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Some functional equations and Picard constants of algebroid surfaces

Dedicated to Professor Mitsuru Nakai on his 60th birthday

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1. Introduction.

When we intend to calculate the Picard constants of algebroid surfaces and study analytic mappings among algebroid surfaces with large Picard constants, we have to consider some functional equations. For example, the first author [7], Ozawa-Sawada [11], [12], [13] and Sawada-Tohge [14], etc. (cf. [10]) considered special cases of the following functional equation:

$$\sum_{\mu=0}^{m} a_{\mu}(z)e^{\mu H(z)} = f(z)\sum_{\nu=0}^{n} b_{\nu}(z)e^{\nu L(z)}.$$

The purpose of this paper is to study the above equation and to give an application.

First we shall prove

THEOREM 1. *Let H and L be non-constant entire functions with $H(0)=L(0)=0$, $a_m=b_n=1$, a_{μ} ($\mu=0, 1, \cdots, m-1$) and b_{ν} ($\nu=0, 1, \cdots, n-1$) meromorphic functions with $a_0 \not\equiv 0$, $b_0 \not\equiv 0$ and f a meromorphic function. Further suppose that*

$$T(r, a_{\mu}) = S(r, e^{H}) \quad \mu = 0, \cdots, m-1, \tag{1.1}$$

$$T(r, b_{\nu}) = S(r, e^{L}) \quad \nu = 0, \cdots, n-1, \tag{1.2}$$

and

$$N(r, 0, f)+N(r, \infty, f) = o(m(r, e^{H})+m(r, e^{L})) \quad r \to \infty \tag{1.3}$$

outside a set of finite measure. If $m \geqq n \geqq 1$, $d=(m, n)$, $m=pd$, $n=qd$ and the identity

$$\sum_{\mu=0}^{m} a_{\mu}(z)e^{\mu H(z)} = f(z)\sum_{\nu=0}^{n} b_{\nu}(z)e^{\nu L(z)} \tag{1.4}$$

holds, then we have one of the following two cases:

(I) $e^{mH(z)+nL(z)}=a_0(z)b_0(z),\quad f(z)=a_0(z)e^{-nL(z)},$

$$a_{jp}(z)=e^{-(j/d)(mH(z)+nL(z))}a_0(z)b_{(d-j)q}(z)\quad \text{for } j=0,1,2,\cdots,d,$$

$$a_\mu(z)\equiv 0\quad \text{for } \mu\neq 0,1p,2p,\cdots,dp=m,$$

$$b_\nu(z)\equiv 0\quad \text{for } \nu\neq 0,1q,2q,\cdots,dq=n;$$

(II) $e^{mH(z)-nL(z)}=a_0(z)/b_0(z),\quad f(z)=e^{mH(z)-nL(z)},$

$$a_{jp}(z)=e^{((d-j)/d)(mH(z)-nL(z))}b_{jq}(z)\quad \text{for } j=0,1,2,\cdots,d,$$

$$a_\mu(z)\equiv 0\quad \text{for } \mu\neq 0,1p,2p,\cdots,dp=m,$$

$$b_\nu(z)\equiv 0\quad \text{for } \nu\neq 0,1q,2q,\cdots,dq=n.$$

Changing some conditions on f in Theorem 1 we also obtain the following:

THEOREM 2. *Let H and L be non-constant entire functions with $H(0)=L(0)=0$, $a_m=b_n=1$ and a_μ ($\mu=0,1,\cdots,m-1$) and b_ν ($\nu=0,1,\cdots,n-1$) meromorphic functions satisfying $a_0\not\equiv 0$, $b_0\not\equiv 0$ and*

$$T(r,a_\mu)=S(r,e^H)\quad \mu=0,\cdots,m-1, \tag{1.5}$$

$$T(r,b_\nu)=S(r,e^L)\quad \nu=0,\cdots,n-1. \tag{1.6}$$

Further suppose that $m\geqq n\geqq 1$, $d=(m,n)$, $m=pd$, $n=qd$ and $F=f_1^m f_2$ where f_1, f_2 are meromorphic functions and f_2 satisfies

$$N(r,0,f_2)+N(r,\infty,f_2)=o(m(r,e^H)+m(r,e^L))\quad r\to\infty \tag{1.7}$$

outside a set of finite measure. If the identity

$$\sum_{\mu=0}^{m}a_\mu(z)e^{\mu H(z)}=F(z)\sum_{\nu=0}^{n}b_\nu(z)e^{\nu L(z)} \tag{1.8}$$

holds, then we have one of the following two cases:

(I) $e^{mH(z)+nL(z)}=a_0(z)b_0(z),\quad F(z)=a_0(z)e^{-nL(z)},$

$$a_{jp}(z)=e^{-(j/d)(mH(z)+nL(z))}a_0(z)b_{(d-j)q}(z)\quad \text{for } j=0,1,2,\cdots,d,$$

$$a_\mu(z)\equiv 0\quad \text{for } \mu\neq 0,1p,2p,\cdots,dp=m,$$

$$b_\nu(z)\equiv 0\quad \text{for } \nu\neq 0,1q,2q,\cdots,dq=n;$$

(II) $e^{mH(z)-nL(z)}=a_0(z)/b_0(z),\quad F(z)=e^{mH(z)-nL(z)},$

$$a_{jp}(z)=e^{((d-j)/d)(mH(z)-nL(z))}b_{jp}(z)\quad \text{for } j=0,1,2,\cdots,d,$$

$$a_\mu(z)\equiv 0\quad \text{for } \mu\neq 0,1p,2p,\cdots,dp=m,$$

$$b_\nu(z)\equiv 0\quad \text{for } \nu\neq 0,1q,2q,\cdots,dq=n,$$

unless $\sum_{\mu=0}^{m} a_\mu(z)e^{\mu H(z)}$ *or* $\sum_{\nu=0}^{n} b_\nu(z)e^{\nu L(z)}$ $(m=n)$ *has the following form*:

$$\beta(z)(e^{\pm K(z)}-\alpha(z))^m, \tag{1.9}$$

where $K(z)=H(z)$ *or* $K(z)=L(z)$ *and* α *and* β *are meromorphic functions satisfying* $T(r, \alpha)=S(r, e^K)$, $T(r, \beta)=S(r, e^K)$.

Next as an application of Theorem 1 we shall consider the Picard constants of algebroid surfaces in § 6.

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and usual notations:

$$T(r, f),\ m(r, f),\ N(r, a, f),\ \bar{N}(r, a, f),\ N_1(r, a, f),\ \text{etc..}$$

In this paper, further, we denote by $S(r, f)$ a quantity satisfying

$$S(r, f) = o(T(r, f)) \quad r\to\infty$$

outside a set of finite measure, a meromorphic function a is called to be small with respect to the function f if $T(r, a)=S(r, f)$ and it is said for an equation with entire or meromorphic coefficients to be irreducible when this equation is irreducible over the field of the single-valued meromorphic functions in the complex plane $\boldsymbol{C}$.

2. Zeros of polynomials in e^H.

First of all we consider the zeros of polynomials in e^H with small meromorphic coefficients. We denote by $N_1^*(r, 0, f)$ the counting function of simple zeros of the referred function f and by $N_0(r, 0; f, g)$ the counting function of common zeros of f and g.

We prove

PROPOSITION 1. *Let* H *be a non-constant entire function and* $a_1, \cdots, a_\mu$ *meromorphic functions satisfying*

$$T(r, a_j) = S(r, e^H) \quad j=1, \cdots, \mu.$$

Then we have

$$N(r, 0, P_\mu(e^H)) = \mu\, m(r, e^H)+S(r, e^H),$$

where

$$P_\mu(X) = X^\mu + a_1(z)X^{\mu-1}+ \cdots + a_\mu(z) \quad (a_\mu \not\equiv 0).$$

Factorizing $P_\mu(X)$ into irreducible polynomial factors in X with meromorphic coefficients, we can deduce Proposition 1 from the following lemma, which is an extension of Lemma 4 in [**5**]:

LEMMA 1. *Let H be a non-constant entire function and $a_1, \cdots, a_\mu$ ($a_\mu \not\equiv 0$) meromorphic functions satisfying*

$$T(r, a_j) = S(r, e^H) \quad j = 1, \cdots, \mu.$$

If the algebraic equation

$$P_\mu(X) := X^\mu + a_1(z)X^{\mu-1} + \cdots + a_\mu(z) = 0$$

is irreducible, then we have

$$N_1^*(r, 0, P_\mu(e^H)) = \mu\, m(r, e^H) + S(r, e^H) \quad and \quad N_1(r, 0, P_\mu(e^H)) = S(r, e^H).$$

PROOF. It follows from the assumption of this lemma that

$$\begin{aligned} R(z, e^H) &:= \frac{d}{dz}\{P_\mu(e^{H(z)})\} / P_\mu(e^{H(z)}) \\ &= \frac{\mu H' e^{\mu H} + \{a_1' + (\mu-1)H' a_1\} e^{(\mu-1)H} + \cdots + a_\mu'}{e^{\mu H} + a_1 e^{(\mu-1)H} + \cdots + a_\mu} \end{aligned}$$

is an irreducible rational function in e^H (in the sense of Laine [2]) with meromorphic coefficients small with respect to e^H. Hence a theorem due to G. Valiron and A. Mokhon'ko (cf. Theorem 2.2.5 in Laine [2]) implies

$$T(r, R(z, e^H)) = \mu\, m(r, e^H) + S(r, e^H).$$

On the other hand, since $R(z, e^H)$ is the logarithmic derivative of $P_\mu(e^H)$, we have

$$T(r, R(z, e^H)) = \bar{N}(r, 0, P_\mu(e^H)) + S(r, e^H).$$

Therefore we obtain

$$\bar{N}(r, 0, P_\mu(e^H)) = \mu\, m(r, e^H) + S(r, e^H)$$

and so

$$N(r, 0, P_\mu(e^H)) \leqq T(r, P_\mu(e^H)) + O(1) = \mu\, m(r, e^H) + S(r, e^H).$$

Hence we have

$$N_1(r, 0, P_\mu(e^H)) = N(r, 0, P_\mu(e^H)) - \bar{N}(r, 0, P_\mu(e^H)) = S(r, e^H).$$

Now we denote by $\bar{N}(r, 0: k)$ the counting function of distinct zeros of $P_\mu(e^H)$ of multiplicity k. Then

$$N_1(r, 0, P_\mu(e^H)) = \sum_{k=2}^{\infty} (k-1)\bar{N}(r, 0: k) \geqq \sum_{k=2}^{\infty} \bar{N}(r, 0: k).$$

Therefore we obtain

$$N_1^*(r, 0, P_\mu(e^H)) = \overline{N}(r, 0:1) = \overline{N}(r, 0, P_\mu(e^H)) - \sum_{k=2}^{\infty} \overline{N}(r, 0: k)$$
$$= \mu\, m(r, e^H) + S(r, e^H),$$

which proves our Lemma 1. q.e.d.

In order to prove Theorem 2 we need the following lemma, which is deduced from Lemma 5 in [**5**]:

LEMMA A. *Let H be a non-constant entire function and $a_1, \cdots, a_\mu, b_1, \cdots, b_\nu$ meromorphic functions satisfying*

$$T(r, a_j) = S(r, e^H) \quad j=1, \cdots, \mu \text{ and } T(r, b_j) = S(r, e^H) \quad j=1, \cdots, \nu.$$

If the equations

$$P_\mu(X) := X^\mu + a_1(z)X^{\mu-1} + \cdots + a_\mu(z) = 0,$$

$$Q_\nu(X) := X^\nu + b_1(z)X^{\nu-1} + \cdots + b_\nu(z) = 0$$

are irreducible and $P_\mu(e^{H(z)}) \not\equiv Q_\nu(e^{H(z)})$, then we have

$$N_0(r, 0; P_\mu(e^H), Q_\nu(e^H)) = S(r, e^H).$$

3. Borel's identity.

In order to prove our theorems we can use Lemma 1 in [**5**], however, we here give its refinement as follows:

PROPOSITION 2. *Let $a_0, a_1, \cdots, a_n$ be meromorphic functions and $g_0, g_1, \cdots, g_n$ entire functions satisfying*

$$T(r, a_j) = S(r, e^{g_\nu - g_0}) \quad j=0, 1, \cdots, n;\ \nu = k, k+1, \cdots, n.$$

If $a_0 \not\equiv 0$ and the identity

$$\sum_{\nu=0}^{n} a_\nu(z) e^{g_\nu(z)} = 0 \tag{3.1}$$

holds, then we have

$$a_0(z)e^{g_0(z)} + \sum_{\nu=1}^{k-1} c_\nu a_\nu(z) e^{g_\nu(z)} = 0, \tag{3.2}$$

where c_ν, $\nu=1, 2, \cdots, k-1$, are suitable constants, not all zero.

Before going to prove this proposition we need the following lemma due to Hiromi-Ozawa [**4**].

LEMMA B. *Let $a_0, a_1, \cdots, a_n$ be meromorphic functions and $g_1, \cdots, g_n$ entire functions. Further suppose that*

$$T(r, a_j) = o\left\{\sum_{\nu=1}^{n} m(r, e^{g_\nu})\right\} \quad (r \to \infty), \quad j = 0, 1, \cdots, n$$

holds outside a set of finite measure. If the identity

$$\sum_{\nu=1}^{n} a_\nu(z) e^{g_\nu(z)} = a_0(z)$$

holds, then we have an identity

$$\sum_{\nu=1}^{n} c_\nu a_\nu(z) e^{g_\nu(z)} = 0,$$

where c_ν, $\nu=1, 2, \cdots, n$, are constants which are not all zero.

PROOF OF PROPOSITION 2. Write (3.1) as

$$\sum_{\nu \in I \cup J} a_\nu(z) e^{g_\nu(z) - g_0(z)} = -a_0(z), \tag{3.3}$$

where sets I and J are defined by

$$I = \{\nu : a_\nu(z) \not\equiv 0,\ 1 \leqq \nu \leqq k-1\} \quad \text{and} \quad J = \{\nu : a_\nu(z) \not\equiv 0,\ k \leqq \nu \leqq n\}.$$

Now we may assume $J \neq \emptyset$, otherwise there is nothing to prove. Then we can apply Lemma B. In fact, for any l in J, then we have

$$m(r, e^{g_l(z) - g_0(z)}) \leqq \sum_{\nu=1}^{n} m(r, e^{g_\nu(z) - g_0(z)})$$

and $T(r, a_j) = S(r, e^{g_l - g_0})$ for any j $(0 \leqq j \leqq n)$. Hence the requirements of Lemma B are fulfilled in the identity (3.3).

Suppose that the set I is empty, that is, (3.3) is reduced to the identity

$$\sum_{\nu \in J} a_\nu(z) e^{g_\nu(z) - g_0(z)} = -a_0(z). \tag{3.4}$$

Then applying Lemma B to (3.4), we obtain

$$\sum_{\nu \in J} c_{1\nu} a_\nu(z) e^{g_\nu(z) - g_0(z)} = 0, \tag{3.5}$$

where $c_{1\nu}$, $\nu \in J$, are constants which are not all zero. Choose a non-zero constant c_{1l} and rewrite (3.5) to

$$a_l(z) e^{g_l(z) - g_0(z)} = -\sum_{\nu \in J \setminus \{l\}} (c_{1\nu}/c_{1l}) a_\nu(z) e^{g_\nu(z) - g_0(z)}.$$

Substituting this identity into (3.4), we have

$$\sum_{\nu \in J \setminus \{l\}} (1 - c_{1\nu}/c_{1l}) a_\nu(z) e^{g_\nu(z) - g_0(z)} = -a_0(z). \tag{3.6}$$

Since $a_0(z)\not\equiv 0$, there is an integer m in J and different from l with $c_{1m}\neq c_{1l}$. Repeating the above procedure, we can reduce (3.6) to the identity of the form

$$c a_\nu(z)e^{g_\nu(z)-g_0(z)} = -a_0(z),$$

where c is a non-zero constant and $\nu\in J$. This is however impossible, because a_ν and a_0 are supposed to be small functions with respect to functions $e^{g_\nu-g_0}$, $k\leqq\nu\leqq n$.

Hence we may assume that the set I is also not empty in the identity (3.3).

Apply Lemma B to (3.3). Then we have

$$\sum_{\nu\in I\cup J} c_{1\nu}a_\nu(z)e^{g_\nu(z)-g_0(z)} = 0, \tag{3.7}$$

where $c_{1\nu}$, $\nu\in I\cup J$, are constants which are not all zero. Choose a non-zero constant c_{1l} and rewrite (3.7) to

$$a_l(z)e^{g_l(z)-g_0(z)} = \sum_{\nu\in(I\cup J)\setminus\{l\}} (c_{1\nu}/c_{1l})a_\nu(z)e^{g_\nu(z)-g_0(z)}.$$

Substituting this into (3.3), we obtain the identity

$$\sum_{\nu\in(I\cup J)\setminus\{l\}} (1-c_{1\nu}/c_{1l})a_\nu(z)e^{g_\nu(z)-g_0(z)} = -a_0(z). \tag{3.8}$$

We here define $I_1=\{\nu\in I: \nu\neq l,\ c_{1\nu}\neq c_{1l}\}$ and $J_1=\{\nu\in J: \nu\neq l,\ c_{1\nu}\neq c_{1l}\}$. Note that $I_1\cup J_1\neq\emptyset$ in view of (3.8) and the assumption $a_0(z)\not\equiv 0$. If $I_1=\emptyset$, then the identity (3.8) reduces to the identity (3.6), which leads to a contradiction by the previous argument. Also the case $J_1=\emptyset$ reduces the identity (3.8) to one which we claim as (3.2). Therefore we assume that neither of the sets I_1 and J_1 is empty and

$$\sum_{\nu\in I_1\cup J_1} c_{2\nu}a_\nu(z)e^{g_\nu(z)-g_0(z)} = -a_0(z)$$

holds with suitable non-zero constants $c_{2\nu}$. Since $I_1\cup J_1$ is a proper subset of $I\cup J$ and further $I_1\neq\emptyset$ and $J_1\neq\emptyset$, we can finally reach the following identity after suitably repeated procedures mentioned above:

$$c_{*\alpha}a_\alpha(z)e^{g_\alpha(z)-g_0(z)}+c_{*\beta}a_\beta(z)e^{g_\beta(z)-g_0(z)} = -a_0(z),$$

where $c_{*\alpha}$ and $c_{*\beta}$ ($\alpha\in I$, $\beta\in J$) are non-zero constants. Writing this identity as

$$a_0(z)e^{g_0(z)-g_\beta(z)}+c_{*\alpha}a_\alpha(z)e^{g_\alpha(z)-g_\beta(z)} = -c_{*\beta}a_\beta(z),$$

and applying Lemma B to this identity, we obtain the desired identity

$$a_0(z)e^{g_0(z)}+c a_\alpha(z)e^{g_\alpha(z)} = 0$$

with a non-zero constant c. q.e.d.

4. Lemmas.

Before going to prove our Theorems, we need some more lemmas.

We can deduce from the impossibility of Borel's identity that

LEMMA C. *Let* $a_1, \cdots, a_n$ *be meromorphic functions,* g *a non-constant entire function and* $\alpha_1, \cdots, \alpha_n$ *distinct constants. Further suppose that*

$$T(r, a_j) = S(r, e^g) \quad j = 1, \cdots, n$$

holds. Then the identity

$$\sum_{j=1}^{n} a_j(z) e^{\alpha_j g(z)} = 0,$$

is impossible unless all $a_1, \cdots, a_n$ *are identically zero.*

We have

LEMMA D ([6]). *Let* g *and* $a_0, a_1, \cdots, a_n$ *be meromorphic functions such that* $a_n \not\equiv 0$ *and*

$$T(r, a_j) = S(r, g) \quad j = 0, \cdots, n$$

holds. If a meromorphic function f *satisfies an equation*

$$a_n(z) f^n + a_{n-1}(z) f^{n-1} + \cdots + a_0(z) = 0,$$

then we have

$$T(r, f) = S(r, g).$$

We prove

LEMMA 2. *Let* α *be a positive number,* μ *and* ν *positive integers and* H *and* L *non-constant entire functions such that* $H(0) = L(0) = 0$. *If*

$$m(r, e^H) = \alpha\, m(r, e^L) + S(r, e^L),$$

then we have

$$m(r, e^{\mu H \pm \nu L}) \geqq \mu |\alpha - \nu/\mu|\, m(r, e^L) + S(r, e^L).$$

PROOF. It follows that

$$\mu\, m(r, e^H) \leqq m(r, e^{\mu H \pm \nu L}) + m(r, e^{\mp \nu L}) = m(r, e^{\mu H \pm \nu L}) + \nu\, m(r, e^L)$$

and hence

$$m(r, e^{\mu H \pm \nu L}) \geqq |\mu\, m(r, e^H) - \nu\, m(r, e^L)| = \mu |\alpha - \nu/\mu|\, m(r, e^L) + S(r, e^L),$$

which proves our Lemma. q.e.d.

5. Proof of Theorems.

PROOF OF THEOREM 1. It follows from (1.4) that $T(r, f)=O(m(r, e^{H})+m(r, e^{L}))$ outside a set of finite measure. Hence from the condition (1.3) we have

$$T(r, f'/f) = o(m(r, e^{H})+m(r, e^{L})) \tag{5.1}$$

outside a set of finite measure. We put

$$U(z):=\sum_{\mu=0}^{m} a_{\mu}(z)e^{\mu H(z)} \quad \text{and} \quad V(z):=\sum_{\nu=0}^{n} b_{\nu}(z)e^{\nu L(z)}.$$

Then Proposition 1 implies

$$N(r, 0, U) = m\, m(r, e^{H})+S(r, e^{H}) \quad \text{and} \quad N(r, 0, V) = n\, m(r, e^{L})+S(r, e^{L}).$$

On the other hand it follows from (1.3) that the identity (1.4) implies $N(r, 0, U)\sim N(r, 0, V)$. Therefore we have

$$m(r, e^{H}) \sim (n/m)\, m(r, e^{L}) \quad r\rightarrow\infty \tag{5.2}$$

outside a set of finite measure.

By differentiating both sides of (1.4) we have

$$U' = fV'+f'V = (U/V)\{V'+(f'/f)V\}$$

that is,

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} A_{\mu\nu}(z)e^{\mu H(z)+\nu L(z)} = 0, \tag{5.3}$$

where

$$A_{\mu\nu} = a'_{\mu}b_{\nu}-a_{\mu}b'_{\nu}+a_{\mu}b_{\nu}\{\mu H'-\nu L'-(f'/f)\},$$

especially, $A_{mn}=mH'-nL'-(f'/f)$ because of $a_{m}=b_{n}=1$. It is clear from (1.1), (1.2), (5.1) and (5.2) that

$$T(r, A_{\mu\nu}) = S(r, e^{L}).$$

Assume that $A_{mn}(z)\not\equiv 0$. Then we apply Proposition 2 to the identity (5.3). Putting $g_{0}=mH+nL$ in Proposition 2 and taking Lemma 2 into account we have

$$\sum_{j=0}^{d} c_{j}A_{(jp)(jq)}(z)e^{j(pH(z)+qL(z))} = 0,$$

where $c_{d}=1$ and the other c_{j} are suitable constants. Hence Lemma C implies

$$T(r, e^{mH+nL}) = dT(r, e^{pH+qL}) = S(r, e^{L}).$$

Therefore, since $\mu H+\nu L=(\mu/m)(mH+nL)+(\nu-(n/m)\mu)L$, (5.3) reduces to

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} B_{\mu\nu}(z)e^{(\nu-(n/m)\mu)L(z)}=0, \tag{5.4}$$

where $B_{\mu\nu}=A_{\mu\nu}e^{(\mu/m)(mH+nL)}$ and they satisfy

$$T(r, B_{\mu\nu})=S(r, e^{L}). \tag{5.5}$$

Since $-n<\nu-(n/m)\mu$ for $0\leqq\mu<m$, $0\leqq\nu\leqq n$ and $n>\nu-(n/m)\mu$ for $0\leqq\mu\leqq m$, $0\leqq\nu<n$, it follows from (5.5) that (5.4) and Lemma C yield

$$B_{m0}=\{-b_0'+b_0(mH'-(f'/f))\}e^{mH+nL}\equiv 0 \quad\text{and}\quad B_{0n}=a_0'-a_0(nL'+(f'/f))\equiv 0$$

and so $f'/f=-nL'+a_0'/a_0=mH'-b_0'/b_0$. Hence we have

$$f(z)=ca_0(z)e^{-nL(z)} \quad\text{and}\quad e^{mH(z)+nL(z)}=c_1a_0(z)b_0(z),$$

where c and c_1 are non-zero constants. Substituting these relations into (1.4) we obtain

$$\sum_{\mu=0}^{m} a_\mu e^{(\mu/m)(mH+nL)}e^{(n-\mu n/m)L}-ca_0\sum_{\nu=0}^{n} b_\nu e^{\nu L}=0.$$

Hence it follows from Lemma C that

$$a_0-ca_0b_n=a_0(1-c)=0,$$

$$a_me^{mH+nL}-ca_0b_0=e^{mH+nL}-ca_0b_0=0,$$

$$a_\mu e^{(\mu/m)(mH+nL)}=ca_0b_\nu \qquad \text{for } \mu \text{ such that } n-\mu n/m=\nu\in \boldsymbol{N},$$

$$a_\mu=0,\quad b_\nu=0 \qquad \text{for the other } \mu \text{ and } \nu,$$

which proves the case (I) of our Theorem 1.

Next we assume that $A_{mn}(z)=mH'(z)-nL'(z)-f'(z)/f(z)\equiv 0$. Then we have

$$f(z)=ce^{mH(z)-nL(z)},$$

where c is a non-zero constant. Substituting this into (1.4) we obtain

$$\sum_{\nu=0}^{n}(b_\nu/b_0)e^{mH-(n-\nu)L}=\sum_{\mu=0}^{m}(a_\mu/cb_0)e^{\mu H},$$

that is,

$$e^{mH-nL}+\sum_{\nu=1}^{n}(b_\nu/b_0)e^{mH-(n-\nu)L}-\sum_{\mu=1}^{m}(a_\mu/cb_0)e^{\mu H}-(a_0/cb_0)=0.$$

We apply Proposition 2 to this identity. Putting $g_0=mH-nL$ in Proposition 2 and using Lemma 2, we have

$$e^{mH(z)-nL(z)}+c_1(a_0(z)/cb_0(z))=0,$$

where c_1 is a non-zero constant. Hence we have

$$T(r, e^{mH-nL}) = S(r, e^L).$$

Therefore we rewrite the identity (1.4) as follows

$$\sum_{\mu=0}^{m} a_\mu e^{(\mu/m)(mH-nL)} e^{(n/m)\mu L} - c e^{mH-nL} \sum_{\nu=0}^{n} b_\nu e^{\nu L} = 0$$

and from Lemma C we have

$$a_0 - c e^{mH-nL} b_0 = 0,$$

$$a_m e^{mH-nL} - c e^{mH-nL} b_n = (1-c) e^{mH-nL} = 0,$$

$$a_\mu e^{(\mu/m)(mH-nL)} = c b_\nu e^{mH-nL} \quad \text{for } \mu \text{ such that } \mu n/m = \nu \in \boldsymbol{N},$$

$$a_\mu = 0, \quad b_\nu = 0 \quad \text{for the other } \mu \text{ and } \nu,$$

which proves the case (II) of our Theorem 1. q.e.d.

PROOF OF THEOREM 2. Let $N_k^*(r, 0, f)$ be the counting function of zeros of f with their multiplicity not greater than k. We put

$$U(z) := \sum_{\mu=0}^{m} a_\mu(z) e^{\mu H(z)} \quad \text{and} \quad V(z) := \sum_{\nu=0}^{n} b_\nu(z) e^{\nu L(z)}.$$

Since U and V are not of the form (1.9), Lemma 1 and Lemma A yield

$$N_{m-1}^*(r, 0, U) = m\, m(r, e^H) + S(r, e^H) \tag{5.6}$$

and further

$$N_{n-1}^*(r, 0, V) = n\, m(r, e^L) + S(r, e^L), \tag{5.7}$$

if V is not of a form $\beta(z)(e^{\pm L(z)} - \alpha(z))^n$, and

$$N_n^*(r, 0, V) = n\, m(r, e^L) + S(r, e^L), \tag{5.8}$$

if $n<m$ and V is of a form $\beta(z)(e^{\pm L(z)} - \alpha(z))^n$.

Hence we can deduce from (1.8), (5.6), (5.7) and (5.8) that

$$N(r, 0, f_1) = o(m(r, e^H) + m(r, e^L)), \quad N(r, \infty, f_1) = o(m(r, e^H) + m(r, e^L))$$

and consequently taking (1.7) into account we obtain

$$N(r, 0, F) = o(m(r, e^H) + m(r, e^L)), \quad N(r, \infty, F) = o(m(r, e^H) + m(r, e^L))$$

outside a set of finite measure.

Thus we have seen that F satisfies the condition (1.3) in our Theorem 1. Therefore Theorem 2 follows Theorem 1. q.e.d.

6. Picard constants of algebroid surfaces.

As an application of Theorem 1 we shall consider the Picard constant of an n-sheeted algebroid surface which is the proper existence domain of an n-valued algebroid function. This is a number defined for an arbitrary Riemann surface R as follows: Let $M(R)$ be the family of non-constant meromorphic functions on R and $P(f)$ the number of the values which are not taken by an element f of $M(R)$. Then the Picard constant $P(R)$ of R is defined by

$$P(R)=\sup\{P(f) \mid f \in M(R)\}.$$

We note that $P(R)$ is a conformal invariant of R, $P(R) \geqq 2$ if R is open, and also $P(R) \leqq 2n$ if R is an n-sheeted algebroid surface (see, for example, Ozawa [**8**]).

Ozawa [**9**], Hiromi-Niino [**3**] and Aogai [**1**] gave a characterization of regularly branched algebroid surfaces with large Picard constants (cf. [**10**]). Recently Ozawa and Sawada [**11**], [**12**], [**13**] have obtained a characterization of three- or four-sheeted algebroid surfaces which are not regularly branched and have large Picard constants. Their results for four-sheeted surfaces are summarized as follows:

THEOREM A. *Let R be a four-sheeted algebroid surface defined by an irreducible equation*

$$y^4-S_1(z)y^3+S_2(z)y^2-S_3(z)y+S_4(z)=0, \tag{6.1}$$

where S_j are entire functions and at least one of them is transcendental.

(I) *Then $P(y)=8$, only if y is defined, up to a linear fractional transformation, by* (6.1) *with*

$$\begin{cases} S_1(z)=\alpha e^{H(z)}+\Phi_1(a_3, a_4, a_5, a_6), \\ S_2(z)=\Phi_1(a_1, a_2)\alpha e^{H(z)}+\Phi_2(a_3, a_4, a_5, a_6), \\ S_3(z)=\Phi_2(a_1, a_2)\alpha e^{H(z)}+\Phi_3(a_3, a_4, a_5, a_6), \\ S_4(z)=\Phi_4(a_3, a_4, a_5, a_6), \end{cases} \tag{6.2}$$

where a_j are distinct non-zero constants, α is a non-zero constant, H is a non-constant entire function with $H(0)=0$, and further $\Phi_k(x_1, x_2, \cdots, x_n)$ denotes the elementary symmetric expression in x_j of degree k.

The discriminant $D(z)$ of this surface is of the form

$$D(z)=\sum_{j=0}^{6} A_j e^{jH(z)} \tag{6.3}$$

for some constants A_j $(j=0, 1, \cdots, 6)$ satisfying $A_0A_6 \neq 0$.

(II) *For every surface R with $P(y)=7$, y is necessarily defined, up to a linear fractional transformation, by* (6.1) *with the following S_j*:

$$S_j(z) = \beta_j e^{L(z)} + \gamma_j \quad (j = 1, 2, 3, 4), \tag{6.4}$$

where L is a non-constant entire function with $L(0)=0$ and the β_j and γ_j are constants such as those in one of the following three cases:

$$\text{Case 1)} \quad \begin{cases} \beta_1 = 0, \\ \beta_2 = \beta, \\ \beta_3 = \Phi_1(b_1)\beta, \\ \beta_4 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \gamma_1 = \Phi_1(b_2, b_3, b_4, b_5), \\ \gamma_2 = \Phi_2(b_2, b_3, b_4, b_5), \\ \gamma_3 = \Phi_3(b_2, b_3, b_4, b_5), \\ \gamma_4 = \Phi_4(b_2, b_3, b_4, b_5), \end{cases}$$

$$\text{Case 2)} \quad \begin{cases} \beta_1 = \beta, \\ \beta_2 = \Phi_1(b_1, b_1)\beta, \\ \beta_3 = \Phi_2(b_1, b_1)\beta, \\ \beta_4 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \gamma_1 = \Phi_1(b_2, b_3, b_4, b_5), \\ \gamma_2 = \Phi_2(b_2, b_3, b_4, b_5), \\ \gamma_3 = \Phi_3(b_2, b_3, b_4, b_5), \\ \gamma_4 = \Phi_4(b_2, b_3, b_4, b_5), \end{cases}$$

$$\text{Case 3)} \quad \begin{cases} \beta_1 = \beta, \\ \beta_2 = \Phi_1(b_1, b_2)\beta, \\ \beta_3 = \Phi_2(b_1, b_2)\beta, \\ \beta_4 = 0, \end{cases} \quad \text{and} \quad \begin{cases} \gamma_1 = \Phi_1(b_3, b_3, b_4, b_5), \\ \gamma_2 = \Phi_2(b_3, b_3, b_4, b_5), \\ \gamma_3 = \Phi_3(b_3, b_3, b_4, b_5), \\ \gamma_4 = \Phi_4(b_3, b_3, b_4, b_5), \end{cases}$$

where β is a non-zero constant and $\Phi_k(x_1, x_2, \cdots, x_n)$ is again the elementary symmetric expression in x_j of degree k.

The discriminant $\Delta(z)$ of R has two forms according to the cases. In the first two cases

$$\Delta(z) = \sum_{j=0}^{5} B_j e^{jL(z)}, \tag{6.5}$$

and in the third case

$$\Delta(z) = \left\{ \sum_{j=0}^{5} B_j e^{jL(z)} \right\} e^{L(z)}, \tag{6.6}$$

where $B_0, B_1, \cdots, B_5$ are constants with $B_0 B_5 \neq 0$.

Using this observation, they have obtained the following characterization under certain conditions.

THEOREM B. *Suppose that L is a polynomial in* (6.4). *Then every surface R given in Cases* 1)-3) *of Theorem A does not admit the maximal Picard constant, unless $B_1=B_2=B_3=B_4=0$ in* (6.5) *or* (6.6). *In other words, $P(R)=7$.*

These results hold similarly for three-sheeted algebroid surfaces ([**11**]). In the case of three-sheeted algebroid surfaces Sawada-Tohge [**14**] proved that there is no need for $L(z)$ in the corresponding Theorem B to be a polynomial. In their proof the lemma, which is due to E. Mues on Weierstrass products, played a central role. This needs to be combined with a kind of Borel's unicity lemma, and so demands a disgusting routine of consideration from us. Applying Theorem 1, however, takes us to a direct goal from the materials given in [**11**].

In this section we shall show this process for the case of four-sheeted instead of three-sheeted algebroid surfaces, that is, we shall prove that

THEOREM 3. *Theorem B remains true without assuming that* $L(z)$ *should be a polynomial.*

To prove this theorem we need the following lemma, which also played a very important role in the proof of Theorem B:

LEMMA E ([**12**]). *Let* R *be a surface listed up in Theorem A, whose Picard constant is at least* 7, *so that we may assume that* R *is defined by the equation* (6.1) *with*

$$S_j(z)=\delta_j e^{M(z)}+\varepsilon_j \quad (j=1,2,3,4),$$

where δ_j *and* ε_j *are suitable constants and* M *is a non-constant entire function satisfying* $M(0)=0$. *Let* f *be a regular function on* R. *Then* f *is representable as*

$$f=F_1+F_2 y+F_3 y^2+F_4 y^3,$$

where F_1, F_2, F_3 *and* F_4 *are meromorphic functions in* $\boldsymbol{C}$, *all of which are regular at any point* z *satisfying* $M'(z)\neq 0$.

PROOF OF THEOREM 3. Let R be a surface with $P(y)=7$, which is defined by (6.1) and (6.4) with β_j and γ_j in one of Cases 1)-3). Suppose that $P(R)=8$ so that there exists a regular function f on R with $P(f)=8$. By Lemma E, f is represented by

$$f=F_{11}+F_{21} y+F_{31} y^2+F_{41} y^3,$$

where F_{j1} $(j=1,2,3,4)$ are meromorphic functions in $\boldsymbol{C}$, and have all the poles possibly at zeros of $L'(z)$. We may assume that f defines the surface S by the equation

$$f^4-U_1(z)f^3+U_2(z)f^2-U_3(z)f+U_4(z)=0$$

with U_j equal to S_j in (6.2).

Now let y_k and f_k are the k-th determinations of y and f, respectively, such that

$$f_k=F_{11}+F_{21} y_k+F_{31} y_k^2+F_{41} y_k^3 \quad (k=1,2,3,4).$$

Then we obtain the equation

$$\begin{pmatrix} f_1\\ f_2\\ f_3\\ f_4\end{pmatrix}=\begin{pmatrix}1 & y_1 & y_1^2 & y_1^3\\ 1 & y_2 & y_2^2 & y_2^3\\ 1 & y_3 & y_3^2 & y_3^3\\ 1 & y_4 & y_4^2 & y_4^3\end{pmatrix}\begin{pmatrix}F_{11}\\ F_{21}\\ F_{31}\\ F_{41}\end{pmatrix}.$$

Since we can also write

$$f_k^j = F_{1j}+F_{2j}y_k+F_{3j}y_k^2+F_{4j}y_k^3 \quad (j = 2, 3\,;\ k = 1, 2, 3, 4)$$

for meromorphic functions F_{ij} (i=1, 2, 3, 4; j=2, 3) having their poles possibly at zeros of $L'(z)$, we further obtain

$$\begin{pmatrix} 1 & f_1 & f_1^2 & f_1^3 \\ 1 & f_2 & f_2^2 & f_2^3 \\ 1 & f_3 & f_3^2 & f_3^3 \\ 1 & f_4 & f_4^2 & f_4^3 \end{pmatrix} = \begin{pmatrix} 1 & y_1 & y_1^2 & y_1^3 \\ 1 & y_2 & y_2^2 & y_2^3 \\ 1 & y_3 & y_3^2 & y_3^3 \\ 1 & y_4 & y_4^2 & y_4^3 \end{pmatrix} \begin{pmatrix} 1 & F_{11} & F_{12} & F_{13} \\ 0 & F_{21} & F_{22} & F_{23} \\ 0 & F_{31} & F_{32} & F_{33} \\ 0 & F_{41} & F_{42} & F_{43} \end{pmatrix}. \tag{6.7}$$

Since f and y are four-valued and the determinant of the second matrix in the right hand side of (6.7) is not identically equal to zero, y is regular on S. Hence we similarly have

$$\begin{pmatrix} 1 & y_1 & y_1^2 & y_1^3 \\ 1 & y_2 & y_2^2 & y_2^3 \\ 1 & y_3 & y_3^2 & y_3^3 \\ 1 & y_4 & y_4^2 & y_4^3 \end{pmatrix} = \begin{pmatrix} 1 & f_1 & f_1^2 & f_1^3 \\ 1 & f_2 & f_2^2 & f_2^3 \\ 1 & f_3 & f_3^2 & f_3^3 \\ 1 & f_4 & f_4^2 & f_4^3 \end{pmatrix} \begin{pmatrix} 1 & G_{11} & G_{12} & G_{13} \\ 0 & G_{21} & G_{22} & G_{23} \\ 0 & G_{31} & G_{32} & G_{33} \\ 0 & G_{41} & G_{42} & G_{43} \end{pmatrix}, \tag{6.8}$$

where G_{ij} (i=1, 2, 3, 4; j=1, 2, 3) are meromorphic functions having their poles possibly at zeros of $H'(z)$.

Discriminants of S and R are $D(z)$ and $\Delta(z)$ given by (6.3) and either of (6.5) and (6.6), respectively. On the other hand

$$D(z) = \begin{vmatrix} 1 & f_1 & f_1^2 & f_1^3 \\ 1 & f_2 & f_2^2 & f_2^3 \\ 1 & f_3 & f_3^2 & f_3^3 \\ 1 & f_4 & f_4^2 & f_4^3 \end{vmatrix}^2 \quad \text{and} \quad \Delta(z) = \begin{vmatrix} 1 & y_1 & y_1^2 & y_1^3 \\ 1 & y_2 & y_2^2 & y_2^3 \\ 1 & y_3 & y_3^2 & y_3^3 \\ 1 & y_4 & y_4^2 & y_4^3 \end{vmatrix}^2,$$

and thus representations (6.7) and (6.8) imply the identity

$$D(z) = \Delta(z)G(z)^2 \tag{6.9}$$

with

$$G(z) = \begin{vmatrix} 1 & F_{11} & F_{12} & F_{13} \\ 0 & F_{21} & F_{22} & F_{23} \\ 0 & F_{31} & F_{32} & F_{33} \\ 0 & F_{41} & F_{42} & F_{43} \end{vmatrix} = \begin{vmatrix} 1 & G_{11} & G_{12} & G_{13} \\ 0 & G_{21} & G_{22} & G_{23} \\ 0 & G_{31} & G_{32} & G_{33} \\ 0 & G_{41} & G_{42} & G_{43} \end{vmatrix}^{-1}. \tag{6.10}$$

Then (6.10) implies

$$\overline{N}(r, \infty, G) \leqq \overline{N}(r, 0, L') \quad \text{and} \quad \overline{N}(r, 0, G) \leqq \overline{N}(r, 0, H').$$

Moreover we have

$$N(r, \infty, G) \leqq 5N(r, 0, L') \tag{6.11}$$

and

$$N(r, 0, G) \leqq 6N(r, 0, H'). \tag{6.12}$$

In fact, we can prove these two inequalities in the same manner as in [**14**]. Write

$$\Delta(z)=B_5e^{\varepsilon L(z)}\prod_{j=1}^{5}(e^{L(z)}-d_j)$$

with $\varepsilon\in\{0, 1\}$ and non-zero constants d_j $(1\leqq j\leqq 5)$. Assume that $G(z)$ has a pole of order p $(\geqq 1)$ at a point z_0. Then $\Delta(z)$ necessarily vanishes there in view of the identity (6.9). Thus there exists an integer j $(1\leqq j\leqq 5)$ such that $e^{L(z_0)}=d_j$. Since z_0 is a zero of $L'(z)$ of order n $(\geqq 1)$, say, the function $e^{L(z)}-d_j$ has z_0 as a zero of order $n+1$. It follows therefore that $5(n+1)\geqq 2p$, and thus

$$p\leqq 5(n+1)/2\leqq 5n,$$

which yields the estimate (6.11). Just a parallel reasoning implies the other (6.12).

We can apply Theorem 1 to the identity (6.9) with (6.3) and either of (6.5) and (6.6). In each case we obtain $B_1=B_2=B_3=B_4=0$ for B_j of $\Delta(z)$. q.e.d.

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