

A CHARACTERIZATION OF CERTAIN WEAKLY PSEUDOCONVEX DOMAINS

Dedicated to Professor Fuichi Uchida on his sixtieth birthday

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Abstract. By making use of well-known extension theorems on holomorphic mappings and CR-mappings and applying Webster's CR-invariant metrics, we give a characterization of certain weakly pseudoconvex domains from the viewpoint of biholomorphic automorphism groups.

Introduction. This is a continuation of our previous paper [11], and we retain the terminology and notation there.

Let D be a bounded domain in \mathbb{C}^n and let $p \in \partial D$. Then we say that *the condition (*) is fulfilled for (D, p)* if

- (*) there exists a compact set K in D , a sequence $\{k_v\}$ in K and a sequence $\{\varphi_v\}$ in $\text{Aut}(D)$ such that $\lim_{v \rightarrow \infty} \varphi_v(k_v) = p$.

Now assume that the condition (*) is fulfilled for (D, p) . Then we may ask if it is possible to determine the global structure of D from the local shape of the boundary ∂D near p . Certainly, it is impossible without any further assumption, as one may see in the examples such as the direct product of the open unit disk in \mathbb{C} and an arbitrary bounded domain in \mathbb{C}^{n-1} . As for this problem, it was shown by Wong [25] that if D is a strictly pseudoconvex domain in \mathbb{C}^n with smooth boundary and the condition (*) is fulfilled for (D, p) for some $p \in \partial D$, then D is biholomorphically equivalent to the open unit ball B^n in \mathbb{C}^n . It was later extended by Rosay [20] to the case where ∂D near p is C^2 -smooth and strictly pseudoconvex. It is natural to see *what happens when p is a weakly (not strictly) pseudoconvex boundary point of D* . It was Greene and Krantz [8] who first dealt with this problem in the category of weakly pseudoconvex domains in \mathbb{C}^n with globally C^{n+1} -smooth boundaries. As a generalization of their result, we obtained in [11] the following characterization of the weakly pseudoconvex domain

$$E(k, \alpha) = \left\{ z \in \mathbb{C}^n \left| \sum_{i=1}^k |z_i|^2 + \left(\sum_{j=k+1}^n |z_j|^2 \right)^\alpha < 1 \right. \right\},$$

where $k \in \mathbb{Z}$ with $1 \leq k \leq n$ and $0 < \alpha \in \mathbb{R}$, and it is understood that $E(k, \alpha) = B^n$ if $k = n$:

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THEOREM K (Kodama [11]). *Let D be a bounded domain in \mathbb{C}^n satisfying the following conditions:*

- (1) $p = (1, 0, \dots, 0) \in \partial D \cap \partial E(k, \alpha)$;
- (2) *there is an open neighborhood U of p in \mathbb{C}^n such that $D \cap U = E(k, \alpha) \cap U$;*
- (3) *the condition (*) is fulfilled for (D, p) .*

Then D is biholomorphically equivalent to the domain $E(k, \alpha)$.

It should be remarked that, in general, $E(k, \alpha)$ is not geometrically convex and, moreover, its boundary is not smooth at every point x of the form $x = (x_1, \dots, x_k, 0, \dots, 0)$. Also, noting the fact that such a boundary point x is an accumulation point of the $\text{Aut}(E(k, \alpha))$ -orbit passing through the origin of \mathbb{C}^n , one sees that exactly the same conclusion in Theorem K remains valid for an arbitrary point $x = (x_1, \dots, x_k, 0, \dots, 0) \in \partial D \cap \partial E(k, \alpha)$ as well as $p = (1, 0, \dots, 0)$. This theorem was later extended by Kodama, Krantz and Ma [15] to a more general domain, called a *generalized complex ellipsoid*,

$$E(n; n_1, \dots, n_s; p_1, \dots, p_s) = \left\{ (z_1, \dots, z_s) \in \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_s} \mid \sum_{i=1}^s |z_i|^{2p_i} < 1 \right\}$$

in $\mathbb{C}^n = \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_s}$, where $0 < p_1, \dots, p_s \in \mathbb{R}$ and $0 < n_1, \dots, n_s \in \mathbb{Z}$ with $n = n_1 + \dots + n_s$, as follows:

THEOREM K-K-M (Kodama, Krantz and Ma [15]). *Let D be a bounded domain in \mathbb{C}^n with a point $p \in \partial D$ and E a generalized complex ellipsoid in \mathbb{C}^n as above. We assume that*

- (1) $p \in \partial E$ and *there is an open neighborhood U of p in \mathbb{C}^n such that $D \cap U = E \cap U$;*
- (2) *the condition (*) is fulfilled for (D, p) and also for (E, p) .*

Then D is biholomorphically equivalent to E . In particular, at least one of the exponents p_i must be equal to 1.

In view of Kodama [12], [13] (in which the structure of generalized complex ellipsoids in \mathbb{C}^n with all $n_i = 1$ was investigated), it would be natural to ask the following questions: In Theorem K-K-M,

- (Q.1) *can we remove the condition (*) for (E, p) ?*
- (Q.2) *can we prove that $D = E$ as sets?*

These cannot be answered in full generality at this moment except when all p_i 's are positive integers, i.e., the boundary ∂E is real-analytic (cf. [14]). Recall that our proofs there relied heavily upon a result on the localization principle of holomorphic automorphisms of generalized complex ellipsoids E with real analytic boundaries due to Dini and Selvaggi Primicerio [5], [6]. A glance at their proof tells us that the real analyticity of ∂E cannot be avoided with their technique.

The main purpose of this paper is to give partial affirmative answers to the questions (Q.1) and (Q.2) when the boundary ∂E is not necessarily smooth. In fact, we

consider here exclusively generalized complex ellipsoids $E(n; k, n-k; 1, \alpha) = E(k, \alpha)$ with arbitrary real numbers $\alpha > 0$ and prove the following theorems, which were announced at the POSTECH International Conference on Several Complex Variables in Pohang, South Korea, 1997:

THEOREM 1. *Let $E_1 = E(k, \alpha)$, $E_2 = E(l, \beta)$ be generalized complex ellipsoids in \mathbb{C}^n with arbitrary real numbers $\alpha, \beta > 0$ and let $p_1 \in \partial E_1$, $p_2 \in \partial E_2$. We assume that*

- (1) $k \leq n-2$ and $l \leq n-2$;
 - (2) *there are open neighborhoods U_1 of p_1 , U_2 of p_2 in \mathbb{C}^n and a biholomorphic mapping $f: U_1 \rightarrow U_2$ such that $f(p_1) = p_2$, $f(U_1 \cap E_1) = U_2 \cap E_2$ and $f(U_1 \cap \partial E_1) = U_2 \cap \partial E_2$.*
- Then f extends to a biholomorphic mapping F from E_1 onto E_2 . In particular, we have $(k, \alpha) = (l, \beta)$.*

Combining this with a result of Bell [2; Theorem 2], we obtain the following:

COROLLARY. *Let $E(k, \alpha)$ and $E(l, \beta)$ be generalized complex ellipsoids in \mathbb{C}^n with $k \leq n-2$, $l \leq n-2$ and assume that $f: E(k, \alpha) \rightarrow E(l, \beta)$ is a proper holomorphic mapping. Then $(k, \alpha) = (l, \beta)$ and f is a biholomorphic automorphism of $E(k, \alpha)$.*

THEOREM 2. *Let D be a bounded domain in \mathbb{C}^n and let $E = E(k, \alpha)$ be a generalized complex ellipsoid in \mathbb{C}^n with $0 < \alpha \in \mathbb{R}$. We assume that*

- (1) *there exist a point $p \in \partial D \cap \partial E$ and an open neighborhood U of p in \mathbb{C}^n such that $D \cap U = E \cap U$;*
- (2) *the condition (*) is fulfilled for (D, p) .*

Then we have $D = E$ as sets.

We would like to remark that the assumption (1) in Theorem 1 is essential. Indeed, consider the generalized complex ellipsoids $E_1 = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid |z|^2 + |w|^{2\alpha} < 1\}$, $E_2 = B^2$ and a branch f of $(z, w) \mapsto (z, w^\alpha)$ defined in a small neighborhood of a point $p_1 = (z_0, w_0) \in \partial E_1$ with $w_0 \neq 0$, where $0 < \alpha \in \mathbb{R}$, $\alpha \neq 1$. Then f gives rise to a biholomorphic equivalence between a neighborhood U_1 of p_1 and a neighborhood U_2 of $p_2 := f(p_1) \in \partial E_2$ satisfying the condition (2) in Theorem 1; however, it is clear that f cannot be continued to a biholomorphic mapping from E_1 onto E_2 . Also, considering the special case $\alpha = \beta = 1$ in the corollary above, we see that every proper holomorphic self-mapping of the unit ball B^n must be a biholomorphic automorphism of B^n . This is just a well-known theorem of Alexander [1].

In Section 1, by making use of Rudin's extension theorem [21; p. 311] on holomorphic mappings defined near boundary points of B^n , we show some properties of generalized complex ellipsoids $E(k, \alpha)$, which will be a key step to the proofs of our theorems. After this preparation, Theorems 1 and 2 will be proved in Sections 2 and 3, respectively. Our proofs here are based on some extension theorems on proper holomorphic mappings and CR-mappings obtained by Forstnerič and Rosay [7], Pinchuk [18], [19], Bell [3], and also on the existence of Webster's CR-invariant metrics

on strictly pseudoconvex real analytic hypersurfaces in \mathbb{C}^n without umbilical points [22], [23].

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1. A key lemma. For later purpose, we prove some facts on the structure of the model spaces $E(k, \alpha)$ with arbitrary real numbers $\alpha > 0$.

Throughout the rest of this paper, we use the following notation: For a point $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and for a domain $E(k, \alpha)$, we set $z' = (z_1, \dots, z_k)$, $z'' = (z_{k+1}, \dots, z_n)$, $E = E(k, \alpha)$ and

$$\partial^*E = \{(z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z''| \neq 0, |z'|^2 + |z''|^{2\alpha} = 1\}$$

which is an open dense subset of ∂E . Then, by using the facts in the previous paper [11; Section 1], the following assertions are easily proved:

(1.1) ∂^*E is a connected, strictly pseudoconvex, real analytic hypersurface in \mathbb{C}^n ; moreover, it is simply connected if $k \leq n-2$ [9; p. 346].

(1.2) $\text{Aut}(E)$ can be regarded as a subgroup of $\text{Aut}(B^k \times \mathbb{C}^{n-k})$.

(1.3) $\text{Aut}(E) \cdot \partial^*E = \partial^*E$ and $\text{Aut}(E)$ acts transitively on ∂^*E as a real analytic CR-automorphism group of ∂^*E .

The following lemma will play a crucial role in our proofs of Theorems 1 and 2.

LEMMA. *Let $E = E(k, \alpha)$ be a generalized complex ellipsoid in \mathbb{C}^n with $k \leq n-2$ and let $p \in \partial^*E$. Assume that there are an open neighborhood U of p in \mathbb{C}^n and a biholomorphic mapping f from U into \mathbb{C}^n such that*

$$U \cap \partial E = U \cap \partial^*E, f(U \cap \partial^*E) = f(U) \cap \partial B^n \quad \text{and} \quad f(U \cap E) = f(U) \cap B^n.$$

Then f extends to a biholomorphic mapping $F: E \rightarrow B^n$. In particular, we have $\alpha = 1$.

PROOF. Since ∂^*E is a connected, strictly pseudoconvex, real analytic hypersurface in \mathbb{C}^n by (1.1), it follows from a result of Pinchuk [18], [19; p. 193] that f can be continued along any path lying in ∂^*E as a locally biholomorphic mapping. Since ∂^*E is now simply connected by our assumption $k \leq n-2$, the monodromy theorem guarantees that f extends to a locally biholomorphic mapping F defined on some connected open neighborhood V of ∂^*E in \mathbb{C}^n such that $F(\partial^*E) \subset \partial B^n$ and $F(V \cap E) \subset B^n$. Now we will proceed in several steps.

(1) F extends to a holomorphic mapping \tilde{F} from E into B^n . To prove this, take an arbitrary r with $0 < r < 1$ and put

$$K_r = \{(z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z'| \leq r, |z'|^2 + |z''|^{2\alpha} = 1\}.$$

Since $K_r \subset \partial^*E \subset V$ and K_r is compact in V , one can choose a small $\varepsilon = \varepsilon(r) > 0$ in such a way that

$$U_{r,\varepsilon} := \{(z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z'| < r, 1 - \varepsilon < |z'|^2 + |z''|^{2\alpha} < 1 + \varepsilon\} \subset V.$$

Clearly, $U_{r,\varepsilon}$ is a bounded Reinhardt domain in \mathbb{C}^n . Moreover, since $k \leq n-2$, we have $U_{r,\varepsilon} \cap \{z \in \mathbb{C}^n \mid z_j = 0\} \neq \emptyset$ for $j=1, \dots, n$. Hence, by a well-known fact [16; p. 15] every component function F_j of F has a holomorphic extension F_j^r to the domain

$$\hat{U}_{r,\varepsilon} = \{(z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z'| < r, |z'|^2 + |z''|^{2\alpha} < 1 + \varepsilon\},$$

the smallest complete Reinhardt domain in \mathbb{C}^n containing $U_{r,\varepsilon}$. In particular, putting

$$E_r = \{(z', z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z'| < r, |z'|^2 + |z''|^{2\alpha} < 1\},$$

we see that $F = (F_1, \dots, F_n)$ has a holomorphic extension $F^r := (F_1^r, \dots, F_n^r)$ to $E_r \cup V$. Note that $E_r \subset E_s$ for $0 < r < s < 1$, $\bigcup_{0 < r < 1} E_r = E$ and that the holomorphic extensions F^r are uniquely determined by the values of F on a small neighborhood of the point $(0, \dots, 0, 1) \in V \cap \partial^* E$. Then, by standard argument, one can define a holomorphic extension $\tilde{F}: E \cup V \rightarrow \mathbb{C}^n$ of $F: V \rightarrow \mathbb{C}^n$.

Now we wish to show that $\tilde{F}(E) \subset B^n$. For this let us fix an arbitrary point $z_o = (z'_o, z''_o) \in E$ and set

$$E(z_o) = \{(z'_o, z'') \in \mathbb{C}^k \times \mathbb{C}^{n-k} \mid |z'_o|^2 + |z''|^{2\alpha} < 1\},$$

which can be regarded as an open ball in \mathbb{C}^{n-k} . Consider the non-constant, continuous plurisubharmonic function $\psi: z'' \mapsto -1 + |\tilde{F}(z'_o, z'')|^2$ defined on some open neighborhood of the closure $\overline{E(z_o)}$ of $E(z_o)$ in \mathbb{C}^{n-k} . Then $\psi(\partial E(z_o)) = 0$ and $\psi(z'') < 0$ on $E(z_o) \cap V$. This, combined with the maximum principle for plurisubharmonic functions, guarantees that $\psi(z'') < 0$, i.e., $\tilde{F}(z_o) \in B^n$ and accordingly $\tilde{F}(E) \subset B^n$.

(2) *There exists a locally injective, real analytic homomorphism $\Phi: \text{Aut}(E) \rightarrow \text{Aut}(B^n)$ such that $\Phi(\sigma) \circ \tilde{F} = \tilde{F} \circ \sigma$ on E for all $\sigma \in \text{Aut}(E)$.* Indeed, take an arbitrary $\sigma \in \text{Aut}(E)$. By virtue of (1.2) and (1.3), one can choose an open neighborhood W of the point $p \in \partial^* E$ so small that $W \cup \sigma(W) \subset V$ and \tilde{F} is injective on W and on $\sigma(W)$. Let us consider the biholomorphic mapping $\Psi := \tilde{F} \circ \sigma \circ (\tilde{F}|_W)^{-1}: \tilde{F}(W) \rightarrow \tilde{F}(\sigma(W))$. By an extension theorem due to Rudin [21; p. 311] we obtain an element $\tilde{\Psi} \in \text{Aut}(B^n)$ such that $\tilde{\Psi}(z) = \Psi(z)$ for all $z \in \tilde{F}(W \cap E)$. Note that $W \cap E$ and $\tilde{F}(W \cap E)$ are non-empty open subsets of E and B^n , respectively. Then, by the principle of analytic continuation, we have that $\tilde{\Psi} \circ \tilde{F} = \tilde{F} \circ \sigma$ on E and $\tilde{\Psi}$ is uniquely determined by σ . Accordingly, one can define a mapping

$$\Phi: \text{Aut}(E) \rightarrow \text{Aut}(B^n)$$

by setting $\Phi(\sigma) = \tilde{\Psi}$ so that $\Phi(\sigma) \circ \tilde{F} = \tilde{F} \circ \sigma$ on E for all $\sigma \in \text{Aut}(E)$.

It is easy to check that Φ is a group homomorphism. Once it is shown that Φ is continuous at the identity element id_E of $\text{Aut}(E)$, it follows that Φ is real analytic on $\text{Aut}(E)$ (cf. [9; p. 117]). Since the topology of $\text{Aut}(E)$ satisfies the second axiom of countability, we have only to show that Φ is sequentially continuous at id_E . For this let us take an arbitrary sequence $\{\sigma_v\}$ in $\text{Aut}(E)$ which converges to id_E and assume that $\{\Phi(\sigma_v)\}$ does not converge to the identity element id_{B^n} of $\text{Aut}(B^n)$. Passing to a

subsequence, we may assume that there is a neighborhood O of id_{B^n} in $\text{Aut}(B^n)$ such that $\Phi(\sigma_v) \notin O$ for all v . Pick an arbitrary point $x \in E$. Then $\lim_{v \rightarrow \infty} \Phi(\sigma_v)(\tilde{F}(x)) = \lim_{v \rightarrow \infty} \tilde{F}(\sigma_v(x)) = \tilde{F}(x) \in B^n$, which implies that $\{\Phi(\sigma_v)(\tilde{F}(x))\}$ lies in a compact subset of B^n . Hence, after taking a subsequence if necessary, we may assume that $\{\Phi(\sigma_v)\}$ converges to some element $g \in \text{Aut}(B^n)$ (cf. [16; p. 82]). Since $g \notin O$, we see that $g \neq \text{id}_{B^n}$. On the other hand, we have $g(\tilde{F}(z)) = \lim_{v \rightarrow \infty} \Phi(\sigma_v)(\tilde{F}(z)) = \lim_{v \rightarrow \infty} \tilde{F}(\sigma_v(z)) = \tilde{F}(z)$ for all $z \in W \cap E$; consequently, $g = \text{id}_{B^n}$ by analytic continuation. This is a contradiction. Therefore, Φ is continuous at id_E , as desired.

Finally we claim that Φ is locally injective. It suffices to prove that Φ is injective in some neighborhood O of id_E . To this end, let us select a small open neighborhood W of the point $p \in \partial^* E$ in C^n and non-empty open subsets W_1, W_2 of $W \cap E$ with the properties: \tilde{F} is injective on W , and W_1 is a relatively compact subset of W_2 . We claim that $O = \{\sigma \in \text{Aut}(E) \mid \sigma(\overline{W_1}) \subset W_2\}$ is what is required. Indeed, it is clear that O is an open neighborhood of id_E in $\text{Aut}(E)$. Moreover, assume that $\Phi(\sigma_1) = \Phi(\sigma_2)$ for $\sigma_1, \sigma_2 \in O$. It follows that $\tilde{F}(\sigma_1(z)) = \Phi(\sigma_1)(\tilde{F}(z)) = \Phi(\sigma_2)(\tilde{F}(z)) = \tilde{F}(\sigma_2(z))$ for all $z \in E$. Since \tilde{F} is injective on $W_2 \subset W$ and since $\sigma_1(z), \sigma_2(z) \in W_2$ for all $z \in W_1$, this says that $\sigma_1 = \sigma_2$ on W_1 ; and hence $\sigma_1 = \sigma_2$ on E by analytic continuation. Therefore, we have shown that Φ is locally injective on $\text{Aut}(E)$.

(3) $\tilde{F}: E \rightarrow B^n$ is locally injective. Set $S = \{z \in E \mid (J\tilde{F})(z) = 0\}$, where $(J\tilde{F})(z)$ denotes the holomorphic Jacobian of \tilde{F} at z . Assume that $S \neq \emptyset$. Then S is a complex analytic subset of E of dimension $n-1$. Once $S \subset \{(z', z'') \in C^k \times C^{n-k} \mid z'' = 0\} \equiv C^k$ is shown, we arrive at a contradiction, since $\dim S = n-1 > k = \dim C^k$ by our assumption. Thus we have only to show that $S \subset C^k \times \{0\}$. To this end, take an arbitrary point $x = (x', x'') \in S$ and assume that $x'' \neq 0$. We may assume that x is a regular point of S . Recall that $\tilde{F} \circ \sigma = \Phi(\sigma) \circ \tilde{F}$ on E for all $\sigma \in \text{Aut}(E)$ by (2). Then

$$(J\tilde{F})(\sigma(x)) \cdot (J\sigma)(x) = (J\Phi(\sigma))(\tilde{F}(x)) \cdot (J\tilde{F})(x) = 0 \quad \text{and} \quad (J\sigma)(x) \neq 0$$

for all $\sigma \in \text{Aut}(E)$. This means that $\text{Aut}(E) \cdot x$, the $\text{Aut}(E)$ -orbit passing through the point x , is contained in S . This is impossible. Indeed, since $x'' \neq 0$, one can show by using the explicit expression of $\text{Aut}(E(k, \alpha))$ as in [11; Section 1] that the orbit $\text{Aut}(E) \cdot x$ is a real analytic submanifold of E of real dimension $2n-1$; on the other hand, S near x is a real analytic submanifold of E of real dimension $2n-2$. Therefore we conclude that $S \subset C^k \times \{0\}$, completing the proof of (3).

Before proceeding further, we need some preparation. First, notice that B^n is homogeneous and each element $g \in \text{Aut}(B^n)$ extends to a biholomorphic mapping defined in an open neighborhood of \bar{B}^n . Thus, shrinking the neighborhood V of $\partial^* E$ and replacing \tilde{F} by a suitable mapping of the form $g \circ \tilde{F}$ with some $g \in \text{Aut}(B^n)$, if necessary, we may assume that the holomorphic mapping $\tilde{F}: E \cup V \rightarrow C^n$ satisfies an additional condition $\tilde{F}(o) = o$, where o stands for the origin of C^n . Next, let us consider the total subgroups T_E and T_{B^n} of $\text{Aut}(E)$ and $\text{Aut}(B^n)$, respectively, induced by the rotations on C^n as follows:

$$(z_1, \dots, z_n) \mapsto ((\exp \sqrt{-1} \theta_1) z_1, \dots, (\exp \sqrt{-1} \theta_n) z_n), \quad (\theta_1, \dots, \theta_n) \in \mathbf{R}^n.$$

Then $\Phi(T_E)(o) = \Phi(T_E)(\tilde{F}(o)) = \tilde{F}(T_E(o)) = \tilde{F}(o) = o$, which says that $\Phi(T_E)$ is contained in the unitary group $U(n)$ of degree n (the isotropy subgroup of $\text{Aut}(B^n)$ at the origin o). Since $\Phi(T_E)$ as well as T_{B^n} is now a maximal torus in $U(n)$ by (2), it is well-known that they are conjugate to each other in $U(n)$, that is, there exists an element $\tau \in U(n)$ such that $\tau \cdot \Phi(T_E) \cdot \tau^{-1} = T_{B^n}$. Thus, considering $\tau \circ \tilde{F}$, $\tau \circ \Phi \circ \tau^{-1}$ instead of \tilde{F} , Φ if necessary, we may further assume that $\Phi(T_E) = T_{B^n}$. Under these assumptions, we claim the following:

(4) $\tilde{F}: E \rightarrow B^n$ is, in fact, a biholomorphic mapping. Thanks to the fact (3) one can choose a small open ball $B_\rho = \{z \in \mathbf{C}^n \mid |z| < \rho\} \subset E$ on which \tilde{F} is injective. Then, since $\tilde{F}(B_\rho) = \tilde{F}(T_E(B_\rho)) = \Phi(T_E)(\tilde{F}(B_\rho)) = T_{B^n}(\tilde{F}(B_\rho))$, we see that $\tilde{F}(B_\rho)$ is a bounded Reinhardt domain in \mathbf{C}^n with center at $\tilde{F}(o) = o$. Therefore, by a well-known theorem of H. Cartan [21; p. 24], the restriction $\tilde{F}|_{B_\rho}: B_\rho \rightarrow \tilde{F}(B_\rho)$ is a linear transformation. So we may assume that $\tilde{F} \in \text{Aut}(\mathbf{C}^n)$. This, combined with the facts that $\tilde{F}(\partial^*E) \subset \partial B^n$ and ∂^*E is dense in ∂E , guarantees that $\tilde{F}(E) = B^n$; and hence $\tilde{F}: E \rightarrow B^n$ is a biholomorphic mapping. Finally, the assertion $\alpha = 1$ follows from a result of Naruki [17]. This completes the proof of the Lemma.

2. Proof of Theorem 1. The proof is divided into three cases as follows:

Case 1. $\alpha = \beta = 1$. We have $E_1 = B^n = E_2$ in this case; hence our theorem follows at once from Rudin's result [21; p. 311].

Case 2. $\alpha \neq 1, \beta = 1$ or $\alpha = 1, \beta \neq 1$. We claim that this case does not occur. Indeed, assume the contrary. Since ∂^*E_1 and ∂^*E_2 are open dense subsets of ∂E_1 and ∂E_2 , respectively, and since $f: U_1 \rightarrow U_2$ is a biholomorphic mapping, we may assume that

$$p_1 \in \partial^*E_1, \quad U_1 \cap \partial E_1 = U_1 \cap \partial^*E_1, \quad \alpha \neq 1 \quad \text{and} \quad \beta = 1.$$

In particular, we have $E_2 = B^n$. As an immediate consequence of the Lemma in Section 1, we now have $\alpha = 1$, a contradiction.

Case 3. $\alpha \neq 1, \beta \neq 1$. Without loss of generality, we may assume that $p_i \in \partial^*E_i$ and $U_i \cap \partial E_i = U_i \cap \partial^*E_i$ for each $i = 1, 2$. Here, we claim that any strictly pseudoconvex real analytic hypersurface ∂^*E_i has no umbilical points in the sense of CR-geometry; hence, Webster's CR-invariant Riemannian metric g_i can be defined on the whole space ∂^*E_i . (For the notion of umbilical points and Webster's CR-invariant metrics in CR-geometry, see [4]; and also, [22], [23], [24].) To prove our claim, assume that there exists an umbilical point on ∂^*E_i . Then, all the points of ∂^*E_i are umbilical, since $\text{Aut}(E_i)$ acts transitively on ∂^*E_i by (1.3). Hence, ∂^*E_i must be locally biholomorphically equivalent to the sphere ∂B^n (see, for example, [22; p. 213]). By the Lemma in Section 1 we conclude that $\alpha = 1$ or $\beta = 1$ according as $i = 1$ or $i = 2$. This is a contradiction, as desired. Moreover, we see that (∂^*E_i, g_i) is complete as a Riemannian manifold, because ∂^*E_i is homogeneous under the CR-automorphism group $\text{Aut}(E_i)$. As a result, each (∂^*E_i, g_i)

is a connected and simply connected, complete real analytic Riemannian manifold. On the other hand, $f: U_1 \cap \partial^* E_1 \rightarrow U_2 \cap \partial^* E_2$ is an isometry with respect to the CR-invariant metrics g_1 and g_2 . By a well-known fact in Riemannian geometry [10; p. 256], f can now be uniquely extended to a global isometry $F: (\partial^* E_1, g_1) \rightarrow (\partial^* E_2, g_2)$. It is easily seen that $F: \partial^* E_1 \rightarrow \partial^* E_2$ is a real analytic CR-diffeomorphism. Accordingly, by a result of Pinchuk [18], [19; p. 186] there are open neighborhoods V_1 of $\partial^* E_1$ and V_2 of $\partial^* E_2$ in \mathbb{C}^n such that $F: \partial^* E_1 \rightarrow \partial^* E_2$ and its inverse $G := F^{-1}: \partial^* E_2 \rightarrow \partial^* E_1$ extend to locally biholomorphic mappings written in the same notation $F: V_1 \rightarrow \mathbb{C}^n$ and $G: V_2 \rightarrow \mathbb{C}^n$ satisfying $F(V_1 \cap E_1) \subset E_2$ and $G(V_2 \cap E_2) \subset E_1$. Hence, in exactly the same way as in (1) of the proof of the Lemma in Section 1, it can be shown that F and G extend to holomorphic mappings $\tilde{F}: E_1 \rightarrow \mathbb{C}^n$ and $\tilde{G}: E_2 \rightarrow \mathbb{C}^n$. Moreover, replacing $\psi(z'')$ by $\psi_1(z'') = \rho_2(\tilde{F}(z'_o, z''))$ in (1) of the proof of the Lemma in Section 1, we can prove that $\tilde{F}(E_1) \subset E_2$, where ρ_2 is the continuous plurisubharmonic function on \mathbb{C}^n defined by $\rho_2(z) = -1 + \sum_{i=1}^l |z_i|^2 + (\sum_{j=l+1}^n |z_j|^2)^\beta$, $z \in \mathbb{C}^n$. Analogously, we see that $\tilde{G}(E_2) \subset E_1$. Since $\tilde{G} \circ \tilde{F} = \text{id}_{E_1}$ near $\partial^* E_1$ and $\tilde{F} \circ \tilde{G} = \text{id}_{E_2}$ near $\partial^* E_2$, we conclude by analytic continuation that $\tilde{G} \circ \tilde{F} = \text{id}_{E_1}$ and $\tilde{F} \circ \tilde{G} = \text{id}_{E_2}$; consequently, $\tilde{F}: E_1 \rightarrow E_2$ is a biholomorphic mapping. Finally the assertion $(k, \alpha) = (l, \beta)$ follows now from Naruki [17], completing the proof of Theorem 1.

3. Proof of Theorem 2. The case $k = n - 1$ is contained in our previous paper [13].

Thus it suffices to prove Theorem 2 when $k \leq n - 2$. We have two cases to consider:

Case 1. *The point $p \in \partial D$ is a strictly pseudoconvex boundary point.* Hence D is biholomorphically equivalent to B^n by a result of Rosay [20]. Fix a biholomorphic mapping $F: D \rightarrow B^n$. Using a theorem on the boundary continuity of proper holomorphic mappings due to Forstnerič and Rosay [7], one sees that F extends to a homeomorphism from a connected open neighborhood M of p in $\partial D \cap \partial E$ onto an open subset M' of ∂B^n . Accordingly, by results of Bell [3; Theorem 2], Pinchuk [19; p. 186], the CR-homeomorphism $F: M \rightarrow M'$ can be extended to a biholomorphism between some open neighborhoods O of M and O' of M' in \mathbb{C}^n . Hence, $E = B^n$ by the Lemma in Section 1 and F extends to a biholomorphic automorphism Φ of B^n by [21; p. 311]. Set $\Psi = \Phi^{-1} \in \text{Aut}(B^n)$. Then, since $\Psi = F^{-1}$ near M' , we have that $\Psi = F^{-1}$ on B^n by analytic continuation. Thus we obtain that $D = F^{-1}(B^n) = \Psi(B^n) = B^n = E$, as desired.

Case 2. *The point $p \in \partial D$ is not a strictly pseudoconvex boundary point.* The point p must be of the form $p = (p_1, \dots, p_k, 0, \dots, 0)$ by (1.1). Therefore, it follows at once by Theorem K in the introduction that there exists a biholomorphic mapping $F: D \rightarrow E$. In exactly the same way as in the proof of [13; Lemma 3], it can be shown that F extends to a homeomorphism from an open subset of $U \cap \partial^* E \cap \partial D$ onto an open subset of $\partial^* E$. By the same reasoning as above, one can now find points $p_1 \in U \cap \partial^* E$, $p_2 \in \partial^* E$, open neighborhoods U_1 of p_1 , U_2 of p_2 in \mathbb{C}^n and a biholomorphic extension $\tilde{F}: U_1 \rightarrow U_2$ of F satisfying all the conditions in (2) of Theorem 1. Thus \tilde{F} extends to a biholomorphic automorphism $\tilde{\Phi}$ of E ; hence, repeating exactly the same arguments as in Case 1, we

can show that $D=E$ as sets. This completes the proof of Theorem 2.

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