenspaces of A. By [27, Lemma 2.4], the ordering $\{V_{d-i}\}_{i=0}^d$ is also standard and no further ordering is standard. A similar result holds for the eigenspaces of A^*. Let $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \le i \le d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*). By [27, Theorem 11.1] the expressions

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad\qquad \frac{\theta_{i-2}^* - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*}$$

are equal and independent of i for $2 \leq i \leq d-1$. We call the sequence $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) for the given standard orderings. See [27, 36, 61, 69, 70] for results on the eigenvalues and dual eigenvalues. By [27, Corollary 5.7], for $0 \leq i \leq d$ the spaces V_i, V^*_i have the same dimension; we denote this common dimension by ρ_i. By [27, Corollaries 5.7, 6.6] the sequence $\{\rho_i\}_{i=0}^d$ is symmetric and unimodal; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$. By [57, Theorem 1.3] we have $\rho_i \leq \rho_0\binom{d}{i}$ for $0 \leq i \leq d$. We call the sequence $\{\rho_i\}_{i=0}^d$ the *shape* of A, A^*. See [28, 38, 44, 45, 53, 57] for results on the shape. The TD pair A, A^* is called *sharp* whenever $\rho_0 = 1$. By [55, Theorem 1.3], if $\mathbb{F}$ is algebraically closed then A, A^* is sharp. In any case A, A^* can be "sharpened" by replacing $\mathbb{F}$ with a certain field extension $\mathbb{K}$ of $\mathbb{F}$ that has index $[\mathbb{K} : \mathbb{F}] = \rho_0$ [38, Theorem 4.12]. Suppose that A, A^* is sharp. Then by [55, Theorem 1.4], there exists a nondegenerate symmetric bilinear form $\langle \, , \, \rangle$ on V such that $\langle Au, v\rangle = \langle u, Av\rangle$ and $\langle A^*u, v\rangle = \langle u, A^*v\rangle$ for all $u, v \in V$. See [1, 53, 65, 66] for results on the bilinear form.

The following special cases of TD pairs have been studied extensively. In [76] the TD pairs of shape $(1, 2, 1)$ are classified and described in detail. A TD pair of shape $(1, 1, \ldots, 1)$ is called a *Leonard pair* [69, Definition 1.1], and these are classified in [69, Theorem 1.9]. This classification yields a correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials and their relatives [4, 72, 74]. This family coincides with the terminating branch of the Askey scheme [40]. See [15, 16, 42, 46–52, 75] and the references therein for results on Leonard pairs. Our TD pair A, A^* is said to have *Krawtchouk type* (resp. *q-geometric type*) whenever $\{d-2i\}_{i=0}^d$ (resp. $\{q^{d-2i}\}_{i=0}^d$) is both an eigenvalue sequence and dual eigenvalue sequence for the pair. In [26, Theorems 1.7, 1.8] Hartwig classified the TD pairs over $\mathbb{F}$ that have Krawtchouk type, provided that $\mathbb{F}$ is algebraically closed with characteristic zero. By [26, Remark 1.9] these TD pairs are in bijection with the finite-dimensional irreducible modules for the three-point loop algebra $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$. See [25, 26, 32, 33, 36] for results on TD pairs of Krawtchouk type. In [30, Theorems 1.6, 1.7] we classified the TD pairs over $\mathbb{F}$ that have q-geometric type, provided that $\mathbb{F}$ is algebraically closed and q is not a root of unity. By [31, Theorems 10.3, 10.4] these TD pairs are in bijection with the type 1, finite-dimensional, irreducible modules for the $\mathbb{F}$-algebra $\boxtimes_q$; this is a q-deformation of $\mathfrak{sl}_2 \otimes \mathbb{F}[t, t^{-1}, (t-1)^{-1}]$ as explained in [31]. See [2, 3, 28–31, 34, 36] for results on q-geometric TD pairs. There is a general family of TD pairs said to have q-Racah type; these have an eigenvalue sequence and dual eigenvalue sequence of the form (11)–(15) below. The Leonard pairs of q-Racah type correspond to the q-Racah polynomials [74, Example 5.3]. In [37, Theorem 3.3] we classified the TD pairs over $\mathbb{F}$ that have q-Racah type, provided that $\mathbb{F}$ is algebraically closed. See [35–37] for results on TD pairs of q-Racah type.

Turning to the present paper, in our main result we classify up to isomorphism the sharp TD pairs. Here is a summary of the argument. In [33, Conjecture 14.6] we conjectured how a classification of all the sharp TD pairs would look; this is the *classification conjecture*. Shortly afterwards we introduced a conjecture, called the *μ-conjecture*, which implies the classification conjecture. The μ-conjecture is roughly described as follows. Start with a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ of scalars taken from $\mathbb{F}$ that satisfy the known constraints on the eigenvalues of a TD pair over $\mathbb{F}$ of diameter d; these are conditions (i), (ii) in Theorem 3.1 below. Following [55, Definition 2.4] we associate with p an $\mathbb{F}$-algebra T defined by generators and relations; see Definition 3.4 for the precise definition. We are interested in the $\mathbb{F}$-algebra $e_0^* T e_0^*$ where e_0^* is a certain idempotent element of T. Let $\{x_i\}_{i=1}^d$ denote mutually commuting indeterminates. Let $\mathbb{F}[x_1, \ldots, x_d]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\{x_i\}_{i=1}^d$ that have all coefficients in $\mathbb{F}$. In [56, Corollary 6.3] we displayed a surjective $\mathbb{F}$-algebra homomorphism $\mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^* T e_0^*$. The *$\mu$-conjecture* [56, Conjecture 6.4] asserts that μ is an isomorphism. By [56, Theorem 10.1] the μ-conjecture implies the classification conjecture. In [56, Theorem 12.1] we showed that the μ-conjecture holds for $d \leq 5$. In [58, Theorem 5.3] we showed that the μ-conjecture holds for the case in which p has q-Racah type. In the present paper we combine this fact with some algebraic geometry to prove the μ-conjecture in general. The μ-conjecture (now a theorem) is given in Theorem 3.9. Theorem 3.9 implies the classification conjecture, and this yields our classification of the sharp TD pairs. The classification is given in Theorem 3.1. As a corollary, we classify up to isomorphism the TD pairs over an algebraically closed field. This result can be found in Corollary 18.1.

Section 3 contains the precise statements of our main results. In Section 2 we review the concepts needed to make these statements.

2 Tridiagonal systems

When working with a TD pair, it is often convenient to consider a closely related object called a TD system. To define a TD system, we recall a few concepts from linear algebra. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{F}$-algebra of all linear transformations from V to V. Let A denote a diagonalizable element of $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. For $0 \leq i \leq d$ define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_i V_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_i E_j = \delta_{i,j} E_i$ $(0 \leq i, j \leq d)$; (iii) $V_i = E_i V$ $(0 \leq i \leq d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Here $\delta_{i,j}$ denotes the Kronecker delta. Note that

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad 0 \leq i \leq d. \tag{3}$$

Observe that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{F}$-subalgebra of $\mathrm{End}(V)$ generated by A. Moreover $\prod_{i=0}^d (A - \theta_i I) = 0$. Now let A, A^* denote a TD pair on V. An ordering of the primitive idempotents of A (resp. A^*) is said to be *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard.

Definition 2.1 [27, Definition 2.1] Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *tridiagonal system* (or *TD system*) on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

that satisfies (i)–(iii) below.

(i) A, A^* is a TD pair on V.

(ii) $\{E_i\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is a standard ordering of the primitive idempotents of A^*.

We say that Φ is *over* $\mathbb{F}$. We call V the *underlying vector space*.

The following result is immediate from lines (1), (2) and Definition 2.1.

Lemma 2.2 *Let* $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ *denote a TD system. Then the following hold for* $0 \leq i, j, k \leq d$.

(i) $E_i^* A^k E_j^* = 0$ *if* $k < |i - j|$;

(ii) $E_i A^{*k} E_j = 0$ *if* $k < |i - j|$.

The notion of isomorphism for TD systems is defined in [53, Section 3].

Definition 2.3 Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with the eigenspace E_iV (resp. E_i^*V). We call $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) the *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of Φ. Observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) are mutually distinct and contained in $\mathbb{F}$. We call Φ *sharp* whenever the TD pair A, A^* is sharp.

The following notation will be useful.

Definition 2.4 Let x denote an indeterminate and let $\mathbb{F}[x]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in x that have all coefficients in $\mathbb{F}$. Let $\{\theta_i\}_{i=0}^d$ and $\{\theta_i^*\}_{i=0}^d$ denote scalars in $\mathbb{F}$. For $0 \leq i \leq d$ define the following polynomials in $\mathbb{F}[x]$:

$$
\begin{aligned}
\tau_i &= (x - \theta_0)(x - \theta_1) \cdots (x - \theta_{i-1}), \\
\eta_i &= (x - \theta_d)(x - \theta_{d-1}) \cdots (x - \theta_{d-i+1}), \\
\tau_i^* &= (x - \theta_0^*)(x - \theta_1^*) \cdots (x - \theta_{i-1}^*), \\
\eta_i^* &= (x - \theta_d^*)(x - \theta_{d-1}^*) \cdots (x - \theta_{d-i+1}^*).
\end{aligned}
$$

Note that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

We now recall the split sequence of a sharp TD system. This sequence was originally defined in [33, Section 5] using the split decomposition [27, Section 4], but in [56] an alternate definition was introduced that is more convenient to our purpose.

Definition 2.5 [56, Definition 2.5] Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a sharp TD system over $\mathbb{F}$, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By [55, Lemma 5.4], for $0 \leq i \leq d$ there exists a unique $\zeta_i \in \mathbb{F}$ such that

$$
E_0^* \tau_i(A) E_0^* = \frac{\zeta_i E_0^*}{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_i^*)}.
$$

Note that $\zeta_0 = 1$. We call $\{\zeta_i\}_{i=0}^d$ the *split sequence* of the TD system.

Definition 2.6 Let Φ denote a sharp TD system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d)$ where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta^*_i\}_{i=0}^d$) is the eigenvalue sequence (resp. dual eigenvalue sequence) of Φ and $\{\zeta_i\}_{i=0}^d$ is the split sequence of Φ.

The following result shows the significance of the parameter array.

Proposition 2.7 [35], [55, Theorem 1.6] *Two sharp TD systems over $\mathbb{F}$ are isomorphic if and only if they have the same parameter array.*

3 Statement of results

In this section we state our main results. The first result below resolves [33, Conjecture 14.6].

Theorem 3.1 *Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \tag{4}$$

denote a sequence of scalars taken from $\mathbb{F}$. Then there exists a sharp TD system Φ over $\mathbb{F}$ with parameter array (4) *if and only if* (i)–(iii) *hold below.*

(i) $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta^*_j$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.

(ii) *The expressions*

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i} \tag{5}$$

are equal and independent of i for $2 \leq i \leq d-1$.

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Suppose (i)–(iii) *hold. Then Φ is unique up to isomorphism of TD systems.*

In [56, Conjecture 6.4] we stated a conjecture called the *μ-conjecture*, and we proved that the μ-conjecture implies Theorem 3.1. To obtain Theorem 3.1 we will prove the μ-conjecture. We now explain this conjecture.

Definition 3.2 Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$. This sequence is called *feasible* whenever it satisfies conditions (i), (ii) of Theorem 3.1.

Definition 3.3 For all integers $d \geq 0$ let $\mathrm{Feas}(d, \mathbb{F})$ denote the set of all feasible sequences $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ of scalars taken from $\mathbb{F}$.

Definition 3.4 [55, Definition 2.4] Fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Let $T = T(p, \mathbb{F})$ denote the $\mathbb{F}$-algebra defined by generators a, $\{e_i\}_{i=0}^d$, a^*, $\{e_i^*\}_{i=0}^d$ and relations

$$e_i e_j = \delta_{i,j} e_i, \qquad e_i^* e_j^* = \delta_{i,j} e_i^* \qquad\qquad 0 \leq i, j \leq d, \tag{6}$$

$$1 = \sum_{i=0}^d e_i, \qquad\qquad 1 = \sum_{i=0}^d e_i^*, \tag{7}$$

$$a = \sum_{i=0}^d \theta_i e_i, \qquad\qquad a^* = \sum_{i=0}^d \theta_i^* e_i^*, \tag{8}$$

$$e_i^* a^k e_j^* = 0 \qquad \text{if } \; k < |i - j| \qquad\qquad 0 \leq i, j, k \leq d, \tag{9}$$

$$e_i a^{*k} e_j = 0 \qquad \text{if } \; k < |i - j| \qquad\qquad 0 \leq i, j, k \leq d. \tag{10}$$

Lemma 3.5 [56, Lemma 4.2] *In the algebra T from Definition 3.4, the elements $\{e_i\}_{i=0}^d$ are linearly independent and the elements $\{e_i^*\}_{i=0}^d$ are linearly independent.*

The algebra T is related to TD systems as follows.

Lemma 3.6 [55, Lemma 2.5] *Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TD system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let T denote the $\mathbb{F}$-algebra from Definition 3.4 corresponding to $(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$. Then there exists a unique T-module structure on V such that a, e_i, a^*, e_i^* acts as A, E_i, A^*, E_i^* respectively. This T-module is irreducible.*

Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let $T = T(p, \mathbb{F})$ denote the corresponding algebra from Definition 3.4. Observe that $e_0^* T e_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity e_0^*.

Lemma 3.7 [55, Theorem 2.6] *With the above notation, the algebra $e_0^* T e_0^*$ is commutative and generated by*

$$e_0^* \tau_i(a) e_0^* \qquad\qquad 1 \leq i \leq d.$$

Corollary 3.8 [56, Corollary 6.3] *With the above notation, there exists a surjective $\mathbb{F}$-algebra homomorphism $\mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^* T e_0^*$ that sends $x_i \mapsto e_0^* \tau_i(a) e_0^*$ for $1 \leq i \leq d$.*

In [56, Conjecture 6.4] we conjectured that the map μ from Corollary 3.8 is an isomorphism. This is the *μ-conjecture*. The following result resolves the μ-conjecture.

Theorem 3.9 *Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let $T = T(p, \mathbb{F})$ denote the corresponding algebra from Definition 3.4. Then the map $\mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^* T e_0^*$ from Corollary 3.8 is an isomorphism.*

We will prove Theorem 3.1 and Theorem 3.9 in Section 17.

4 The q-Racah case

Our proof of Theorem 3.9 will use the fact that the theorem is known to be true in a special case called q-Racah [58, Theorem 5.3]. In this section we describe the q-Racah case. We start with some comments about the feasible sequences from Definition 3.2.

Lemma 4.1 *Assume $\mathbb{F}$ is infinite. Then for all integers $d \geq 0$ the set* $\mathrm{Feas}(d, \mathbb{F})$ *is nonempty.*

Proof: Consider the polynomial $\prod_{n=1}^{d}(x^n - 1)$ in $\mathbb{F}[x]$. Since $\mathbb{F}$ is infinite there exists a nonzero $\vartheta \in \mathbb{F}$ that is not a root of this polynomial. Define $\theta_i = \vartheta^i$ and $\theta^*_i = \vartheta^i$ for $0 \leq i \leq d$. Then $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ satisfies the conditions (i), (ii) of Theorem 3.1 and is therefore feasible. The result follows. $\square$

Fix an integer $d \geq 0$ and a sequence $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. This sequence must satisfy condition (ii) in Theorem 3.1. For this constraint the "most general" solution is

$$\theta_i = \alpha + bq^{2i-d} + cq^{d-2i} \qquad 0 \leq i \leq d, \tag{11}$$

$$\theta^*_i = \alpha^* + b^*q^{2i-d} + c^*q^{d-2i} \qquad 0 \leq i \leq d, \tag{12}$$

$$q,\ \alpha,\ b,\ c,\ \alpha^*,\ b^*,\ c^*\ \in \overline{\mathbb{F}}, \tag{13}$$

$$q \neq 0, \quad q^2 \neq 1, \quad q^2 \neq -1. \tag{14}$$

We have a few comments about this solution. For the moment define $\beta = q^2 + q^{-2}$, and observe that $\beta+1$ is the common value of (5). We have $\beta-2 = (q-q^{-1})^2$ and $\beta+2 = (q+q^{-1})^2$. Therefore $\beta \neq 2$, $\beta \neq -2$ in view of (14). Using (11), (12) we obtain

$$bc = \frac{(\theta_0 - \theta_1)^2 - \beta(\theta_0 - \theta_1)(\theta_1 - \theta_2) + (\theta_1 - \theta_2)^2}{(\beta - 2)^2(\beta + 2)},$$

$$b^*c^* = \frac{(\theta^*_0 - \theta^*_1)^2 - \beta(\theta^*_0 - \theta^*_1)(\theta^*_1 - \theta^*_2) + (\theta^*_1 - \theta^*_2)^2}{(\beta - 2)^2(\beta + 2)}$$

provided $d \geq 2$. We will focus on the case

$$bb^*cc^* \neq 0. \tag{15}$$

Definition 4.2 [37, Definition 3.1] Let d denote a nonnegative integer and let $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ denote a sequence of scalars taken from $\mathbb{F}$. We call this sequence *q-Racah* whenever the following (i), (ii) hold:

(i) $\theta_i \neq \theta_j$, $\theta^*_i \neq \theta^*_j$ if $i \neq j$ $(0 \leq i, j \leq d)$;

(ii) there exist $q, \alpha, b, c, \alpha^*, b^*, c^*$ that satisfy (11)–(15).

Definition 4.3 For all integers $d \geq 0$ let $\mathrm{Rac}(d, \mathbb{F})$ denote the set of all q-Racah sequences $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ of scalars taken from $\mathbb{F}$.

Observe that the set $\mathrm{Rac}(d, \mathbb{F})$ from Definition 4.3 is contained in the set $\mathrm{Feas}(d, \mathbb{F})$ from Definition 3.3. In the next lemma we characterize $\mathrm{Rac}(d, \mathbb{F})$ as a subset of $\mathrm{Feas}(d, \mathbb{F})$. To avoid trivialities we assume $d \geq 3$.

Lemma 4.4 *Fix an integer $d \geq 3$ and a sequence $(\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ in* $\mathrm{Feas}(d, \mathbb{F})$*. Let $\beta + 1$ denote the common value of (5). Then the sequence is in* $\mathrm{Rac}(d, \mathbb{F})$ *if and only if each of the following hold:*

(i) $\beta^2 \neq 4$;

(ii) $(\theta_0 - \theta_1)^2 - \beta(\theta_0 - \theta_1)(\theta_1 - \theta_2) + (\theta_1 - \theta_2)^2 \neq 0$;

(iii) $(\theta^*_0 - \theta^*_1)^2 - \beta(\theta^*_0 - \theta^*_1)(\theta^*_1 - \theta^*_2) + (\theta^*_1 - \theta^*_2)^2 \neq 0$.

Proof: Use the comments above Definition 4.2. $\square$

Proposition 4.5 *Assume $\mathbb{F}$ is infinite and pick an integer $d \geq 3$. Let h denote a polynomial in $2d + 2$ mutually commuting indeterminates that has all coefficients in $\mathbb{F}$. Suppose that $h(p) = 0$ for all $p \in \mathrm{Rac}(d, \mathbb{F})$. Then $h(p) = 0$ for all $p \in \mathrm{Feas}(d, \mathbb{F})$.*

Proof: Let $\flat, \{t_i\}_{i=0}^2, \{t^*_i\}_{i=0}^2$ denote mutually commuting indeterminates. Consider the $\mathbb{F}$-algebra $\mathbb{F}[\flat, t_0, t_1, t_2, t^*_0, t^*_1, t^*_2]$ consisting of the polynomials in $\flat, t_0, t_1, t_2, t^*_0, t^*_1, t^*_2$ that have all coefficients in $\mathbb{F}$. For $3 \leq i \leq d$ define $t_i, t^*_i \in \mathbb{F}[\flat, t_0, t_1, t_2, t^*_0, t^*_1, t^*_2]$ by

$$\begin{aligned} 0 &= t_i - (\flat + 1)t_{i-1} + (\flat + 1)t_{i-2} - t_{i-3}, \\ 0 &= t^*_i - (\flat + 1)t^*_{i-1} + (\flat + 1)t^*_{i-2} - t^*_{i-3}. \end{aligned}$$

Define a polynomial $f \in \mathbb{F}[\flat, t_0, t_1, t_2, t^*_0, t^*_1, t^*_2]$ to be the composition

$$f = h(t_0, t_1, \ldots, t_d, t^*_0, t^*_1, \ldots, t^*_d).$$

We mention one significance of f. Given a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$, let $\beta + 1$ denote the common value of (5). Define the sequence $s = (\beta, \theta_0, \theta_1, \theta_2, \theta^*_0, \theta^*_1, \theta^*_2)$. Observe that $\theta_i = t_i(s)$ and $\theta^*_i = t^*_i(s)$ for $0 \leq i \leq d$. Therefore

$$f(s) = h(p). \tag{16}$$

We show $f = 0$. Instead of working directly with f, it will be convenient to work with the product $\Psi = f\xi\xi^*\omega\omega^*(\flat^2 - 4)$, where

$$\xi = \prod_{0 \leq i < j \leq d} (t_i - t_j), \tag{17}$$

$$\xi^* = \prod_{0 \leq i < j \leq d} (t^*_i - t^*_j), \tag{18}$$

$$\omega = (t_0 - t_1)^2 - \flat(t_0 - t_1)(t_1 - t_2) + (t_1 - t_2)^2, \tag{19}$$

$$\omega^* = (t^*_0 - t^*_1)^2 - \flat(t^*_0 - t^*_1)(t^*_1 - t^*_2) + (t^*_1 - t^*_2)^2. \tag{20}$$

Each of ξ, ξ^* is nonzero by Lemma 4.1 and since $\mathbb{F}$ is infinite. Each of $\omega, \omega^*, \flat^2 - 4$ is nonzero by construction. To show $f = 0$ we will show that $\Psi = 0$ and invoke the fact that $\mathbb{F}[\flat, t_0, t_1, t_2, t^*_0, t^*_1, t^*_2]$ is a domain [64, p. 129]. We now show that $\Psi = 0$. Since $\mathbb{F}$ is infinite it suffices to show that $\Psi(s) = 0$ for all sequences $s = (\beta, \theta_0, \theta_1, \theta_2, \theta^*_0, \theta^*_1, \theta^*_2)$ of scalars taken from $\mathbb{F}$ [64, Proposition 6.89]. Let s be given. For $3 \leq i \leq d$ define $\theta_i = t_i(s)$, $\theta^*_i = t^*_i(s)$ and put $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$. We may assume $\{\theta_i\}_{i=0}^d$ are mutually distinct; otherwise $\xi(s) = 0$ so $\Psi(s) = 0$. We may assume $\{\theta^*_i\}_{i=0}^d$ are mutually distinct; otherwise $\xi^*(s) = 0$ so $\Psi(s) = 0$. By construction p satisfies condition (ii) of Theorem 3.1, with $\beta+1$ the common value of (5). Therefore p is feasible by Definition 3.2. For the moment assume that $p \in \mathrm{Rac}(d, \mathbb{F})$. Then $f(s) = 0$ by (16) and since $h(p) = 0$. Therefore $\Psi(s) = 0$. Next assume that $p \notin \mathrm{Rac}(d, \mathbb{F})$. Then the product $\omega\omega^*(\flat^2 - 4)$ vanishes at s in view of Lemma 4.4. The product $\omega\omega^*(\flat^2 - 4)$ is a factor of Ψ so $\Psi(s) = 0$. By the above comments $\Psi(s) = 0$ for all sequences of scalars $s = (\beta, \theta_0, \theta_1, \theta_2, \theta^*_0, \theta^*_1, \theta^*_2)$ taken from $\mathbb{F}$. Therefore $\Psi = 0$ so $f = 0$. Now consider any sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Then $h(p) = 0$ by (16) and since $f = 0$. $\square$

5 The algebra $\check{T}$

In order to prove Theorem 3.9 we will need some detailed results about the algebra T from Definition 3.4. In order to obtain these results it is helpful to first consider the following algebra $\check{T}$.

Definition 5.1 Fix an integer $d \geq 0$. Let $\check{T} = \check{T}(d, \mathbb{F})$ denote the $\mathbb{F}$-algebra defined by generators $\{\epsilon_i\}_{i=0}^d$, $\{\epsilon^*_i\}_{i=0}^d$ and relations

$$\epsilon_i \epsilon_j = \delta_{i,j} \epsilon_i, \qquad \epsilon^*_i \epsilon^*_j = \delta_{i,j} \epsilon^*_i \qquad 0 \leq i, j \leq d. \tag{21}$$

Definition 5.2 Referring to Definition 5.1, we call $\{\epsilon_i\}_{i=0}^d$ and $\{\epsilon^*_i\}_{i=0}^d$ the *idempotent generators* for $\check{T}$. We say that the $\{\epsilon^*_i\}_{i=0}^d$ are *starred* and the $\{\epsilon_i\}_{i=0}^d$ are *nonstarred*.

Definition 5.3 A pair of idempotent generators for $\check{T}$ is called *alternating* whenever one of them is starred and the other is nonstarred. For an integer $n \geq 0$, by a *word of length* n in $\check{T}$ we mean a product $g_1 g_2 \cdots g_n$ such that $\{g_i\}_{i=1}^n$ are idempotent generators for $\check{T}$ and g_{i-1}, g_i are alternating for $2 \leq i \leq n$. We interpret the word of length 0 to be the identity of $\check{T}$. We call this word *trivial*.

Proposition 5.4 *The* $\mathbb{F}$*-vector space* $\check{T}$ *has a basis consisting of its words.*

Proof: Let S denote the set of words in $\check{T}$. By construction S spans $\check{T}$. We show that S is linearly independent. To this end we introduce some indeterminates $\{f_i\}_{i=0}^d$; $\{f^*_i\}_{i=0}^d$ called *formal idempotents*. We call the $\{f^*_i\}_{i=0}^d$ *starred* and the $\{f_i\}_{i=0}^d$ *nonstarred*. A pair of formal idempotents is said to be *alternating* whenever one of them is starred and the other is nonstarred. For an integer $n \geq 0$, by a *formal word of length* n we mean a sequence $(y_1, y_2, \ldots, y_n)$ such that $\{y_i\}_{i=1}^n$ are formal idempotents and y_{i-1}, y_i are alternating for $2 \leq i \leq n$. The formal word of length 0 is called *trivial* and denoted by 1. Let $\mathcal{S}$ denote the set of all formal words. Let V denote the vector space over $\mathbb{F}$ consisting of the $\mathbb{F}$-linear combinations of $\mathcal{S}$ that have finitely many nonzero coefficients. The set $\mathcal{S}$ is a basis for V. For $0 \leq i \leq d$ we define linear transformations $F_i : V \to V$ and $F^*_i : V \to V$. To do this we give the action of F_i and F_i^* on $\mathcal{S}$. We define $F_i.1 = f_i$ and $F_i^*.1 = f_i^*$. Pick a nontrivial formal word $y = (y_1, y_2, \ldots, y_n)$. For the moment assume that y_1 is starred. We define $F_i.y = (f_i, y_1, y_2, \ldots, y_n)$. Also $F_i^*.y = y$ if $y_1 = f_i^*$ and $F_i^*.y = 0$ if $y_1 \neq f_i^*$. Next assume that y_1 is nonstarred. We define $F_i.y = y$ if $y_1 = f_i$ and $F_i.y = 0$ if $y_1 \neq f_i$. Also $F_i^*.y = (f_i^*, y_1, y_2, \ldots, y_n)$. The linear transformations $F_i : V \to V$ and $F_i^* : V \to V$ are now defined. By construction $F_i F_j = \delta_{i,j} F_i$ and $F_i^* F_j^* = \delta_{i,j} F_i^*$ for $0 \leq i, j \leq d$. Therefore V has a $\check{T}$-module structure such that ϵ_i (resp. ϵ_i^*) acts on V as F_i (resp. F_i^*) for $0 \leq i \leq d$. Consider the linear transformation $\gamma : \check{T} \to V$ that sends $z \mapsto z.1$ for all $z \in \check{T}$. For each word $g_1 g_2 \cdots g_n$ in $\check{T}$ we find $\gamma(g_1 g_2 \cdots g_n) = (g_1', g_2', \ldots, g_n')$, where $\epsilon_i' = f_i$ and $\epsilon_i^{*\prime} = f_i^*$ for $0 \leq i \leq d$. Therefore the restriction of γ to S gives a bijection $S \mapsto \mathcal{S}$. The set $\mathcal{S}$ is linearly independent and γ is linear so S is linearly independent. We have shown that S is a basis for $\check{T}$. $\square$

Let u, v denote words in $\check{T}$. Then their product uv is either 0 or a word in $\check{T}$.

6 The algebras D and D^*

Throughout this section we fix an integer $d \geq 0$ and consider the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1.

Definition 6.1 Let D (resp. D^*) denote the subspace of $\check{T}$ with a basis $\{\epsilon_i\}_{i=0}^d$ (resp. $\{\epsilon_i^*\}_{i=0}^d$).

We mention some notation. For subsets Y, Z of $\check{T}$ let YZ denote the subspace of $\check{T}$ spanned by $\{yz \mid y \in Y,\ z \in Z\}$.

Lemma 6.2 *In the $\mathbb{F}$-vector space $\check{T}$ the following sum is direct:*

$$\check{T} = \mathbb{F}1 \ + \ D \ + \ D^* \ + \ DD^* \ + \ D^*D \ + \ DD^*D \ + \ D^*DD^* \ + \ \cdots$$

Moreover the $\mathbb{F}$-algebra $\check{T}$ is generated by D, D^*.

Proof: The first assertion is immediate from Proposition 5.4. The last assertion is clear. $\square$

Lemma 6.3 *The space D (resp. D^*) is an $\mathbb{F}$-algebra with multiplicative identity $\sum_{i=0}^d \epsilon_i$ (resp. $\sum_{i=0}^d \epsilon_i^*$).*

Proof: Use (21). $\square$

We emphasize that D and D^* are not subalgebras of $\check{T}$, since their multiplicative identities do not equal the multiplicative identity 1 of $\check{T}$. However we do have the following.

Lemma 6.4 *The spaces $D + \mathbb{F}1$ and $D^* + \mathbb{F}1$ are $\mathbb{F}$-subalgebras of $\check{T}$.*

Proof: These subspaces are closed under multiplication and contain the identity 1 of $\check{T}$. $\square$

7 The homogeneous components of $\check{T}$

Throughout this section we fix an integer $d \geq 0$ and consider the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1.

Definition 7.1 Let $w = g_1 g_2 \cdots g_n$ denote a nontrivial word in $\check{T}$. We say that w *begins* with g_1 and *ends* with g_n. We write

$$g_1 = \text{begin}(w), \qquad\qquad g_n = \text{end}(w).$$

Example 7.2 Assume $d = 2$. In the table below we display some nontrivial words w in $\check{T}$. For each word w we give begin(w) and end(w).

w	begin(w)	end(w)
ϵ_1	ϵ_1	ϵ_1
$\epsilon_1\epsilon_2^*$	ϵ_1	ϵ_2^*
$\epsilon_1^*\epsilon_0\epsilon_2^*$	ϵ_1^*	ϵ_2^*

Definition 7.3 We define a binary relation $\sim$ on the set of words in $\check{T}$. With respect to $\sim$ the trivial word in $\check{T}$ is related to itself and no other word in $\check{T}$. For nontrivial words u, v in $\check{T}$ we define $u \sim v$ whenever each of the following holds:

$$\text{length}(u) = \text{length}(v), \qquad\quad \text{begin}(u) = \text{begin}(v), \qquad\quad \text{end}(u) = \text{end}(v).$$

Observe that $\sim$ is an equivalence relation.

Definition 7.4 Let Λ denote the set of equivalence classes for the relation $\sim$ in Definition 7.3. An element of Λ is called a *type*. For $\lambda \in \Lambda$ the words in λ are said to have *type* λ.

Definition 7.5 For $\lambda \in \Lambda$ let length(λ) denote the common length of each word of type λ.

Definition 7.6 There exists a unique type in Λ that has length 0. This type consists of the trivial word 1 and nothing else. We call this type *trivial*.

Definition 7.7 For all nontrivial $\lambda \in \Lambda$,

(i) let begin(λ) denote the common beginning of each word of type λ;

(ii) let end(λ) denote the common ending of each word of type λ.

Definition 7.8 For $\lambda \in \Lambda$ let $\check{T}_\lambda$ denote the subspace of $\check{T}$ with a basis consisting of the words of type λ.

Proposition 7.9 *The $\mathbb{F}$-vector space $\check{T}$ decomposes as*

$$\check{T} = \sum_{\lambda \in \Lambda} \check{T}_\lambda \qquad\qquad \text{(direct sum).} \tag{22}$$

Proof: Immediate from Proposition 5.4 and Definition 7.8. □

Definition 7.10 For $\lambda \in \Lambda$ we call $\check{T}_\lambda$ the λ-*homogeneous component* of $\check{T}$. Elements of $\check{T}_\lambda$ are said to be λ-*homogeneous*. An element of $\check{T}$ is called *homogeneous* whenever it is λ-homogeneous for some $\lambda \in \Lambda$.

8 The zigzag words in $\check{T}$

Throughout this section we fix an integer $d \geq 0$ and consider the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1. We have been discussing the words in $\check{T}$. We now focus our attention on a special kind of word said to be zigzag.

Definition 8.1 Given an ordered pair of integers i, j and an integer m we say that m is *between* i, j whenever $i \geq m > j$ or $i \leq m < j$.

Definition 8.2 For an idempotent generator ϵ_i or ϵ_i^* of $\check{T}$, we call i the *index* of the generator. For an idempotent generator g of $\check{T}$ let $\overline{g}$ denote the index of g.

Definition 8.3 A word $g_1 g_2 \cdots g_n$ in $\check{T}$ is said to be *zigzag* whenever both

(i) $\overline{g}_i$ is not between $\overline{g}_{i-1}$, $\overline{g}_{i+1}$ for $2 \leq i \leq n-1$;

(ii) at least one of $\overline{g}_{i-1}$, $\overline{g}_i$ is not between $\overline{g}_{i-2}$, $\overline{g}_{i+1}$ for $3 \leq i \leq n-1$.

We now describe the zigzag words in $\check{T}$. We will use the following notion. Two integers m, m' are said to have *opposite sign* whenever $mm' \leq 0$.

Proposition 8.4 [57, Theorem 7.7] *Let $g_1 g_2 \cdots g_n$ denote a word in $\check{T}$. Then this word is zigzag if and only if both*

(i) *$\overline{g}_{i-1} - \overline{g}_i$ and $\overline{g}_i - \overline{g}_{i+1}$ have opposite sign for $2 \leq i \leq n-1$;*

(ii) *for $2 \leq i \leq n-1$, if $|\overline{g}_{i-1} - \overline{g}_i| < |\overline{g}_i - \overline{g}_{i+1}|$ then*

$$0 < |\overline{g}_1 - \overline{g}_2| < |\overline{g}_2 - \overline{g}_3| < \cdots < |\overline{g}_i - \overline{g}_{i+1}|.$$

Definition 8.5 A word $g_1 g_2 \cdots g_n$ in $\check{T}$ is said to be *constant* whenever the index $\overline{g}_i$ is independent of i for $1 \leq i \leq n$. Note that the trivial word is constant, and each constant word is zigzag.

Proposition 8.6 [57, Theorem 7.9] *Let $g_1 g_2 \cdots g_n$ denote a nonconstant zigzag word in $\check{T}$. Then there exists a unique integer κ $(2 \leq \kappa \leq n)$ such that both*

(i) *$0 < |\overline{g}_1 - \overline{g}_2| < \cdots < |\overline{g}_{\kappa-1} - \overline{g}_\kappa|$;*

(ii) *$|\overline{g}_{\kappa-1} - \overline{g}_\kappa| \geq |\overline{g}_\kappa - \overline{g}_{\kappa+1}| \geq \cdots \geq |\overline{g}_{n-1} - \overline{g}_n|$.*

Definition 8.7 For $\lambda \in \Lambda$ let Z_λ denote the subspace of $\check{T}$ with a basis consisting of the zigzag words of type λ. Note that $Z_\lambda \subseteq \check{T}_\lambda$.

9 The algebra $\epsilon_0^*\check{T}\epsilon_0^*$

Throughout this section we fix an integer $d \geq 0$ and consider the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1. Observe that $\epsilon_0^*\check{T}\epsilon_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity ϵ_0^*.

Lemma 9.1 *The $\mathbb{F}$-vector space $\epsilon_0^*\check{T}\epsilon_0^*$ has a basis consisting of the nontrivial words in $\check{T}$ that begin and end with ϵ_0^*.*

Proof: Let U denote the subspace of $\check{T}$ with a basis consisting of the nontrivial words in $\check{T}$ that begin and end with ϵ_0^*. We show that $\epsilon_0^*\check{T}\epsilon_0^* = U$. We first show that $\epsilon_0^*\check{T}\epsilon_0^* \subseteq U$. Recall that $\check{T}$ is spanned by its words. For all words w in $\check{T}$ the product $\epsilon_0^* w \epsilon_0^*$ is either zero, or a nontrivial word in $\check{T}$ that begins and ends with ϵ_0^*. In either case $\epsilon_0^* w \epsilon_0^* \in U$, and therefore $\epsilon_0^*\check{T}\epsilon_0^* \subseteq U$. Next we show that $U \subseteq \epsilon_0^*\check{T}\epsilon_0^*$. Let w denote a nontrivial word in $\check{T}$ that begins and ends with ϵ_0^*. We have $w = \epsilon_0^* w \epsilon_0^*$ since $\epsilon_0^{*2} = \epsilon_0^*$, so $w \in \epsilon_0^*\check{T}\epsilon_0^*$. Therefore $U \subseteq \epsilon_0^*\check{T}\epsilon_0^*$. We have shown that $\epsilon_0^*\check{T}\epsilon_0^* = U$ and the result follows. $\square$

Definition 9.2 Let Λ_0 denote the set of types in Λ that begin and end with ϵ_0^*.

Our next goal is to describe Λ_0.

Definition 9.3 Let $g_1 g_2 \cdots g_n$ denote a word in $\check{T}$. By the *star-length* (resp. *nonstar-length*) of this word we mean the number of terms in the sequence $(g_1, g_2, \ldots, g_n)$ that are starred (resp. nonstarred). Note that the star-length plus the nonstar-length is equal to the length n. For $\lambda \in \Lambda$, by the *star-length* (resp. *nonstar-length*) of λ we mean the common star-length (resp. nonstar-length) of each word of type λ.

Definition 9.4 For an integer $n \geq 0$ let $[n]$ denote the unique type in Λ_0 that has nonstar-length n. Observe that $[n]$ has star-length $n+1$ and length $2n+1$.

The next two lemmas follow immediately from Definition 9.2 and Definition 9.4.

Lemma 9.5 *The map $n \mapsto [n]$ gives a bijection from the set of nonnegative integers to the set Λ_0.*

Lemma 9.6 *Let m and n denote nonnegative integers. Let u and v denote words in $\check{T}$ of type $[m]$ and $[n]$ respectively. Then uv is a word in $\check{T}$ of type $[m+n]$.*

Proposition 9.7 *The $\mathbb{F}$-vector space $\epsilon_0^*\check{T}\epsilon_0^*$ decomposes as*

$$\epsilon_0^*\check{T}\epsilon_0^* = \sum_{n=0}^{\infty} \check{T}_{[n]} \qquad \text{(direct sum).} \tag{23}$$

Moreover $\check{T}_{[m]} \cdot \check{T}_{[n]} \subseteq \check{T}_{[m+n]}$ for all integers $m, n \geq 0$.

Proof: By Lemma 9.1 and Definition 9.2 we have $\epsilon_0^* \check{T} \epsilon_0^* = \sum_{\lambda \in \Lambda_0} \check{T}_\lambda$ (direct sum). Combining this with Lemma 9.5 we obtain (23). The last assertion follows from Lemma 9.6. $\square$

We turn our attention to the zigzag words in $\check{T}$ that begin and end with ϵ_0^*.

Proposition 9.8 *Pick an integer $n \geq 0$ and a word $g_1 g_2 \cdots g_{2n+1}$ in $\check{T}$ of type $[n]$. This word is zigzag if and only if both*

(i) $\overline{g}_i = 0$ *for all odd* i $(1 \leq i \leq 2n+1)$;

(ii) $\overline{g}_i \geq \overline{g}_{i+2}$ *for all even* i $(2 \leq i \leq 2n-2)$.

Proof: The type $[n]$ begins and ends with ϵ_0^*, so $\overline{g}_1 = 0$ and $\overline{g}_{2n+1} = 0$. The result follows from this and Propositions 8.4, 8.6. $\square$

10 The elements a, a^*

Recall that the algebra T from Definition 3.4 is defined using relations (6)–(10). So far we have investigated relation (6). We now prepare to bring in relations (8)–(10).

Throughout this section we fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Recall the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1.

Definition 10.1 Define $a = a(p)$ and $a^* = a^*(p)$ in $\check{T}$ by

$$a = \sum_{i=0}^d \theta_i \epsilon_i, \qquad a^* = \sum_{i=0}^d \theta_i^* \epsilon_i^*. \tag{24}$$

Observe that $a \in D$ and $a^* \in D^*$, where D, D^* are from Definition 6.1.

Lemma 10.2 *For $0 \leq i \leq d$,*

$$a\epsilon_i = \epsilon_i a = \theta_i \epsilon_i, \qquad a^* \epsilon_i^* = \epsilon_i^* a^* = \theta_i^* \epsilon_i^*. \tag{25}$$

Proof: Use (21) and (24). $\square$

Note 10.3 We will be considering powers of the elements a, a^* from Definition 10.1. We wish to clarify the meaning of a^0 and a^{*0}. We always interpret

$$a^0 = \sum_{i=0}^d \epsilon_i, \qquad a^{*0} = \sum_{i=0}^d \epsilon_i^*. \tag{26}$$

This is justified by Lemma 6.3. We mention some related notational conventions. Consider the $\mathbb{F}$-algebra homomorphism $\mathbb{F}[x] \to D$ that sends $x \mapsto a$. By definition this homomorphism sends the identity 1 of $\mathbb{F}[x]$ to the identity $\sum_{i=0}^d \epsilon_i$ of D. For $f \in \mathbb{F}[x]$ the image of f under this homomorphism will be denoted $f(a)$. Writing $f = \sum_{i=0}^n c_i x^i$ we have $f(a) = \sum_{i=0}^n c_i a^i$, with the $i = 0$ summand interpreted using the equation on the left in (26). A similar comment applies to a^*.

The above notational conventions are illustrated in the following lemma.

Lemma 10.4 *For $f \in \mathbb{F}[x]$,*

$$f(a) = \sum_{i=0}^{d} f(\theta_i)\epsilon_i, \qquad f(a^*) = \sum_{i=0}^{d} f(\theta^*_i)\epsilon^*_i.$$

In particular for an integer $k \geq 0$,

$$a^k = \sum_{i=0}^{d} \theta_i^k \epsilon_i, \qquad a^{*k} = \sum_{i=0}^{d} \theta_i^{*k} \epsilon^*_i. \tag{27}$$

Proof: Use (21) and (24). □

11 The algebra $\tilde{T}$

In our study of the algebra T we now bring in relations (8)–(10). We do this in a compact way.

Definition 11.1 Fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$ in $\text{Feas}(d, \mathbb{F})$. Let $\tilde{T} = \tilde{T}(p, \mathbb{F})$ denote the $\mathbb{F}$-algebra with generators $\{\varepsilon_i\}_{i=0}^d$, $\{\varepsilon^*_i\}_{i=0}^d$ and relations

$$\varepsilon_i \varepsilon_j = \delta_{i,j}\varepsilon_i, \qquad \varepsilon^*_i \varepsilon^*_j = \delta_{i,j}\varepsilon^*_i \qquad 0 \leq i, j \leq d, \tag{28}$$

$$0 = \sum_{\ell=0}^{d} \theta_\ell^k \varepsilon^*_i \varepsilon_\ell \varepsilon^*_j, \qquad 0 = \sum_{\ell=0}^{d} \theta_\ell^{*k} \varepsilon_i \varepsilon^*_\ell \varepsilon_j \qquad 0 \leq i, j \leq d, \qquad 0 \leq k < |i - j|. \tag{29}$$

Many of the concepts that apply to $\check{T}$ also apply to $\tilde{T}$. We emphasize a few such concepts in the following definitions.

Definition 11.2 Referring to Definition 11.1, we call $\{\varepsilon_i\}_{i=0}^d$ and $\{\varepsilon^*_i\}_{i=0}^d$ the *idempotent generators* for $\tilde{T}$. We say that the $\{\varepsilon^*_i\}_{i=0}^d$ are *starred* and the $\{\varepsilon_i\}_{i=0}^d$ are *nonstarred.* A pair of idempotent generators for $\tilde{T}$ will be called *alternating* whenever one of them is starred and the other is nonstarred.

Definition 11.3 For an integer $n \geq 0$, by a *word of length n* in $\tilde{T}$ we mean a product $g_1 g_2 \cdots g_n$ such that $\{g_i\}_{i=1}^n$ are idempotent generators for $\tilde{T}$ and g_{i-1}, g_i are alternating for $2 \leq i \leq n$. We interpret the word of length 0 to be the identity of $\tilde{T}$. We call this word *trivial.* Let $g_1 g_2 \cdots g_n$ denote a nontrivial word in $\tilde{T}$. We say that this word *begins* with g_1 and *ends* with g_n.

Referring to Definition 11.3, observe that $\tilde{T}$ is spanned by its words.

From the construction we have canonical $\mathbb{F}$-algebra homomorphisms $\check{T} \to \tilde{T} \to T$. We will investigate these homomorphisms in the following sections.

12 The homomorphism $\varphi : \check{T} \to \tilde{T}$

From now until the end of Lemma 12.18 the following notation will be in effect. Fix an integer $d \geq 0$ and let the algebra $\check{T} = \check{T}(d, \mathbb{F})$ be as in Definition 5.1. Fix a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$ and let the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ be as in Definition 11.1. We now consider how $\check{T}$ and $\tilde{T}$ are related.

Definition 12.1 Let $R = R(p)$ denote the two-sided ideal of $\check{T}$ generated by the elements

$$\epsilon_i^* a^k \epsilon_j^*, \qquad \epsilon_i a^{*k} \epsilon_j, \qquad\qquad 0 \leq i, j \leq d, \qquad 0 \leq k < |i - j|, \tag{30}$$

where $a = a(p)$ and $a^* = a^*(p)$ are from Definition 10.1.

Lemma 12.2 *There exists a surjective $\mathbb{F}$-algebra homomorphism $\varphi : \check{T} \to \tilde{T}$ that sends $\epsilon_i \mapsto \varepsilon_i$ and $\epsilon_i^* \mapsto \varepsilon_i^*$ for $0 \leq i \leq d$. The kernel of φ coincides with the ideal R.*

Proof: Compare the defining relations for $\check{T}$ and $\tilde{T}$. □

Our next goal is to display a spanning set for R. To this end we introduce a type of element in $\check{T}$ called a relator.

Definition 12.3 Let C (resp. C^*) denote the set of three-tuples (u, v, k) such that:

(i) each of u, v is a nontrivial word in $\check{T}$;

(ii) $\mathrm{end}(u)$ and $\mathrm{begin}(v)$ are both nonstarred (resp. both starred);

(iii) k is an integer such that $0 \leq k < |\overline{\mathrm{end}(u)} - \overline{\mathrm{begin}(v)}|$.

Observe that $C \cap C^* = \emptyset$.

Definition 12.4 With reference to Definition 12.3, to each element in $C \cup C^*$ we associate an element of $\check{T}$ called its *relator*. For $(u, v, k) \in C$ the corresponding relator is $ua^{*k}v$, where $a^* = a^*(p)$ is from Definition 10.1. For $(u, v, k) \in C^*$ the corresponding relator is $ua^k v$, where $a = a(p)$ is from Definition 10.1.

Lemma 12.5 *The $\mathbb{F}$-vector space R is spanned by the relators in $\check{T}$.*

Proof: By Definition 12.1 and since $\check{T}$ is spanned by its words. □

Lemma 12.6 *With reference to Definition 12.3, for $(u, v, k) \in C \cup C^*$ the corresponding relator is λ-homogeneous, where*

$$\mathrm{begin}(\lambda) = \mathrm{begin}(u), \qquad\qquad \mathrm{end}(\lambda) = \mathrm{end}(v),$$
$$\mathrm{length}(\lambda) = \mathrm{length}(u) + \mathrm{length}(v) + 1.$$

Proof: Use (27) and Definition 12.4. □

Definition 12.7 For $\lambda \in \Lambda$ let $R_\lambda = R_\lambda(p)$ denote the subspace of $\check{T}$ spanned by the λ-homogeneous relators. Observe that $R_\lambda \subseteq \check{T}_\lambda$.

Lemma 12.8 *The $\mathbb{F}$-vector space R decomposes as*

$$R = \sum_{\lambda \in \Lambda} R_\lambda \qquad \text{(direct sum).}$$

Proof: By Lemmas 12.5, 12.6 and Definition 12.7 we obtain $R = \sum_{\lambda \in \Lambda} R_\lambda$. The sum $\sum_{\lambda \in \Lambda} R_\lambda$ is direct by Proposition 7.9 and since $R_\lambda \subseteq \check{T}_\lambda$ for all $\lambda \in \Lambda$. □

Corollary 12.9 *For $\lambda \in \Lambda$ we have $R_\lambda = R \cap \check{T}_\lambda$.*

Proof: Observe that $R_\lambda \subseteq R$ by Lemma 12.8 and $R_\lambda \subseteq \check{T}_\lambda$ by Definition 12.7, so $R_\lambda \subseteq R \cap \check{T}_\lambda$. To obtain the reverse inclusion, we pick any $v \in R \cap \check{T}_\lambda$ and show $v \in R_\lambda$. By Lemma 12.8 there exists $r \in R_\lambda$ such that $v - r \in \sum_{\chi \in \Lambda \setminus \lambda} R_\chi$. We have $v - r \in \check{T}_\lambda$ by construction and the last sentence in Definition 12.7. Similarly $v - r \in \sum_{\chi \in \Lambda \setminus \lambda} \check{T}_\chi$. So $v - r$ is contained in the intersection of $\check{T}_\lambda$ and $\sum_{\chi \in \Lambda \setminus \lambda} \check{T}_\chi$. Now $v = r$ in view of Proposition 7.9, so $v \in R_\lambda$. We have shown $R_\lambda \supseteq R \cap \check{T}_\lambda$ and the result follows. □

Definition 12.10 For $\lambda \in \Lambda$ let $\tilde{T}_\lambda$ denote the image of $\check{T}_\lambda$ under the homomorphism φ from Lemma 12.2.

Proposition 12.11 *The $\mathbb{F}$-vector space $\tilde{T}$ decomposes as*

$$\tilde{T} = \sum_{\lambda \in \Lambda} \tilde{T}_\lambda \qquad \text{(direct sum).}$$

Proof: Recall the map $\varphi : \check{T} \to \tilde{T}$ from Lemma 12.2. To get $\tilde{T} = \sum_{\lambda \in \Lambda} \tilde{T}_\lambda$, apply φ to each side of (22) and evaluate the result using Definition 12.10 and the surjectivity of φ. To see that the sum $\sum_{\lambda \in \Lambda} \tilde{T}_\lambda$ is direct, we pick any $\lambda \in \Lambda$ and show that $\tilde{T}_\lambda$ has zero intersection with $\sum_{\chi \in \Lambda \setminus \lambda} \tilde{T}_\chi$. To this end we fix u in the intersection and show $u = 0$. By Definition 12.10 and since $u \in \tilde{T}_\lambda$, there exists $v \in \check{T}_\lambda$ such that $\varphi(v) = u$. By Definition 12.10 and since $u \in \sum_{\chi \in \Lambda \setminus \lambda} \tilde{T}_\chi$, there exists $v' \in \sum_{\chi \in \Lambda \setminus \lambda} \check{T}_\chi$ such that $\varphi(v') = u$. Observe that $\varphi(v - v') = 0$ so $v - v' \in R$. By Lemma 12.8 there exists $r \in R_\lambda$ and $r' \in \sum_{\chi \in \Lambda \setminus \lambda} R_\chi$ such that $v - v' = r - r'$. Observe that $v - r = v' - r'$. We have $v - r \in \check{T}_\lambda$ by construction and the last sentence of Definition 12.7. Similarly $v' - r' \in \sum_{\chi \in \Lambda \setminus \lambda} \check{T}_\chi$. Now $v = r$ and $v' = r'$ in view of Proposition 7.9. In the equation $v = r$ we apply φ to each side and get $u = 0$, as desired. We have shown that the sum $\sum_{\lambda \in \Lambda} \tilde{T}_\lambda$ is direct. □

Definition 12.12 For $\lambda \in \Lambda$ we call $\tilde{T}_\lambda$ the λ-*homogeneous component* of $\tilde{T}$. Elements of $\tilde{T}_\lambda$ are said to be λ-*homogeneous*. An element of $\tilde{T}$ is called *homogeneous* whenever it is λ-*homogeneous* for some $\lambda \in \Lambda$.

Proposition 12.13 [57, Theorem 8.1] *For all $\lambda \in \Lambda$ the map φ sends Z_λ onto $\tilde{T}_\lambda$.*

Proof: In [57, Theorem 8.1] it is proved that for an integer $n \geq 1$ and idempotent generators y, z of T the following sets have the same span:

(i) The words of length n in T that begin with y and end with z.

(ii) The zigzag words of length n in T that begin with y and end with z.

In that proof the relations (7) were never used; consequently the verbatim proof applies to $\tilde{T}$ as well, provided that we interpret things using Note 10.3. The result follows. $\square$

Lemma 12.14 *For $\lambda \in \Lambda$,*

$$\check{T}_\lambda = R_\lambda + Z_\lambda. \tag{31}$$

Proof: The space $\check{T}_\lambda$ contains R_λ by Definition 12.7, and it contains Z_λ by Definition 8.7. Consider the map $\varphi : \check{T} \to \tilde{T}$ from Lemma 12.2. By Definition 12.10 $\tilde{T}_\lambda$ is the image of $\check{T}_\lambda$ under φ. By Corollary 12.9 R_λ is the kernel of φ on $\check{T}_\lambda$. By Proposition 12.13 φ sends Z_λ onto $\tilde{T}_\lambda$. The result follows. $\square$

We conjecture that the sum (31) is direct for all $\lambda \in \Lambda$. For our present purpose the following weaker result will suffice. As part of our proof of Theorem 3.9 we will show that the sum (31) is direct for all $\lambda \in \Lambda_0$. We will say more about this in the next section. For the rest of this section we discuss some aspects of (31) that apply to all $\lambda \in \Lambda$.

Lemma 12.15 *The following hold for all $\lambda \in \Lambda$.*

(i) $\dim \check{T}_\lambda = \dim R_\lambda + \dim Z_\lambda - \dim (R_\lambda \cap Z_\lambda)$.

(ii) $\dim \tilde{T}_\lambda = \dim Z_\lambda - \dim (R_\lambda \cap Z_\lambda)$.

Proof: (i) By Lemma 12.14 and elementary linear algebra.
(ii) The action of φ on Z_λ is onto $\tilde{T}_\lambda$ and has kernel $R_\lambda \cap Z_\lambda$. $\square$

Corollary 12.16 *The following hold for all $\lambda \in \Lambda$.*

(i) $\dim \check{T}_\lambda \leq \dim R_\lambda + \dim Z_\lambda$.

(ii) $\dim \tilde{T}_\lambda \leq \dim Z_\lambda$.

Proof: Immediate from Lemma 12.15. $\square$

Pick $\lambda \in \Lambda$ and consider when is the sum (31) direct. Recall the sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ from the first paragraph of this section. Since R_λ depends on p, it is conceivable that the sum (31) is direct for some values of p but not others. It is also conceivable that the field $\mathbb{F}$ matters. The following definition will facilitate our discussion of these issues.

Definition 12.17 For $\lambda \in \Lambda$, we say that λ is $(p, \mathbb{F})$-*direct* whenever the sum (31) is direct.

Lemma 12.18 *For $\lambda \in \Lambda$ the following are equivalent:*

(i) $R_\lambda \cap Z_\lambda = 0$;

(ii) λ *is* $(p, \mathbb{F})$*-direct;*

(iii) *equality holds in Corollary 12.16(i);*

(iv) *equality holds in Corollary 12.16(ii);*

(v) *the restriction of* φ *to* Z_λ *is injective.*

Proof: (i) $\Leftrightarrow$ (ii) By Definition 12.17.
(i) $\Leftrightarrow$ (iii) By Lemma 12.15(i).
(i) $\Leftrightarrow$ (iv) By Lemma 12.15(ii).
(i) $\Leftrightarrow$ (v) The restriction of φ to Z_λ has kernel $R_\lambda \cap Z_\lambda$. $\square$

Proposition 12.19 *Assume* $\mathbb{F}$ *is infinite and pick an integer* $d \geq 3$*. Suppose we are given a type* $\lambda \in \Lambda$ *that is* $(p, \mathbb{F})$*-direct for all sequences* $p \in \mathrm{Rac}(d, \mathbb{F})$*. Then* λ *is* $(p, \mathbb{F})$*-direct for all sequences* $p \in \mathrm{Feas}(d, \mathbb{F})$*.*

Proof: For notational convenience abbreviate $d_\lambda = \dim \check{T}_\lambda - \dim Z_\lambda$. Recall the relators of $\check{T}$ from Definition 12.4. In the definition of a relator an element $p \in \mathrm{Feas}(d, \mathbb{F})$ is involved, so that relator can be viewed as a function of p. We adopt this point of view throughout the proof. Let $\mathcal{R}_\lambda$ denote the set of all λ-homogeneous relators in $\check{T}$. By Definition 12.7 we have $R_\lambda(p) = \mathrm{Span}\{\varrho(p) | \varrho \in \mathcal{R}_\lambda\}$ for all $p \in \mathrm{Feas}(d, \mathbb{F})$. We assume that there exists $p' \in \mathrm{Feas}(d, \mathbb{F})$ such that λ is not $(p', \mathbb{F})$-direct, and get a contradiction. By Lemma 12.18(ii),(iii) we have $\dim R_\lambda(p') \geq d_\lambda + 1$. By our above comments $R_\lambda(p') = \mathrm{Span}\{\varrho(p') | \varrho \in \mathcal{R}_\lambda\}$. Therefore there exists a subset $H \subseteq \mathcal{R}_\lambda$ such that (i) H has cardinality $d_\lambda + 1$; and (ii) the set $\{\varrho(p')\}_{\varrho \in H}$ is linearly independent. Pick any $p \in \mathrm{Feas}(d, \mathbb{F})$. Recall by Definition 7.8 that $\check{T}_\lambda$ has a basis consisting of the words of type λ. For $\varrho \in H$ write $\varrho(p)$ as a linear combination of these words, and let $M = M(p)$ denote the corresponding coefficient matrix. The rows of M are indexed by the words of type λ, and the columns of M are indexed by H. Each entry of M is a power of some θ_i or θ^*_i, where $p = (\{\theta_i\}_{i=0}^d; \{\theta^*_i\}_{i=0}^d)$. The matrix $M(p')$ has full rank $d_\lambda + 1$ since $\{\varrho(p')\}_{\varrho \in H}$ are linearly independent. Therefore there exists a set L consisting of words of type λ such that (i) L has cardinality $d_\lambda + 1$; and (ii) the rows of $M(p')$ indexed by L are linearly independent. For $p \in \mathrm{Feas}(d, \mathbb{F})$ let $N = N(p)$ denote the submatrix of $M(p)$ obtained by deleting all rows not indexed by L. By construction N is $(d_\lambda + 1) \times (d_\lambda + 1)$, and its determinant is a polynomial in p that has all coefficients in $\mathbb{F}$. Denote this polynomial by h. By construction $N(p')$ is nonsingular so $h(p') \neq 0$. We will obtain a contradiction by showing that $h(p') = 0$. To this end we will show that $h(p) = 0$ for all $p \in \mathrm{Rac}(d, \mathbb{F})$, and invoke Proposition 4.5. Pick any $p \in \mathrm{Rac}(d, \mathbb{F})$. By assumption λ is $(p, \mathbb{F})$-direct. So $\dim R_\lambda(p) = d_\lambda$ in view of Lemma 12.18(ii),(iii). Now $N(p)$ is singular and hence $h(p) = 0$. We have shown that $h(p) = 0$ for all $p \in \mathrm{Rac}(d, \mathbb{F})$. Now by Proposition 4.5, $h(p) = 0$ for all $p \in \mathrm{Feas}(d, \mathbb{F})$. In particular $h(p') = 0$, for a contradiction. The result follows. $\square$

13 The algebra $\varepsilon_0^*\tilde{T}\varepsilon_0^*$

Throughout this section fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Recall the algebra $\check{T} = \check{T}(d, \mathbb{F})$ from Definition 5.1 and the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ from Definition 11.1. Observe that $\varepsilon_0^*\tilde{T}\varepsilon_0^*$ is an $\mathbb{F}$-algebra with multiplicative identity ε_0^*.

Recall the map $\varphi : \check{T} \to \tilde{T}$ from Lemma 12.2.

Lemma 13.1 *The restriction of φ to $\epsilon_0^*\check{T}\epsilon_0^*$ gives a surjective $\mathbb{F}$-algebra homomorphism $\epsilon_0^*\check{T}\epsilon_0^* \to \varepsilon_0^*\tilde{T}\varepsilon_0^*$.*

Proof: By Lemma 12.2 the map $\varphi : \check{T} \to \tilde{T}$ is a surjective $\mathbb{F}$-algebra homomorphism that sends $\epsilon_0^* \mapsto \varepsilon_0^*$. The result follows. $\square$

Proposition 13.2 *The $\mathbb{F}$-vector space $\varepsilon_0^*\tilde{T}\varepsilon_0^*$ decomposes as*

$$\varepsilon_0^*\tilde{T}\varepsilon_0^* = \sum_{n=0}^{\infty} \tilde{T}_{[n]} \qquad \text{(direct sum)}. \tag{32}$$

Moreover $\tilde{T}_{[m]} \cdot \tilde{T}_{[n]} \subseteq \tilde{T}_{[m+n]}$ for all integers $m, n \geq 0$.

Proof: To get $\varepsilon_0^*\tilde{T}\varepsilon_0^* = \sum_{n=0}^{\infty} \tilde{T}_{[n]}$, apply φ to each side of (23) and evaluate the result using Definition 12.10 and Lemma 13.1. The sum $\sum_{n=0}^{\infty} \tilde{T}_{[n]}$ is direct by Proposition 12.11. The last assertion follows from the last assertion of Proposition 9.7. $\square$

Lemma 13.3 *The elements $\{\varepsilon_0^*\varepsilon_i\varepsilon_0^*\}_{i=0}^d$ mutually commute.*

Proof: In [54, Theorem 2.4] it was proved that $\{e_0^*e_ie_0^*\}_{i=0}^d$ commute in T. In that proof the relations (7) were never used. Consequently the verbatim proof applies to the elements $\{\varepsilon_0^*\varepsilon_i\varepsilon_0^*\}_{i=0}^d$ of $\tilde{T}$, provided that we interpret things using Note 10.3. $\square$

Let $\{x_i\}_{i=0}^d$ denote mutually commuting indeterminates. Let $\mathbb{F}[x_0, \ldots, x_d]$ denote the $\mathbb{F}$-algebra consisting of the polynomials in $\{x_i\}_{i=0}^d$ that have all coefficients in $\mathbb{F}$. We abbreviate $P = \mathbb{F}[x_0, \ldots, x_d]$.

Corollary 13.4 *There exists an $\mathbb{F}$-algebra homomorphism $\tilde{\nu} : P \to \varepsilon_0^*\tilde{T}\varepsilon_0^*$ that sends $x_i \mapsto \varepsilon_0^*\varepsilon_i\varepsilon_0^*$ for $0 \leq i \leq d$.*

Proof: Immediate from Lemma 13.3. $\square$

In Section 18 we will show that the map $\tilde{\nu}$ from Corollary 13.4 is an isomorphism. For the time being, our goal is to show that $\tilde{\nu}$ is an isomorphism if and only if $[n]$ is $(p, \mathbb{F})$-direct for all integers $n \geq 0$.

As we discuss the algebra P the following notation will be helpful. We call $\{x_i\}_{i=0}^d$ the *generators* for P. For a generator x_i of P we call i the *index* of x_i. For a generator y of P let

$\overline{y}$ denote the index of y. For an integer $n \geq 0$, by a *monomial of degree n* in P we mean an element $y_1 y_2 \cdots y_n$ such that y_i is a generator of P for $1 \leq i \leq n$. For notational convenience we always order the factors such that $\overline{y}_{i-1} \geq \overline{y}_i$ for $2 \leq i \leq n$. We interpret the monomial of degree 0 to be the identity of P. Observe that the $\mathbb{F}$-vector space P has a basis consisting of its monomials. For $n \geq 0$ let P_n denote the subspace of P with a basis consisting of the monomials of degree n. We have

$$P = \sum_{n=0}^{\infty} P_n \qquad \text{(direct sum)}. \tag{33}$$

Moreover $P_m P_n = P_{m+n}$ for all $m, n \geq 0$. We call P_n the *nth homogeneous component* of P.

Definition 13.5 We define an $\mathbb{F}$-linear map $\natural : P \to \epsilon_0^* \check{T} \epsilon_0^*$. To do this we give the action of $\natural$ on the monomial basis for P. By definition $\natural$ sends $1 \mapsto \epsilon_0^*$. For each nontrivial monomial $y_1 y_2 \cdots y_n$ in P the image under $\natural$ is $\epsilon_0^* y_1' \epsilon_0^* y_2' \epsilon_0^* \cdots \epsilon_0^* y_n' \epsilon_0^*$, where $x_i' = \epsilon_i$ for $0 \leq i \leq d$.

We caution the reader that $\natural$ is not an algebra homomorphism in general.

Lemma 13.6 *For an integer $n \geq 0$ the map $\natural$ from Definition 13.5 induces a bijection between the following two sets:*

(i) *the monomials in P that have degree n;*

(ii) *the zigzag words in $\check{T}$ of type $[n]$.*

Proof: Compare Proposition 9.8 and Definition 13.5. □

Lemma 13.7 *The map $\natural$ from Definition 13.5 is an injection. For $n \geq 0$ the image of P_n under $\natural$ is equal to $Z_{[n]}$.*

Proof: The monomials in P form a basis for P. The zigzag words in $\check{T}$ are linearly independent. By these comments and Lemma 13.6 the map $\natural$ is injective. To get the last assertion, note that in Lemma 13.6 the set (i) is a basis for P_n and the set (ii) is a basis for $Z_{[n]}$. □

Lemma 13.8 *Let φ' denote the restriction of φ to $\epsilon_0^* \check{T} \epsilon_0^*$. Then the following diagram commutes:*

$$\begin{tikzcd}
P \arrow[r, "\mathrm{id}"] \arrow[d, "\natural"'] & P \arrow[d, "\tilde{\nu}"] \\
\epsilon_0^* \check{T} \epsilon_0^* \arrow[r, "\varphi'"'] & \varepsilon_0^* \tilde{T} \varepsilon_0^*
\end{tikzcd}$$

Proof: The map $\natural$ is from Definition 13.5 and the map $\tilde{\nu}$ is from Corollary 13.4. The map φ is from Lemma 12.2. The result is a routine consequence of how these maps are defined. □

Lemma 13.9 *For an integer $n \geq 0$ the image of P_n under $\tilde{\nu}$ is $\tilde{T}_{[n]}$.*

Proof: By Lemma 13.8 the composition $\varphi' \circ \natural$ equals $\tilde{\nu}$. By Lemma 13.7 the image of P_n under $\natural$ is $Z_{[n]}$. By Lemma 12.13 the space $\tilde{T}_{[n]}$ is the image of $Z_{[n]}$ under φ and hence φ'. The result follows. $\square$

Lemma 13.10 *The map $\tilde{\nu}$ from Corollary 13.4 is surjective.*

Proof: In the equation (33) apply $\tilde{\nu}$ to each side, and evaluate the result using Lemma 13.9 and then Proposition 13.2. $\square$

Lemma 13.11 *For an integer $n \geq 0$ the following are equivalent:*

(i) *the restriction $\tilde{\nu}$ to P_n is injective;*

(ii) *the type $[n]$ is $(p, \mathbb{F})$-direct in the sense of Definition 12.17.*

Proof: Consider the commuting diagram in Lemma 13.8. By Lemma 13.7 the map $\natural$ is an injection that sends P_n onto $Z_{[n]}$. Therefore the restriction of $\tilde{\nu}$ to P_n is injective if and only if the restriction of φ to $Z_{[n]}$ is injective. By Lemma 12.18 the restriction of φ to $Z_{[n]}$ is injective if and only if $[n]$ is $(p, \mathbb{F})$-direct. The result follows. $\square$

Lemma 13.12 *The kernel of $\tilde{\nu}$ decomposes as follows:*

$$\ker(\tilde{\nu}) = \sum_{n=0}^{\infty} \big(\ker(\tilde{\nu}) \cap P_n\big). \tag{34}$$

Proof: The inclusion $\supseteq$ is clear, so consider the inclusion $\subseteq$. Pick $h \in \ker(\tilde{\nu})$. By (33) there exists an integer $m \geq 0$ and a sequence $\{h_n\}_{n=0}^{m}$ such that $h_n \in P_n$ for $0 \leq n \leq m$ and $h = \sum_{n=0}^{m} h_n$. In this equation we apply $\tilde{\nu}$ to each term and get $0 = \sum_{n=0}^{m} \tilde{\nu}(h_n)$. By Lemma 13.9 we have $\tilde{\nu}(h_n) \in \tilde{T}_{[n]}$ for $0 \leq n \leq m$. By these comments and (32) we obtain $\tilde{\nu}(h_n) = 0$ for $0 \leq n \leq m$. So for $0 \leq n \leq m$ the polynomial h_n is contained in the n-summand on the right in (34). Therefore h is contained in the sum on the right in (34). We have verified the inclusion $\subseteq$ and the result follows. $\square$

Proposition 13.13 *The following are equivalent:*

(i) *the map $\tilde{\nu}$ from Corollary 13.4 is an isomorphism;*

(ii) *for all integers $n \geq 0$ the type $[n]$ is $(p, \mathbb{F})$-direct in the sense of Definition 12.17.*

Proof: The map $\tilde{\nu}$ is surjective by Lemma 13.10, so $\tilde{\nu}$ is an isomorphism if and only if $\tilde{\nu}$ is injective. By Lemma 13.12 the map $\tilde{\nu}$ is injective if and only if its restriction to P_n is injective for all $n \geq 0$. The result follows from these comments and Lemma 13.11. $\square$

14 A central element of $\tilde{T}$

Recall that the algebra T from Definition 3.4 is defined using relations (6)–(10). So far we have investigated all these relations except (7). We now prepare to bring in the relations (7).

Throughout this section we fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Recall the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ from Definition 11.1.

Definition 14.1 Define $\Delta \in \tilde{T}$ and $\Delta^* \in \tilde{T}$ by

$$\Delta = 1 - \sum_{i=0}^{d} \varepsilon_i, \qquad \Delta^* = 1 - \sum_{i=0}^{d} \varepsilon_i^*.$$

The elements Δ and Δ^* are nonzero by Proposition 12.11.

Lemma 14.2 *We have $\Delta^2 = \Delta$ and $\Delta^{*2} = \Delta^*$. Moreover*

$$\varepsilon_i \Delta = \Delta \varepsilon_i = 0, \qquad \varepsilon_i^* \Delta^* = \Delta^* \varepsilon_i^* = 0 \qquad 0 \leq i \leq d. \tag{35}$$

Proof: Line (35) follows from (28) and Definition 14.1. To obtain $\Delta^2 = \Delta$, observe that $\Delta(1 - \Delta) = \sum_{i=0}^{d} \Delta \varepsilon_i = 0$. The equation $\Delta^{*2} = \Delta^*$ is similarly obtained. □

Lemma 14.3 *For $0 \leq i, j \leq d$ with $i \neq j$,*

$$\varepsilon_i^* \Delta \varepsilon_j^* = 0, \qquad \varepsilon_i \Delta^* \varepsilon_j = 0.$$

Proof: By Definition 14.1 and since $\varepsilon_i^* \varepsilon_j^* = 0$, we find $\varepsilon_i^* \Delta \varepsilon_j^* = -\sum_{\ell=0}^{d} \varepsilon_i^* \varepsilon_\ell \varepsilon_j^*$. Setting $k = 0$ in the equation on the left in (29) we find $0 = \sum_{\ell=0}^{d} \varepsilon_i^* \varepsilon_\ell \varepsilon_j^*$. Therefore $\varepsilon_i^* \Delta \varepsilon_j^* = 0$. The equation $\varepsilon_i \Delta^* \varepsilon_j = 0$ is similarly obtained. □

Definition 14.4 Define $\psi \in \tilde{T}$ by

$$\psi = (\Delta - \Delta^*)^2, \tag{36}$$

where Δ, Δ^* are from Definition 14.1.

An element of an algebra is called *central* whenever it commutes with everything in the algebra. Our next goal is to show that ψ is central.

Lemma 14.5 *The element ψ coincides with each of the following:*

(i) $\Delta + \Delta^* - \Delta\Delta^* - \Delta^*\Delta$;

(ii) $\Delta(\varepsilon_0^* + \varepsilon_1^* + \cdots + \varepsilon_d^*) + \Delta^*(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_d)$;

(iii) $(\varepsilon_0^* + \varepsilon_1^* + \cdots + \varepsilon_d^*)\Delta + (\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_d)\Delta^*$.

Proof: (i) Multiply out the right-hand side of (36), and simplify the result using $\Delta^2 = \Delta$ and $\Delta^{*2} = \Delta^*$.
(ii) In the given expression eliminate $\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_d$ and $\varepsilon_0^* + \varepsilon_1^* + \cdots + \varepsilon_d^*$ using Definition 14.1, and compare the result with (i) above.
(iii) Similar to the proof of (ii) above. $\square$

Lemma 14.6 *The following hold for $0 \le i \le d$.*

(i) *Each of $\varepsilon_i\psi$, $\psi\varepsilon_i$ is equal to $\varepsilon_i\Delta^*\varepsilon_i$.*

(ii) *Each of $\varepsilon_i^*\psi$, $\psi\varepsilon_i^*$ is equal to $\varepsilon_i^*\Delta\varepsilon_i^*$.*

Proof: (i) Evaluating $\varepsilon_i\psi$ using Lemma 14.5(ii) and $\varepsilon_i\Delta = 0$, we obtain $\varepsilon_i\psi = \varepsilon_i\Delta^*(\varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_d)$. By Lemma 14.3, for $0 \le j \le d$ we have $\varepsilon_i\Delta^*\varepsilon_j = 0$ provided $i \ne j$. By these comments $\varepsilon_i\psi = \varepsilon_i\Delta^*\varepsilon_i$. Using Lemma 14.5(iii) we similarly find $\psi\varepsilon_i = \varepsilon_i\Delta^*\varepsilon_i$.
(ii) Similar to the proof of (i) above. $\square$

Corollary 14.7 *The element ψ is central in $\tilde{T}$.*

Proof: The elements $\{\varepsilon_i\}_{i=0}^d$, $\{\varepsilon_i^*\}_{i=0}^d$ together generate $\tilde{T}$, and each of these elements commutes with ψ by Lemma 14.6. $\square$

For later use we summarize Lemma 14.3 and Lemma 14.6.

Lemma 14.8 *For $0 \le i, j \le d$ we have*

$$\varepsilon_i\Delta^*\varepsilon_j = \delta_{i,j}\psi\varepsilon_i, \qquad \varepsilon_i^*\Delta\varepsilon_j^* = \delta_{i,j}\psi\varepsilon_i^*.$$

15 The homomorphism $\pi : \tilde{T} \to T$

Throughout this section fix an integer $d \ge 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Recall the algebras $T = T(p, \mathbb{F})$ from Definition 3.4 and $\tilde{T} = \tilde{T}(p, \mathbb{F})$ from Definition 11.1.

Definition 15.1 Let J denote the two-sided ideal of $\tilde{T}$ generated by the elements Δ, Δ^* from Definition 14.1.

Lemma 15.2 *There exists a surjective $\mathbb{F}$-algebra homomorphism $\pi : \tilde{T} \to T$ that sends $\varepsilon_i \mapsto e_i$ and $\varepsilon_i^* \mapsto e_i^*$ for $0 \le i \le d$. The kernel of π coincides with the ideal J.*

Proof: Compare the defining relations for $\tilde{T}$ and T. $\square$

We will be discussing the action of π on $\varepsilon_0^*\tilde{T}\varepsilon_0^*$.

Lemma 15.3 *The restriction of π to $\varepsilon_0^* \tilde{T} \varepsilon_0^*$ gives a surjective $\mathbb{F}$-algebra homomorphism $\varepsilon_0^* \tilde{T} \varepsilon_0^* \to e_0^* T e_0^*$.*

Proof: The map π is a surjective $\mathbb{F}$-algebra homomorphism that sends $\varepsilon_0^* \mapsto e_0^*$. □

Proposition 15.4 *The following are equal:*

(i) *the kernel of π on $\varepsilon_0^* \tilde{T} \varepsilon_0^*$;*

(ii) *the intersection of J and $\varepsilon_0^* \tilde{T} \varepsilon_0^*$;*

(iii) *$\varepsilon_0^* J \varepsilon_0^*$;*

(iv) *the ideal of $\varepsilon_0^* \tilde{T} \varepsilon_0^*$ generated by $\varepsilon_0^* \Delta \varepsilon_0^*$.*

Proof: The spaces (i), (ii) are equal by the last assertion of Lemma 15.2. We now show that the spaces (ii)–(iv) are equal. Let J' denote the ideal of $\varepsilon_0^* \tilde{T} \varepsilon_0^*$ generated by $\varepsilon_0^* \Delta \varepsilon_0^*$.
$J \cap \varepsilon_0^* \tilde{T} \varepsilon_0^* \subseteq \varepsilon_0^* J \varepsilon_0^*$: For $u \in \varepsilon_0^* \tilde{T} \varepsilon_0^*$ we have $u = \varepsilon_0^* u \varepsilon_0^*$ since $\varepsilon_0^{*2} = \varepsilon_0^*$.
$\varepsilon_0^* J \varepsilon_0^* \subseteq J'$: Let J_1 (resp. J_2) denote the two-sided ideal of $\tilde{T}$ generated by Δ (resp. Δ^*). By construction $J = J_1 + J_2$, so $\varepsilon_0^* J \varepsilon_0^* = \varepsilon_0^* J_1 \varepsilon_0^* + \varepsilon_0^* J_2 \varepsilon_0^*$. We now show that $\varepsilon_0^* J_1 \varepsilon_0^* \subseteq J'$. The space $\varepsilon_0^* J_1 \varepsilon_0^*$ is spanned by elements of the form $u \Delta v$ where u (resp. v) is a nontrivial word in $\tilde{T}$ that begins with ε_0^* (resp. ends with ε_0^*). We show that such an element $u\Delta v$ is contained in J'. Suppose for the moment that u ends with a nonstarred idempotent generator, which we denote by ε_i. Then $u\Delta = 0$ since $\varepsilon_i \Delta = 0$. Therefore we may assume that u ends with a starred idempotent generator, which we denote ε_i^*. Note that $u = u\varepsilon_i^*$ since $\varepsilon_i^{*2} = \varepsilon_i^*$. Suppose for the moment that v begins with a nonstarred idempotent generator, which we denote by ε_j. Then $\Delta v = 0$ since $\Delta \varepsilon_j = 0$. Therefore we may assume that v begins with a starred idempotent generator, which we denote by ε_j^*. Note that $v = \varepsilon_j^* v$ since $\varepsilon_j^{*2} = \varepsilon_j^*$. We may now argue

$$\begin{aligned} u\Delta v &= u\varepsilon_i^* \Delta \varepsilon_j^* v \\ &= \delta_{i,j} u \psi \varepsilon_i^* v && \text{(by Lemma 14.8)} \\ &= \delta_{i,j} \psi u \varepsilon_i^* v && \text{(by Corollary 14.7)} \\ &= \delta_{i,j} \psi u v. \end{aligned}$$

Since u begins with ε_0^* and $\varepsilon_0^{*2} = \varepsilon_0^*$ we find $u = \varepsilon_0^* u$. Also $\psi \varepsilon_0^* = \varepsilon_0^* \Delta \varepsilon_0^*$ by Lemma 14.8. Therefore $\psi u v = \varepsilon_0^* \Delta \varepsilon_0^* u v$. Since u begins with ε_0^* and v ends with ε_0^* we find $uv \in \varepsilon_0^* \tilde{T} \varepsilon_0^*$. Consequently J' contains $\psi u v$ and hence $u \Delta v$. We have shown $\varepsilon_0^* J_1 \varepsilon_0^* \subseteq J'$. One similarly shows that $\varepsilon_0^* J_2 \varepsilon_0^* \subseteq J'$.
$J' \subseteq J \cap \varepsilon_0^* \tilde{T} \varepsilon_0^*$: Observe that $J' \subseteq J$ since $\Delta \in J$, and $J' \subseteq \varepsilon_0^* \tilde{T} \varepsilon_0^*$ by construction. □

16 Comparing $\varepsilon_0^*\tilde{T}\varepsilon_0^*$ and $e_0^*Te_0^*$

Throughout this section fix an integer $d \geq 0$ and a sequence $p = (\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Feas}(d, \mathbb{F})$. Recall the algebras $T = T(p, \mathbb{F})$ from Definition 3.4 and $\tilde{T} = \tilde{T}(p, \mathbb{F})$ from Definition 11.1. We will be comparing the map $\mu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^*Te_0^*$ from Corollary 3.8, and the map $\tilde{\nu} : P \to \varepsilon_0^*\tilde{T}\varepsilon_0^*$ from Corollary 13.4. We will show that μ is an isomorphism if and only if $\tilde{\nu}$ is an isomorphism.

In order to compare μ and $\tilde{\nu}$ it is helpful to introduce the following map.

Definition 16.1 Define an $\mathbb{F}$-algebra homomorphism $\nu : \mathbb{F}[x_1, \ldots, x_d] \to e_0^*Te_0^*$ that sends $x_i \to e_0^*e_ie_0^*$ for $1 \leq i \leq d$.

Our next goal is to compare μ and ν. After that, we will compare ν and $\tilde{\nu}$.

Definition 16.2 Define an $\mathbb{F}$-algebra homomorphism $\phi : \mathbb{F}[x_1, \ldots, x_d] \to \mathbb{F}[x_1, \ldots, x_d]$ that sends $x_i \to \sum_{j=1}^d \tau_i(\theta_j)x_j$ for $1 \leq i \leq d$. (The τ_i are from Definition 2.4).

Lemma 16.3 *The map ϕ from Definition 16.2 is an isomorphism.*

Proof: Consider the $d \times d$ matrix that has (i, j)-entry $\tau_i(\theta_j)$ for $1 \leq i, j \leq d$. This matrix is upper triangular and has all diagonal entries nonzero. Therefore the matrix is invertible. The result follows. □

Lemma 16.4 *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{F}[x_1, \ldots, x_d] & \xrightarrow{\phi} & \mathbb{F}[x_1, \ldots, x_d] \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ e_0^*Te_0^* & \xrightarrow[\mathrm{id}]{} & e_0^*Te_0^* \end{array}$$

Proof: For $1 \leq i \leq d$ we chase x_i around the diagram. The image of x_i under the composition $\nu \circ \phi$ is $\sum_{j=1}^d \tau_i(\theta_j)e_0^*e_je_0^*$. The image of x_i under μ is $e_0^*\tau_i(a)e_0^*$, and this is equal to $\sum_{j=0}^d \tau_i(\theta_j)e_0^*e_je_0^*$. In this sum the $j = 0$ summand is zero; indeed $\tau_i(\theta_0) = 0$ since $i \geq 1$. Therefore x_i has the same image under $\nu \circ \phi$ and μ. The result follows. □

Corollary 16.5 *The map ν is surjective.*

Proof: The map μ is surjective by Corollary 3.8. The result follows from this and Lemma 16.4. □

Proposition 16.6 *The following are equivalent:*

(i) *the map μ is an isomorphism;*

(ii) *the map ν is an isomorphism.*

Proof: Combine Lemma 16.3 and Lemma 16.4. $\square$

Our next goal is to compare ν and $\tilde{\nu}$.

Definition 16.7 Let K denote the ideal of P generated by $1 - \sum_{i=0}^{d} x_i$.

We identify $\mathbb{F}[x_1, \ldots, x_d]$ with the $\mathbb{F}$-subalgebra of P generated by $\{x_i\}_{i=1}^{d}$.

Lemma 16.8 *The $\mathbb{F}$-vector space P decomposes as*

$$P = K + \mathbb{F}[x_1, \ldots, x_d] \qquad \text{(direct sum)}. \tag{37}$$

Proof: Let K_0 denote the ideal of P generated by x_0. Observe that the $\mathbb{F}$-vector space P decomposes as

$$P = K_0 + \mathbb{F}[x_1, \ldots, x_d] \qquad \text{(direct sum)}. \tag{38}$$

Define an $\mathbb{F}$-algebra homomorphism $\sigma : P \to P$ that sends $x_0 \mapsto 1 - \sum_{i=0}^{d} x_i$ and fixes x_j for $1 \leq j \leq d$. The composition of σ with itself is the identity, so σ is an isomorphism. To obtain (37), apply σ to each side of (38) and note that σ sends K_0 to K while leaving $\mathbb{F}[x_1, \ldots, x_d]$ invariant. $\square$

Lemma 16.9 *We have $K \cap P_n = 0$ for $n \geq 0$.*

Proof: For notational convenience abbreviate $y = \sum_{i=0}^{d} x_i$. We assume that there exists a nonzero $f \in K \cap P_n$ and get a contradiction. Since $f \in K$ there exists $h \in P$ such that $f = (1-y)h$. Observe that $h \neq 0$ since $f \neq 0$. By (33) there exists an integer $m \geq 0$ and polynomials $\{h_i\}_{i=0}^{m}$ in P such that $h_i \in P_i$ for $0 \leq i \leq m$ and $h = \sum_{i=0}^{m} h_i$. Without loss we may assume $h_m \neq 0$. We define some polynomials $\{h'_i\}_{i=0}^{m+1}$ as follows:

$$h'_0 = h_0, \qquad h'_i = h_i - y h_{i-1} \ \ (1 \leq i \leq m), \qquad h'_{m+1} = -y h_m.$$

Note that $h'_i \in P_i$ for $0 \leq i \leq m+1$, and $f = \sum_{i=0}^{m+1} h'_i$. Observe that $h'_{m+1} \neq 0$ since P is a domain and each of y, h_m is nonzero. By these comments and since $f \in P_n$ we find $n = m+1$, $f = h'_{m+1}$, and $h'_i = 0$ for $0 \leq i \leq m$. Since the $\{h'_i\}_{i=0}^{m}$ are all zero we have $h_0 = 0$ and $h_i = y h_{i-1}$ for $1 \leq i \leq m$. Therefore $h_i = 0$ for $0 \leq i \leq m$. In particular $h_m = 0$, for a contradiction. The result follows. $\square$

Lemma 16.10 *The following are equal:*

(i) *the image of K under $\tilde{\nu}$;*

(ii) *the ideal of $\varepsilon_0^* \tilde{T} \varepsilon_0^*$ generated by $\varepsilon_0^* \Delta \varepsilon_0^*$.*

Proof: By Corollary 13.4 and Definition 14.1, the image of $1 - \sum_{i=0}^{d} x_i$ under $\tilde{\nu}$ is equal to $\varepsilon_0^* \Delta \varepsilon_0^*$. By Lemma 13.10 the map $\tilde{\nu}$ is surjective. The result follows from these comments and Definition 16.7. $\square$

Lemma 16.11 *Let π' denote the restriction of π to $\varepsilon_0^* \tilde{T} \varepsilon_0^*$. Let $\iota : \mathbb{F}[x_1, \ldots, x_d] \to P$ denote the inclusion map. Then the following diagram commutes:*

$$\begin{tikzcd} P \arrow[d, "\tilde{\nu}"'] & \mathbb{F}[x_1, \ldots, x_d] \arrow[l, "\iota"'] \arrow[d, "\nu"] \\ \varepsilon_0^* \tilde{T} \varepsilon_0^* \arrow[r, "\pi'"'] & e_0^* T e_0^* \end{tikzcd}$$

Proof: For $1 \leq i \leq d$ we chase x_i around the diagram. The image of x_i under the composition $\tilde{\nu} \circ \iota$ is $\varepsilon_0^* \varepsilon_i \varepsilon_0^*$, and the image of this under π' is $e_0^* e_i e_0^*$. The image of x_i under ν is $e_0^* e_i e_0^*$. The result follows. $\square$

Proposition 16.12 *The following are equivalent:*

(i) *the map ν is an isomorphism;*

(ii) *the map $\tilde{\nu}$ is an isomorphism.*

Proof: (i) $\Rightarrow$ (ii) The map $\tilde{\nu}$ is surjective by Lemma 13.10. We show that $\tilde{\nu}$ is injective. By Lemma 13.12 it suffices to show that $\tilde{\nu}$ is injective on P_n for all integers $n \geq 0$. Let n be given, and pick any $f \in P_n$ such that $\tilde{\nu}(f) = 0$. We show $f = 0$. Invoking Lemma 16.8 we write $f = k + h$ with $k \in K$ and $h \in \mathbb{F}[x_1, \ldots, x_d]$. In the equation $f = k + h$ we apply the composition $\pi \circ \tilde{\nu}$ to each term. The image of f under $\pi \circ \tilde{\nu}$ is zero since $\tilde{\nu}(f) = 0$. The image of k under $\pi \circ \tilde{\nu}$ is zero by Lemma 16.10 and Proposition 15.4(i),(iv). The image of h under $\pi \circ \tilde{\nu}$ is $\nu(h)$ by Lemma 16.11. By these comments $\nu(h) = 0$. We assume ν is an isomomorphism so $h = 0$. Therefore $f = k \in K$. We have $f \in K$ and $f \in P_n$, so $f = 0$ in view of Lemma 16.9.
(ii) $\Rightarrow$ (i) The map ν is surjective by Corollary 16.5. We show that ν is injective. Suppose we are given $h \in \mathbb{F}[x_1, \ldots, x_d]$ such that $\nu(h) = 0$. We show $h = 0$. By Lemma 16.11 the composition $\pi \circ \tilde{\nu}$ sends $h \mapsto 0$. Therefore $\tilde{\nu}(h)$ is in the kernel of π. By this and Proposition 15.4(i),(iv) we see that $\tilde{\nu}(h)$ is in the ideal of $\varepsilon_0^* \tilde{T} \varepsilon_0^*$ generated by $\varepsilon_0^* \Delta \varepsilon_0^*$. Now $h \in K$ by Lemma 16.10 and since $\tilde{\nu}$ is an isomorphism. We have $h \in K$ and $h \in \mathbb{F}[x_1, \ldots, x_d]$, so $h = 0$ in view of Lemma 16.8. $\square$

Corollary 16.13 *The following are equivalent:*

(i) *the map μ from Corollary 3.8 is an isomorphism;*

(ii) *the map $\tilde{\nu}$ from Corollary 13.4 is an isomorphism.*

Proof: Combine Proposition 16.6 and Proposition 16.12. $\square$

17 The proof of Theorem 3.1 and Theorem 3.9

In this section we prove Theorem 3.1 and Theorem 3.9.

Throughout this section fix an integer $d \geq 0$. Recall the sets $\mathrm{Feas}(d, \mathbb{F})$ from Definition 3.3 and $\mathrm{Rac}(d, \mathbb{F})$ from Definition 4.3.

Definition 17.1 Pick any sequence $p \in \mathrm{Feas}(d, \mathbb{F})$ and consider the ordered pair $(p, \mathbb{F})$. This pair is said to be *confirmed* whenever Theorem 3.9 is true for that p and $\mathbb{F}$.

Lemma 17.2 [56, Theorem 12.1] *Assume $d \leq 5$. Then $(p, \mathbb{F})$ is confirmed for all $p \in \mathrm{Feas}(d, \mathbb{F})$.*

Lemma 17.3 [58, Theorem 5.3] *The pair $(p, \mathbb{F})$ is confirmed for all $p \in \mathrm{Rac}(d, \mathbb{F})$.*

Lemma 17.4 [58, Theorem 5.2] *Pick any $p \in \mathrm{Feas}(d, \mathbb{F})$. If there exists a field extension $\mathbb{K}$ of $\mathbb{F}$ such that $(p, \mathbb{K})$ is confirmed, then $(p, \mathbb{F})$ is confirmed.*

Proof of Theorem 3.9: We will confirm the pair $(p, \mathbb{F})$ in the sense of Definition 17.1. We may assume $d \geq 3$; otherwise $(p, \mathbb{F})$ is confirmed by Lemma 17.2. Abbreviate $\mathbb{K} = \overline{\mathbb{F}}$ for the algebraic closure of $\mathbb{F}$, and note that $\mathbb{K}$ is infinite. By Lemma 17.3 the pair $(p', \mathbb{K})$ is confirmed for all sequences $p' \in \mathrm{Rac}(d, \mathbb{K})$. Now by Proposition 13.13 and Corollary 16.13, the type $[n]$ is $(p', \mathbb{K})$-direct for all integers $n \geq 0$ and all sequences $p' \in \mathrm{Rac}(d, \mathbb{K})$. Now by Proposition 12.19 and since $\mathbb{K}$ is infinite, the type $[n]$ is $(p', \mathbb{K})$-direct for all integers $n \geq 0$ and all sequences $p' \in \mathrm{Feas}(d, \mathbb{K})$. In particular the type $[n]$ is $(p, \mathbb{K})$-direct for all integers $n \geq 0$. Now by Proposition 13.13 and Corollary 16.13, the pair $(p, \mathbb{K})$ is confirmed. Now by Lemma 17.4 the pair $(p, \mathbb{F})$ is confirmed. $\square$

Proof of Theorem 3.1: Immediate from Theorem 3.9 and [56, Theorem 10.1]. $\square$

18 Comments

In the previous section we proved Theorem 3.1 and Theorem 3.9. In this section we list some related results that might be of independent interest. We also mention a conjecture.

The following is a corollary to Theorem 3.1.

Corollary 18.1 *Assume the field $\mathbb{F}$ is algebraically closed. Let d denote a nonnegative integer and let*

$$(\{\theta_i\}_{i=0}^d; \{\theta_i^*\}_{i=0}^d; \{\zeta_i\}_{i=0}^d) \tag{39}$$

denote a sequence of scalars taken from $\mathbb{F}$. Then there exists a TD system Φ over $\mathbb{F}$ with parameter array (39) *if and only if* (i)–(iii) *hold below.*

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.

(ii) *The expressions*

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$.

(iii) $\zeta_0 = 1$, $\zeta_d \neq 0$, *and*

$$0 \neq \sum_{i=0}^{d} \eta_{d-i}(\theta_0)\eta^*_{d-i}(\theta^*_0)\zeta_i.$$

Suppose (i)–(iii) *hold. Then Φ is unique up to isomorphism of TD systems.*

Proof: By Theorem 3.1 and since every tridiagonal system over an algebraically closed field is sharp [55, Theorem 1.3]. □

Theorem 18.2 *Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let the algebra $T = T(p, \mathbb{F})$ be as in Definition 3.4. Then the corresponding map $\nu : \mathbb{F}[x_1, \ldots, x_d] \to e^*_0 T e^*_0$ from Definition 16.1 is an isomorphism.*

Proof: Combine Theorem 3.9 and Proposition 16.6. □

Theorem 18.3 *Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ be as in Definition 11.1. Then the corresponding map $\tilde{\nu} : P \to \varepsilon^*_0 \tilde{T} \varepsilon^*_0$ from Corollary 13.4 is an isomorphism.*

Proof: Combine Theorem 3.9 and Corollary 16.13. □

Theorem 18.4 *Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Then for all integers $n \geq 0$ the type $[n]$ is $(p, \mathbb{F})$-direct in the sense of Definition 12.17.*

Proof: Combine Proposition 13.13 and Theorem 18.3. □

Theorem 18.5 *Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ be as in Definition 11.1. Then the $\mathbb{F}$-vector space $\tilde{T}$ is spanned by its zigzag words.*

Proof: Combine Proposition 12.11 and Proposition 12.13. □

Below Lemma 12.14 we conjectured that the sum (31) is always direct. In the context of $\tilde{T}$ this conjecture can be expressed as follows.

Conjecture 18.6 Fix an integer $d \geq 0$ and a sequence $p \in \mathrm{Feas}(d, \mathbb{F})$. Let the algebra $\tilde{T} = \tilde{T}(p, \mathbb{F})$ be as in Definition 11.1. Then the $\mathbb{F}$-vector space $\tilde{T}$ has a basis consisting of its zigzag words.

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