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A GROUP-THEORETIC CHARACTERIZATION OF THE DIRECT PRODUCT OF A BALL AND PUNCTURED PLANES

Jisoo Byun, Akio Kodama and Satoru Shimizu

Abstract. Employing the same technique as in our previous papers, we establish an intrinsic characterization of the direct product of a complex Euclidean ball and punctured planes in the category of Stein manifolds from the viewpoint of holomorphic automorphism group.

1. Introduction. This is a continuation of our previous papers [1, 5], and the same terminology and notation will be used.

Let M be a connected complex manifold and let $\mathrm{Aut}(M)$ be the group of all holomorphic automorphisms of M equipped with the compact-open topology. Then one of the fundamental problems in complex geometric analysis is to determine the complex analytic structure of M by the topological group structure of $\mathrm{Aut}(M)$. Of course, this is impossible without any further assumptions on M, since there exist many domains in $\mathbf{C}^n$ that are not biholomorphically equivalent although their groups of automorphisms are isomorphic as topological groups. Moreover, it seems worth noting here that $\mathrm{Aut}(M)$ cannot have the structure of a Lie group, in general. Indeed, consider the holomorphic automorphism group $\mathrm{Aut}(\mathbf{C}^n)$ of $\mathbf{C}^n$ with $n \geq 2$, for example. Then it is terribly big and cannot have the structure of a Lie group with respect to the compact-open topology. However, there already exist several articles solving the problem affirmatively. For instance, Byun-Kodama-Shimizu [1], Isaev [2], Isaev-Kruzhilin [3] and Kodama-Shimizu [4, 5, 6] investigated the problem in the case when the manifolds M are some special domains in $\mathbf{C}^n$, and characterized such domains by their holomorphic automorphism groups. In particular, in the previous paper [1], by looking at some topological subgroups with Lie group structures of $\mathrm{Aut}(B^k \times \mathbf{C}^l)$, we succeeded in characterizing the space $B^k \times \mathbf{C}^l$ from the viewpoint of the holomorphic automorphism group, where B^k denotes the open unit ball in $\mathbf{C}^k$. In view of this, it would be naturally expected that the same conclusion is also valid for the space $B^k \times (\mathbf{C}^*)^l$, where $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ the punctured plane. Recall that,

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in the proof of our characterization theorem of $B^k \times \mathbf{C}^l$ in [1], the crucial fact is that $B^k \times \mathbf{C}^l$ admits an effective continuous action of the direct product $U(k) \times U(l)$ of unitary groups by holomorphic automorphisms and this fact simplified many arguments especially in the case where $k+l \geq 3$. But, $B^k \times (\mathbf{C}^*)^l$ no longer admits such an action of $U(k) \times U(l)$, except when $l = 1$. This causes many new difficulties to characterize the space $B^k \times (\mathbf{C}^*)^l$.

The main purpose of this paper is to overcome these difficulties and establish the following group-theoretic characterization of the space $B^k \times (\mathbf{C}^*)^l$:

Theorem. *Let M be a connected Stein manifold of dimension n. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$ *as topological groups for some integer k with $0 \leq k \leq n$. Then M is biholomorphically equivalent to $B^k \times (\mathbf{C}^*)^{n-k}$.*

In the special case of $n = 2$ and $k = 1$, this was already verified in [8] by using the same method as in Section 3.2 of this paper. Moreover, when $k = 0$, our result is a special case of [5, Theorem 2]. On the other hand, in the case of $k = n$, this was shown in [1] and [2]. Therefore, taking these into account, we will carry out the proof of Theorem in the case where $0 < k < n$.

The main idea of the proof of Theorem is as follows. Firstly, we realize M as a Reinhardt domain D in $\mathbf{C}^n$ by using the assumption of Theorem. Since M is a Stein manifold, the Reinhardt domain D is pseudoconvex. Secondly, using the pseudoconvexity of D, we list up all the possible cases where $\mathrm{Aut}(D)$ is isomorphic to $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$ as topological groups. We essentially use here the assumption that M is Stein, because, if D is not necessarily pseudoconvex, then we cannot classify D explicitly in contrast with the argument in [5], where M is only holomorphically separable and admitting a smooth envelope of holomorphy. Finally, by comparing carefully the structure of suitable subgroups of $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$ and $\mathrm{Aut}(D)$ that are isomorphic to each other under the given isomorphism $\Phi\colon \mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k}) \to \mathrm{Aut}(D)$, we eliminate all the possibilities except for the case where D is biholomorphically equivalent to the model domain $B^k \times (\mathbf{C}^*)^{n-k}$. As a typical example of this, we illustrate the following: Let Γ be a topological subgroup of $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$ and put $\Lambda = \Phi(\Gamma)$. Let $C(\Gamma)$ be the centralizer of Γ and $Z(\Gamma)$ its commutator group in $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$. Similarly, we denote by $C(\Lambda)$ and $Z(\Lambda)$ the subgroups of $\mathrm{Aut}(D)$ relative to Λ. Hence, $Z(\Gamma)$ and $Z(\Lambda)$ are isomorphic under the isomorphism Φ. With these notations, if one of the cases where D is not biholomorphically equivalent to $B^k \times (\mathbf{C}^*)^{n-k}$ occurs, then one can find a topological subgroup Γ of $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^{n-k})$ such that $Z(\Gamma)$

is non-abelian, while $Z(\Lambda)$ is abelian, a contradiction. Making use of these kind of arguments, we obtain the conclusion of Theorem.

Combining Riemann's extension theorem with the proof of [1, Corollary], we obtain the following fundamental fact:

COROLLARY. *If two pairs (k, l) and (k', l') of non-negative integers do not coincide, then the groups* $\mathrm{Aut}(B^k \times (\mathbf{C}^*)^l)$ *and* $\mathrm{Aut}(B^{k'} \times (\mathbf{C}^*)^{l'})$ *are not isomorphic as topological groups.*

This paper is organized as follows. In Section 2, we collect some preliminary facts. In particular, two main tools for our study are given. One is a special subgroup $\mathcal{G}(D)$ of $GL(n, \mathbf{Z})$ which, in some sense, measures the complexity of the algebraic automorphism group $\mathrm{Aut}_{\mathrm{alg}}(D)$ of a given Reinhardt domain D, and the other is a tool for the standardization of compact group actions on complex manifolds. After that, employing the same techniques as in the previous papers [1, 5], we prove Theorem in Section 3.

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2. Preliminaries. Throughout this paper, we use the following notation: For the given integer k with $0 \leq k \leq n$ and a point $(z_1, \ldots, z_n) \in \mathbf{C}^n$, we set

$$\begin{aligned} &l = n - k, \quad \Omega^*_{k,l} = B^k \times (\mathbf{C}^*)^l, \quad z = (z_1, \ldots, z_k), \\ &z' = (z_2, \ldots, z_k) \quad \text{and} \quad w = (w_1, \ldots, w_l) = (z_{k+1}, \ldots, z_n). \end{aligned}$$

For non-negative integers p, q, positive integers r, s, and a ring R, we denote by

$$\begin{aligned} &X_{p,q} \;=\; \mathbf{C}^p \times (\mathbf{C}^*)^q, \;\; X^\sharp \;=\; X_{1,0} \text{ or } X_{0,1}, \text{ and} \\ &M(r, s, R) \text{ the set of all } r \times s \text{ matrices over } R. \end{aligned}$$

Let D be an arbitrary Reinhardt domain in $\mathbf{C}^n$. Then we have important topological subgroups $T(D)$ and $\Pi(D)$ of $\mathrm{Aut}(D)$ induced by the standard actions of the n-dimensional torus $T^n = (U(1))^n$ on $\mathbf{C}^n$ and the multiplicative group $(\mathbf{C}^*)^n$ on $\mathbf{C}^n$ respectively, and also the topological subgroup $\mathrm{Aut}_{\mathrm{alg}}(D)$ of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that each component of φ is given by a Laurent monomial, that is, setting $\varphi = (\varphi_1, \ldots, \varphi_n)$ by coordinates, φ_i are given by

$$\varphi_i(z) = \alpha_i z_1^{a_{i1}} \cdots z_n^{a_{in}}, \quad 1 \leq i \leq n, \tag{2.1}$$

where $(a_{ij}) \in GL(n, \mathbf{Z})$ and $(\alpha_i) \in (\mathbf{C}^*)^n$. We call $\mathrm{Aut}_{\mathrm{alg}}(D)$ the *algebraic automorphism group of* D and each element of $\mathrm{Aut}_{\mathrm{alg}}(D)$ is called an *algebraic automorphism of* D. It is known [4] that these groups are related in the following manner: *The centralizer of the torus* $T(D)$ *in* $\mathrm{Aut}(D)$ *is given by* $\Pi(D)$, *while the normalizer of* $T(D)$ *in* $\mathrm{Aut}(D)$ *is given by* $\mathrm{Aut}_{\mathrm{alg}}(D)$.

Here consider the mapping $\varpi\colon \mathrm{Aut}_{\mathrm{alg}}(D) \to GL(n, \mathbf{Z})$ that sends an element φ of $\mathrm{Aut}_{\mathrm{alg}}(D)$ written in the form (2.1) into the element $(a_{ij}) \in GL(n, \mathbf{Z})$. Then it is easy to see that ϖ is a group homomorphism with $\ker \varpi = \Pi(D)$; and hence it induces a group isomorphism

$$\mathrm{Aut}_{\mathrm{alg}}(D)/\Pi(D) \xrightarrow{\cong} \mathcal{G}(D) := \varpi(\mathrm{Aut}_{\mathrm{alg}}(D)) \subset GL(n, \mathbf{Z}). \tag{2.2}$$

More precisely, combining this with the proof of [5, p. 660, Sublemma], one can see the following:

LEMMA 2.1. *Let* D_1 *and* D_2 *be Reinhardt domains in* $\mathbf{C}^n$. *Assume that there exists a topological group isomorphism* $\Psi\colon \mathrm{Aut}(D_1) \to \mathrm{Aut}(D_2)$ *such that* $\Psi(T(D_1)) = T(D_2)$. *Then* Ψ *induces a group isomorphism* $\tilde{\Psi}\colon \mathcal{G}(D_1) \to \mathcal{G}(D_2)$ *between the groups* $\mathcal{G}(D_1)$ *and* $\mathcal{G}(D_2)$. *Moreover, there exists an element* $L \in GL(n, \mathbf{Z})$ *such that* $\tilde{\Psi}$ *is given by* $\tilde{\Psi}(N) = LNL^{-1}$ *for* $N \in \mathcal{G}(D_1)$.

The following standardization of compact group actions on complex manifolds is important for our proof:

LEMMA 2.2 (Generalized Standardization Theorem [7]). *Let* M *be a connected complex manifold of dimension* n *that is holomorphically separable and admits a smooth envelope of holomorphy and let* K *be a connected compact Lie group of rank* n. *Assume that an injective continuous group homomorphism* ρ *of* K *into* $\mathrm{Aut}(M)$ *is given. Then there exists a biholomorphic mapping* F *of* M *onto a Reinhardt domain* D *in* $\mathbf{C}^n$ *such that*

$$F\rho(K)F^{-1} = U(n_1) \times \cdots \times U(n_s) \subset \mathrm{Aut}(D),$$

where each $U(n_j)$ *is the unitary group of degree* n_j *and* $\sum_{j=1}^{s} n_j = n$.

Now let us consider the special case where D is our model space $\Omega^*_{k,l} = B^k \times (\mathbf{C}^*)^l$. Then, concerning the algebraic automorphism group of $\Omega^*_{k,l}$, we can see the following fact (cf. [9, Section 4]):

LEMMA 2.3. *The group* $\mathrm{Aut}_{\mathrm{alg}}(\Omega^*_{k,l})$ *consists of all elements having the form*

$$(z,w) \longmapsto \left(\alpha_1 z_{\sigma(1)}, \dots, \alpha_k z_{\sigma(k)}, \beta_1 w_1^{a_{11}} \cdots w_l^{a_{1l}}, \dots, \beta_l w_1^{a_{l1}} \cdots w_l^{a_{ll}}\right)$$

where $(\alpha_1, \dots, \alpha_k) \in T^k$, $(\beta_1, \dots, \beta_l) \in (\mathbf{C}^*)^l$, σ *are permutations of* $\{1, \dots, k\}$ *and* $(a_{ij}) \in GL(l, \mathbf{Z})$. *In particular, the group* $\mathcal{G}(\Omega^*_{k,l})$ *is isomorphic to the direct product* $S_k \times GL(l, \mathbf{Z})$, *where* S_k *is the linear symmetric group of degree* k.

We finish this section by the following fact, which can be proved by direct computations:

LEMMA 2.4. (1) *Let* $P \in M(r, n, \mathbf{Z})$, $Q \in M(n, s, \mathbf{Z})$ *and assume that* $PAQ = 0$ *for all* $A \in GL(n, \mathbf{Z})$. *If* $P \neq 0$, *then we have* $Q = 0$.

(2) *Let* $R, S \in M(1, t, \mathbf{Z})$ *and assume that* $RNS = 0$ *for all* $N \in M(t, 1, \mathbf{Z})$. *If* $R \neq 0$, *then we have* $S = 0$.

3. Proof of Theorem. Throughout this section, we write $\Omega^* = \Omega^*_{k,l}$ for the sake of simplicity.

As mentioned in the introduction, we shall prove Theorem in the case where $0 < k = n - l < n$. Moreover, once the proof of Theorem for $l \geq 2$ is accomplished, then that for $l = 1$ follows by a simple modification of it. Indeed, when $l = 1$, we have only one vector $\mathbf{q}_1$ in (3.5) below, and there are only two possibilities: $\mathbf{q}_1 = \mathbf{0}$ or $\mathbf{q}_1 \neq \mathbf{0}$. Hence, applying (a properly adjusted form of) the method used in CASES I or II in the proof for $l \geq 2$, one may obtain the proof required here. Therefore, we have only to prove Theorem under the assumption that $k \geq 1$ and $l \geq 2$. The proof will be divided into two cases where $k \geq 2$ and $k = 1$, and will be carried out in the following two subsections.

Now let M be a connected Stein manifold of dimension n, and assume that there exists a topological group isomorphism $\Phi \colon \mathrm{Aut}(\Omega^*) \to \mathrm{Aut}(M)$. Note that $T^n \subset U(k) \times T^l \subset \mathrm{Aut}(\Omega^*)$. Hence, by Lemma 2.2 we may assume that M is a Reinhardt domain D in $\mathbf{C}^n$ and we have a topological group isomorphism $\Phi \colon \mathrm{Aut}(\Omega^*) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega^*)) = T(D)$. It then follows from Lemma 2.1 that the groups $\mathcal{G}(\Omega^*)$ and $\mathcal{G}(D)$ defined in (2.2) are isomorphic. Moreover, by the same reasoning as in the proof of [5, Theorem 1], we may assume that

$$\Phi(U(k) \times T^l) = U(k) \times T^l, \quad \text{and so} \quad \Phi(SU(k)) = SU(k)$$

under the identification given by $SU(k) = SU(k) \times \{(1, \dots, 1)\} \subset U(k) \times T^l$.

Now, notice that the centralizer $C_{\Omega^*}(U(k) \times T^l)$ of $U(k) \times T^l$ in $\mathrm{Aut}(\Omega^*)$ is given by $\{\gamma E_k \,;\, |\gamma| = 1\} \times (\mathbf{C}^*)^l$ and it is isomorphic to the centralizer $C_D(U(k) \times T^l) \subset$

$\{\gamma E_k\,;\gamma\in\mathbf{C}^*\}\times(\mathbf{C}^*)^l$ of $U(k)\times T^l$ in $\mathrm{Aut}(D)$ under the isomorphism Φ. Therefore Φ induces a continuous topological group isomorphism, denoted again by Φ,

$$(3.1)\quad \Phi\colon (U(k)\times T^l)C_{\Omega^*}(U(k)\times T^l) = U(k)\times(\mathbf{C}^*)^l \to (U(k)\times T^l)C_D(U(k)\times T^l).$$

For later purpose, we here look into this isomorphism more closely. To this end, recall the description of the isomorphism Φ of $U(k)\times T^l$ onto itself given in [5, (2.3)]. Then, by the same reasoning as in that paper, one can choose a $(2l+2)\times(2l+1)$ real matrix

$$(3.2)\qquad \begin{pmatrix} M_1 & M_2 \\ 0 & M_3 \end{pmatrix} = \begin{pmatrix} a & b_1 & \cdots & b_l & r_1 & \cdots & r_l \\ c_1 & d_{11} & \cdots & d_{1l} & p_{11} & \cdots & p_{1l} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ c_l & d_{l1} & \cdots & d_{ll} & p_{l1} & \cdots & p_{ll} \\ 0 & \dots & \dots & 0 & s_1 & \cdots & s_l \\ \vdots & & & \vdots & q_{11} & \cdots & q_{1l} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & q_{l1} & \cdots & q_{ll} \end{pmatrix},$$

where $M_1\in GL(l+1,\mathbf{Z})$, M_2, $M_3\in M(l+1,l,\mathbf{R})$ with $\det(d_{ij})\neq 0$, $\operatorname{rank} M_3 = l$, such that the isomorphism Φ in (3.1) can be expressed as follows: For an arbitrary element $g\in U(k)\times(\mathbf{C}^*)^l$ written in the form

$$(3.3)\qquad g=\left(e^{2\pi i\theta}A, e^{2\pi i(\theta_1+i\phi_1)},\dots,e^{2\pi i(\theta_l+i\phi_l)}\right)$$

with $A\in SU(k)$ and $\theta,\theta_j,\phi_j\in\mathbf{R}$, $1\le j\le l$, the isomorphism Φ is given by

$$(3.4)\qquad \begin{aligned} \Phi(g) = \Big(&e^{2\pi i\{(a\theta+\sum_{j=1}^l b_j\theta_j)+\sum_{j=1}^l (r_j+is_j)\phi_j\}}\Phi_s(A), \\ &e^{2\pi i\{(c_1\theta+\sum_{j=1}^l d_{1j}\theta_j)+\sum_{j=1}^l (p_{1j}+iq_{1j})\phi_j\}}, \\ &\dots, e^{2\pi i\{(c_l\theta+\sum_{j=1}^l d_{lj}\theta_j)+\sum_{j=1}^l (p_{lj}+iq_{lj})\phi_j\}}\Big), \end{aligned}$$

where Φ_s is the restriction of Φ to $SU(k)$ and, without loss of generality, we may assume that $\Phi_s(A) = A$ or $\Phi_s(A) = \bar{A}$ for every $A\in SU(k)$. Note that the expression in the form (3.3) of a given element $g\in U(k)\times(\mathbf{C}^*)^l$ is not unique. However, the right hand side of (3.4) does not depend on the choice of representation of g as in (3.3) (cf. [5]).

The following vectors will be important in our proof:

$$(3.5)\qquad \mathbf{s} = (s_1,\dots,s_l),\quad \mathbf{q}_j = (q_{j1},\dots,q_{jl}),\ \ 1\le j\le l,$$

where s_i and q_{jk} are the components of the matrix M_3 in (3.2).

3.1. Proof of the Theorem in the case $k \geq 2$. We now proceed to define the following subgroup Γ of $\mathrm{Aut}(\Omega^*)$ and consider its image $\Lambda := \Phi(\Gamma)$ under Φ:

$$\begin{aligned}\Gamma &= \left\{(E_k, e^{-2\pi\phi_1}, \dots, e^{-2\pi\phi_l}) \,;\, \phi_j \in \mathbf{R},\ 1 \leq j \leq l\right\} \subset U(k) \times (\mathbf{C}^*)^l,\\ \Lambda &= \Big\{\big(e^{2\pi i \sum_j (r_j + i s_j)\phi_j} E_k, e^{2\pi i \sum_j (p_{1j} + i q_{1j})\phi_j}, \dots,\\ &\qquad e^{2\pi i \sum_j (p_{lj} + i q_{lj})\phi_j}\big) \,;\, \phi_j \in \mathbf{R},\ 1 \leq j \leq l\Big\} \subset GL(k, \mathbf{C}) \times (\mathbf{C}^*)^l.\end{aligned}$$

Since D is a pseudoconvex Reinhardt domain in $\mathbf{C}^n$ and since $\dim \Pi(D) = \dim \Pi(\Omega^*) = k + 2l < 2n$, we see that $D \cap (\mathbf{C}^*)^n$ is a proper subset of $(\mathbf{C}^*)^n$. Thus there exists a point $p_0 = (z_0, w_1^0, \dots, w_l^0) \in \partial D \cap (\mathbf{C}^*)^n$. Since the subgroup Λ of $\mathrm{Aut}(D)$ can be regarded as a subgroup of $\mathrm{Aut}(\mathbf{C}^n)$, the orbit $\Lambda \cdot p_0$ of Λ passing through the point p_0 must lie in the boundary ∂D. Take a point $(z, w) \in \Lambda \cdot p_0$ arbitrarily. Then there is a point $\phi = (\phi_1, \dots, \phi_l) \in \mathbf{R}^l$ such that

$$(\|z\|, |w_1|, \dots, |w_l|) = \left(e^{-2\pi \mathbf{s}\cdot\phi}\|z_0\|, e^{-2\pi \mathbf{q}_1\cdot\phi}|w_1^0|, \dots, e^{-2\pi \mathbf{q}_l\cdot\phi}|w_l^0|\right), \tag{3.6}$$

where $\mathbf{a} \cdot \mathbf{b}$ is the Euclidean inner product of $\mathbf{a}, \mathbf{b} \in \mathbf{R}^l$ and $\mathbf{s}, \mathbf{q}_j$ are the vectors appearing in (3.5). We now have two cases to consider.

Case I. $\{\mathbf{q}_1, \dots, \mathbf{q}_l\}$ is linearly dependent in $\mathbf{R}^l$.

Since $\mathrm{rank}\, M_3 = l$ by (3.2), we may assume that $\{\mathbf{q}_1, \dots, \mathbf{q}_{l-1}\}$ is linearly independent, so that $\mathbf{q}_l$ can be written uniquely as

$$\mathbf{q}_l = \lambda_1 \mathbf{q}_1 + \cdots + \lambda_{l-1}\mathbf{q}_{l-1} \quad \text{with} \quad \lambda_j \in \mathbf{R},\ 1 \leq j \leq l-1.$$

Thus $|w_1|^{-\lambda_1} \cdots |w_{l-1}|^{-\lambda_{l-1}}|w_l| = |w_1^0|^{-\lambda_1} \cdots |w_{l-1}^0|^{-\lambda_{l-1}}|w_l^0|$ on $\Lambda \cdot p_0$ by (3.6).

Lemma 3.1. Case I *does not occur.*

Proof. Assuming that this case occurs, we shall derive a contradiction. We have now two cases to consider.

Case (I-1). $\mathbf{q}_l = \mathbf{0}$: For each $\phi \in \mathbf{R}^l$ and for each $p_0 \in \partial D \cap (\mathbf{C}^*)^n$, we have

$$\|z\| = e^{-2\pi \mathbf{s}\cdot\phi}\|z_0\|, \quad |w_j| = e^{-2\pi \mathbf{q}_j\cdot\phi}|w_j^0|, \ 1 \leq j \leq l-1, \quad |w_l| = |w_l^0|.$$

Moreover, $\{\mathbf{s}, \mathbf{q}_1, \dots, \mathbf{q}_{l-1}\}$ is lnearly independent and $D \cap \{z = 0\} \neq \emptyset$, since D is a pseudoconvex Reinhardt domain and D is invariant under the action of $U(k) \times T^l$ with $k \geq 2$. Thus, after a linear change of coordinates, if necessary, D may be described as $D = \mathbf{C}^k \times X_{a,b} \times W$, where $0 \le a, b \in \mathbf{Z}$, $a + b = l - 1$, and W is one of the domains

$$\{|w_l| < 1\}, \quad \{|w_l| > 1\}, \quad \{0 < |w_l| < 1\}, \quad \text{and} \quad \{r < |w_l| < 1\} \ (0 < r < 1).$$

in $\mathbf{C}$. However this is impossible. Indeed, observe that the centralizer $\Pi(\Omega^*)$ of T^n in $\mathrm{Aut}(\Omega^*)$ is isomorphic to the real Lie group $T^k \times (\mathbf{C}^*)^l$ of dimension $k + 2l$. On the other hand, the centralizer $\Pi(D)$ of T^n in $\mathrm{Aut}(D)$ is isomorphic to the Lie group $(\mathbf{C}^*)^{n-1} \times T^1$ of dimension $2k + 2l - 1$ in any cases. Since these groups are isomorphic, we have $k = 1$, which contradicts our assumption $k \ge 2$, as desired.

Case (I-2). $\mathbf{q}_l \ne \mathbf{0}$: After relabeling the indices, if necessary, we may assume that $\lambda_1 \cdots \lambda_s \ne 0$, $\lambda_{s+1} = \cdots = \lambda_{l-1} = 0$, so that D can be expressed as

$$D \ = \ \mathbf{C}^k \times X_{a,b} \times Y$$

for

$$Y \ = \ \begin{cases} \{|w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} |w_l| < 1, \ \ w_j, w_l \in X^\sharp\}, \\ \{|w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} |w_l| > 1, \ \ w_j \in X^\sharp, \ w_l \in \mathbf{C}^*\}, \\ \{r < |w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} |w_l| < 1, \ \ w_j, \ w_l \in \mathbf{C}^*\} \ \ (0 < r < 1), \end{cases}$$

where $0 \le a, b \in \mathbf{Z}$, $a + b = l - s - 1$. (Recall that $X^\sharp = \mathbf{C}$ or $\mathbf{C}^*$.) This is also impossible. Indeed, notice that $\Pi(D)$ is isomorphic to the Lie group

$$(\mathbf{C}^*)^{k+l-s-1} \times \big\{(\gamma_1, \ldots, \gamma_s, \gamma_l) \in (\mathbf{C}^*)^{s+1} \, ; |\gamma_1|^{-\lambda_1} \cdots |\gamma_s|^{-\lambda_s} |\gamma_l| = 1\big\}$$

of dimension $2k + 2l - 1$. Then, since $\dim \Pi(\Omega^*) = k + 2l$ as above, we have $k = 1$, contradicting our assumption $k \ge 2$.

As a result, we have shown that Case I does not occur. $\square$

Case II. $\{\mathbf{q}_1, \ldots, \mathbf{q}_l\}$ is linearly independent in $\mathbf{R}^l$.

In this case, the vector $\mathbf{s}$ can be written uniquely in the form

$$\mathbf{s} \ = \ \lambda_1 \mathbf{q}_1 + \cdots + \lambda_l \mathbf{q}_l \quad \text{with} \quad \lambda_j \in \mathbf{R}, \ 1 \le j \le l.$$

Then, by the same procedure as in Case I, one can see that $\|z\| |w_1|^{-\lambda_1} \cdots |w_l|^{-\lambda_l}$ is constant on each orbit $\Lambda \cdot p_0$ in ∂D. We have again two cases to consider.

Case (II-1). $\mathbf{s} = \mathbf{0}$: By repeating the same arguement as in Case I, we may assume that D has the form

$$D = \{\|z\| < 1\} \times X_{a,b} \ \text{ for some } \ 0 \le a, b \in \mathbf{Z}, \ a + b = l. \tag{3.7}$$

Lemma 3.2. *In Case* (II-1), *D is biholomorphically equivalent to Ω^*.*

Proof. It suffices to show that $a = 0$ in (3.7). First we assert that $b > 0$. Indeed, assume not. Then $D = \{\|z\| < 1\} \times \mathbf{C}^l$ and so $\mathcal{G}(D) = S_k \times S_l$ is a finite group. On the other hand, $\mathcal{G}(\Omega^*) = S_k \times GL(l, \mathbf{Z})$ is infinite. Since $\mathcal{G}(D)$ is isomorphic to $\mathcal{G}(\Omega^*)$ by Lemma 2.1, this is a contradiction. Thus we have $b > 0$, as asserted.

Next, assume that $a > 0$. Then, by using obvious notation, the groups $\mathcal{G}(D)$ and $\mathcal{G}(\Omega^*)$ can be described as

$$\mathcal{G}(D) = \begin{pmatrix} S_k & 0 & 0 \\ 0 & S_a & M(a,b,\mathbf{Z}) \\ 0 & 0 & GL(b,\mathbf{Z}) \end{pmatrix} \quad \text{and} \quad \mathcal{G}(\Omega^*) = \begin{pmatrix} S_k & 0 \\ 0 & GL(l,\mathbf{Z}) \end{pmatrix}.$$

Moreover, by Lemma 2.1 one can find an element $L \in GL(n, \mathbf{Z})$ such that the group isomorphism $\tilde{\Phi}\colon \mathcal{G}(\Omega^*) \to \mathcal{G}(D)$ is given by $\tilde{\Phi}(N) = LNL^{-1}$ for $N \in \mathcal{G}(\Omega^*)$. Thus, putting $L = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right)$ and $L^{-1} = \left(\begin{smallmatrix} P^* & Q^* \\ R^* & S^* \end{smallmatrix}\right)$, where $P, P^* \in M(k,k,\mathbf{Z})$, $Q, Q^* \in M(k,l,\mathbf{Z})$, $R, R^* \in M(l,k,\mathbf{Z})$ and $S, S^* \in M(l,l,\mathbf{Z})$, we see that

$$\tilde{\Phi}\left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right) = \begin{pmatrix} PAP^* + QBR^* & PAQ^* + QBS^* \\ RAP^* + SBR^* & RAQ^* + SBS^* \end{pmatrix} \tag{3.8}$$

belongs to $\mathcal{G}(D)$ for every $A \in S_k$ and for every $B \in GL(l, \mathbf{Z})$. Hence we have $PAQ^* = 0$, $QBS^* = 0$ for all $A \in S_k$ and for all $B \in GL(l, \mathbf{Z})$.

If $Q = 0$, we have $Q^* = 0$ and $S^* = S^{-1}$. Thus it follows from (3.8) that

$$\begin{pmatrix} S_a & M(a,b,\mathbf{Z}) \\ 0 & GL(b,\mathbf{Z}) \end{pmatrix} = SGL(l,\mathbf{Z})S^{-1} = GL(l,\mathbf{Z}).$$

If $Q \neq 0$, we have $S^* = 0$ by Lemma 2.4. Thus, (3.8) implies that

$$\begin{pmatrix} S_a & M(a,b,\mathbf{Z}) \\ 0 & GL(b,\mathbf{Z}) \end{pmatrix} = \{RAQ^*\,; A \in S_k\}.$$

Therefore, in any cases we arrive at a contradiction, since $a > 0$ and $b > 0$.

As a result, we conclude that $a = 0$, as desired. □

Case (II-2). $\mathbf{s} \neq \mathbf{0}$: We wish to prove that this case does not occur. Once this is done, our proof of Theorem for $k \geq 2$ follows from Lemmas 3.1 and 3.2.

Renaming the indices if necessary, we may assume that

$$\lambda_1 \cdots \lambda_s \neq 0,\ \lambda_{s+1} = \cdots = \lambda_l = 0 \quad \text{for some } 1 \leq s \leq l.$$

After a change of coordinates by an algebraic automorphism, there exist only three possibilities as follows:

(A.1) $D \cap (\mathbf{C}^*)^n = \{\|z\||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} < 1\} \cap (\mathbf{C}^*)^n$,

(A.2) $D \cap (\mathbf{C}^*)^n = \{\|z\||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} > 1\} \cap (\mathbf{C}^*)^n$,

(A.3) $D \cap (\mathbf{C}^*)^n = \{r < \|z\||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} < 1\} \cap (\mathbf{C}^*)^n \quad (0 < r < 1)$.

Since $D \cap \{z = 0\} \neq \emptyset$, the cases (A.2) and (A.3) do not occur. So it is enough to consider only the case (A.1). In this case, if $\lambda_{j_0} > 0$ for a j_0, then $D \cap \{w_{j_0} = 0\} = \emptyset$. Indeed, assume that there exists a point $p = (z^*, w_1^*, \dots, w_l^*) \in D$ with $w_{j_0}^* = 0$. By taking a suitable nearby point if necessary, we may assume that $z^* \neq 0$, $w_j^* \neq 0$ for all $j \neq j_0$. Then, taking the limit $(z, w) \to p$ through $D \cap (\mathbf{C}^*)^n$, we obtain a contradiction: $1 \geq \lim_{(z,w)\to p} \|z\||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} = \infty$. Therefore, after a change of coordinates induced by the correspondence $w_j \to w_j^{\pm 1}$, $1 \leq j \leq s$, if necessary, one may assume that D is of the form

$$D = \left\{(z, w_1, \dots, w_l)\,;\, \|z\||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} < 1,\ z \in \mathbf{C}^k,\ w_j \in X^\sharp\right\} \tag{3.9}$$

with $\lambda_j < 0$, $1 \leq j \leq s$.

Lemma 3.3. *In (3.9), all λ_j's are rational numbers.*

Proof. Assuming that $\lambda_{j_0} \notin \mathbf{Q}$ for some j_0, we shall show that the commutator group $Z_{\Omega^*}(\Gamma)$ of the centralizer $C_{\Omega^*}(\Gamma)$ of Γ in $\mathrm{Aut}(\Omega^*)$ is non-abelian, while the commutator group $Z_D(\Lambda)$ of the centralizer $C_D(\Lambda)$ of Λ in $\mathrm{Aut}(D)$ is abelian. Once this is verified, we arrive at a contradiction, since $Z_{\Omega^*}(\Gamma)$ is isomorphic to $Z_D(\Lambda)$.

Now, put $H(z, w) = (h(z), w)$ for $h \in \mathrm{Aut}(B^k)$. Then $H \in C_{\Omega^*}(\Gamma)$; accordingly, one may regard $\mathrm{Aut}(B^k)$ as a subgroup of $C_{\Omega^*}(\Gamma)$. This, together with the fact that $\mathrm{Aut}(B^k)$ is a simple Lie group, guarantees that $Z_{\Omega^*}(\Gamma)$ is a non-abelian group containing $\mathrm{Aut}(B^k)$ as its subgroup.

Let us now study the structure of $Z_D(\Lambda)$. For this purpose, take an arbitrary element $F = (F_1, \dots, F_n) \in C_D(\Lambda)$ and let $F_a(z, w) = \sum A_{\mu\nu} z^\mu w^\nu$ be the Laurent expansion of F_a $(1 \leq a \leq k)$ on D. Then

$$\begin{aligned} e^{2\pi i \sum_j (r_j + i s_j)\phi_j} F_a(z, w) \;=\; & F_a\Big(e^{2\pi i \sum_j (r_j + i s_j)\phi_j} z_1, \dots, \\ & e^{2\pi i \sum_j (r_j + i s_j)\phi_j} z_k, e^{2\pi i \sum_j (p_{1j} + i q_{1j})\phi_j} w_1, \dots, e^{2\pi i \sum_j (p_{lj} + i q_{lj})\phi_j} w_l\Big) \end{aligned}$$

for all $\phi_j \in \mathbf{R}$. Therefore, whenever $A_{\mu\nu} \neq 0$, we have

$$(|\mu| - 1)\sum_{j=1}^{l}(r_j + i s_j)\phi_j + \nu_1 \sum_{j=1}^{l}(p_{1j} + i q_{1j})\phi_j + \cdots + \nu_l \sum_{j=1}^{l}(p_{lj} + i q_{lj})\phi_j = 0$$

for all $\phi_j \in \mathbf{R}$. Hence

$$(|\mu|-1)r_j + \nu_1 p_{1j} + \cdots + \nu_l p_{lj} = 0, \quad (|\mu|-1)s_j + \nu_1 q_{1j} + \cdots + \nu_l q_{lj} = 0$$

for all j. In particular, it follows from the second equality that $(|\mu|-1)\mathbf{s} + \nu_1\mathbf{q}_1 + \cdots + \nu_l\mathbf{q}_l = \mathbf{0}$. Since $\{\mathbf{q}_1, \ldots, \mathbf{q}_l\}$ is linearly independent and $\mathbf{s} = \lambda_1\mathbf{q}_1 + \cdots + \lambda_s\mathbf{q}_s$, this means that $(|\mu|-1)\lambda_j + \nu_j = 0$, $1 \le j \le s$, and $\nu_j = 0$, $s+1 \le j \le l$. Consequently, $|\mu| = 1$ and $\nu_j = 0$ for all j, since $\lambda_{j_0} \notin \mathbf{Q}$. Therefore F_a has the form $F_a(z,w) = \sum_{j=1}^k \alpha_{aj} z_j$ with $\alpha_{aj} \in \mathbf{C}$.

Analogously, one can show that the component function F_{k+b} $(1 \le b \le l)$ can be written in the form $F_{k+b}(z,w) = \gamma_b w_b$ with $\gamma_b \in \mathbf{C}^*$.

Therefore each $F \in C_D(\Lambda)$ has the form $F(z,w) = (Az, \gamma_1 w_1, \ldots, \gamma_l w_l)$, where $A \in GL(k,\mathbf{C})$ and $\gamma_j \in \mathbf{C}^*$ (think of z as column vector), and hence, $Z_D(\Lambda)$ can be expressed as $Z_D(\Lambda) = \{(z,w) \mapsto (Az,w)\,;A \in G\}$ with a certain closed subgroup G of $SL(k,\mathbf{C})$. More strongly, we here claim that G is contained in $SU(k)$ and so $Z_D(\Lambda)$ is a real Lie group of dimension $\le k^2-1$. Once this is shown, we obtain a contradiction: $k^2 - 1 \ge \dim Z_D(\Lambda) = \dim Z_{\Omega^*}(\Gamma) \ge \dim \mathrm{Aut}(B^k) = k^2+2k$. Thus, to complete the proof of Lemma 3.3, we have only to prove that G is a subgroup of $SU(k)$. To this end, choose an element $F(z,w) = (Az,w)$ of $Z_D(\Lambda)$ arbitrarily and take a point $(z_0,w_0) = (z_0, w_1^0, \ldots, w_l^0) \in \partial D \cap (\mathbf{C}^*)^n$. Then $(\|z_0\|u, w_0) \in \partial D$ or $(A(\|z_0\|u), w_0) \in \partial D$ for every $u \in \mathbf{C}^k$, $\|u\| = 1$, so that $\|Au\| = \|A(\|z_0\|u)\||w_1^0|^{-\lambda_1}\cdots|w_s^0|^{-\lambda_s} = 1$. This means that $A \in U(k)$ and hence G is a subgroup of $SU(k)$, as asserted. $\square$

Our domain D is now of the form

$$D = \left\{(z,w_1,\ldots,w_l)\,;\|z\||w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s} < 1,\, z \in \mathbf{C}^k,\, w_j \in X^\sharp\right\}, \tag{3.10}$$

where $\lambda_j \in \mathbf{Q}$ and $\lambda_j < 0$ for every $1 \le j \le s$.

Lemma 3.4. *In* (3.10), *if all λ_j's are integers and $D \cap \{w_1 \cdots w_s = 0\} = \emptyset$, then D is biholomorphically equivalent to Ω^*.*

Proof. Consider the algebraic automorphism $\varphi(z,w) = (z^*,w^*)$ defined by

$$(z_1^*, \ldots, z_k^*, w^*) = (z_1 w_1^{-\lambda_1}\cdots w_s^{-\lambda_s}, \ldots, z_k w_1^{-\lambda_1}\cdots w_s^{-\lambda_s}, w).$$

Note that D is now contained in $\mathbf{C}^k \times (\mathbf{C}^*)^s \times \mathbf{C}^{l-s}$, the domain of φ. Thus it gives a biholomorphic equivalence between D and $\varphi(D) = \{\|z^*\| < 1\} \times X_{a,b}$ for some non-negative integers a, b. Hence, D is biholomorphically equivalent to Ω^* by Lemma 3.2. $\square$

Thanks to Lemma 3.4, we may assume that D has the form as in (3.10) and furthermore D satisfies the following:

$$(3.11)\qquad (\lambda_1,\dots,\lambda_s)\notin \mathbf{Z}^s \quad \text{or} \quad D\cap\{w_1\cdots w_s=0\}\neq\emptyset.$$

As mentioned before, in order to complete the proof of Theorem for $k\geq 2$, we have only to verify the following:

LEMMA 3.5. *Case* (II-2) *does not occur.*

For every subgroup G of $\mathrm{Aut}(\Omega^*)$, we know that $Z_D(\Phi(G))$ and $Z_{\Omega^*}(G)$ are isomorphic. Thus, in what follows we assume that Case (II-2) occurs, and we shall derive a contradiction by showing that the group $Z_D(\Phi(\Gamma))$ is abelian, while the group $Z_{\Omega^*}(\Gamma)$ is not, for a certain subgroup Γ of $\mathrm{Aut}(\Omega^*)$. Our proof of this will be divided into two cases where $k\geq 3$ and $k=2$.

PROOF OF LEMMA 3.5 FOR $k\geq 3$: In this case, under the identification given by

$$SU(k-1)=\left\{\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix};A\in SU(k-1)\right\}\subset SU(k),$$

we consider the following subgroup Γ of $\mathrm{Aut}(\Omega^*)$ and its image $\Lambda=\Phi(\Gamma)$:

$$\begin{aligned}\Gamma&=\left\{(A,e^{-2\pi\phi_1},\dots,e^{-2\pi\phi_l});A\in SU(k-1),\ \phi_j\in\mathbf{R},\ 1\leq j\leq l\right\},\\ \Lambda&=\Big\{\big(e^{2\pi i\sum_j(r_j+is_j)\phi_j}A,e^{2\pi i\sum_j(p_{1j}+iq_{1j})\phi_j},\dots,\\ &\qquad\qquad e^{2\pi i\sum_j(p_{lj}+iq_{lj})\phi_j}\big);A\in SU(k-1),\ \phi_j\in\mathbf{R},\ 1\leq j\leq l\Big\}.\end{aligned}$$

SUBLEMMA 3.6. *The group $Z_{\Omega^*}(\Gamma)$ is a non-abelian group.*

PROOF. Let $\Delta=\{u\in\mathbf{C};|u|<1\}$ be the unit disc in $\mathbf{C}$. Then, since every $\alpha\in\mathrm{Aut}(\Delta)$ extends to an element $\tilde\alpha\in\mathrm{Aut}(\Omega^*)$ written in the form $\tilde\alpha(z,w)=(\alpha(z_1),\beta(z_1)z',w)$, $(z,w)=(z_1,z',w)\in\Omega^*$, the group $C_{\Omega^*}(\Gamma)$ contains a subgroup G isomorphic to the simple Lie group $\mathrm{Aut}(\Delta)$. Thus, the commutator group $Z_{\Omega^*}(\Gamma)$ of $C_{\Omega^*}(\Gamma)$ also contains G. Consequently, it is non-abelian, as desired. $\square$

SUBLEMMA 3.7. *The group $Z_D(\Lambda)$ is an abelian group.*

PROOF. Take an element $F=(F_1,\dots,F_n)\in C_D(\Lambda)$ arbitrarily, and notice that $\Phi_s(SU(k-1))=SU(k-1)$. We then have

$$\begin{aligned}e^{2\pi i\sum_j(r_j+is_j)\phi_j}F_1(z,w)&=F_1\big(e^{2\pi i\sum_j(r_j+is_j)\phi_j}z_1,\\ &\quad e^{2\pi i\sum_j(r_j+is_j)\phi_j}Az',e^{2\pi i\sum_j(p_{1j}+iq_{1j})\phi_j}w_1,\dots,e^{2\pi i\sum_j(p_{lj}+iq_{lj})\phi_j}w_l\big)\end{aligned}$$

for all $\phi_j \in \mathbf{R}$ and for all $A \in SU(k-1)$. Since $k-1 \geq 2$, this implies that F_1 does not depend on the variable z'. So F_1 has the form $F_1(z,w) = F_1(z_1,w)$. Let $F_1(z_1,w) = \sum A_{\mu\nu} z_1^\mu w^\nu$ be the Laurent expansion of F_1 and let $A_{\mu\nu} \neq 0$. Then

$$(\mu-1)\sum_{j=1}^{l}(r_j+is_j)\phi_j + \nu_1\sum_{j=1}^{l}(p_{1j}+iq_{1j})\phi_j + \cdots + \nu_l\sum_{j=1}^{l}(p_{lj}+iq_{lj})\phi_j = 0$$

for all $\phi_j \in \mathbf{R}$. Hence

$$(\mu-1)r_j + \nu_1 p_{1j} + \cdots + \nu_l p_{lj} \;=\; 0, \;\; (\mu-1)s_j + \nu_1 q_{1j} + \cdots + \nu_l q_{lj} \;=\; 0 \tag{3.12}$$

for all j. Thus, putting

$$M = \begin{pmatrix} r_1 & p_{11} & \cdots & p_{l1} \\ \vdots & \vdots & & \vdots \\ r_l & p_{1l} & \cdots & p_{ll} \\ s_1 & q_{11} & \cdots & q_{l1} \\ \vdots & \vdots & & \vdots \\ s_l & q_{1l} & \cdots & q_{ll} \end{pmatrix}, \tag{3.13}$$

we have

$$M \cdot {}^t(\mu-1, \nu_1, \ldots, \nu_l) \;=\; 0, \tag{3.14}$$

where ${}^t\mathfrak{a}$ denotes the transpose of a given column vector $\mathfrak{a}$. On the other hand, by the second equality in (3.12), we have $(\mu-1)\mathbf{s} + \nu_1\mathbf{q}_1 + \cdots + \nu_l\mathbf{q}_l = \mathbf{0}$. Accordingly

$$\nu_j = -\lambda_j(\mu-1), \; 1 \leq j \leq s, \;\; \nu_{s+1} = \cdots = \nu_l = 0.$$

Now consider the mapping $F' := (F_2, \ldots, F_k)$. Since $F'(z_1, Az', w) = AF'(z,w)$ for all $A \in SU(k-1)$ and $k-1 \geq 2$, F' can be written in the form $F'(z,w) = \beta(z_1,w)z'$, where β is a holomorphic function. It then follows that

$$\beta\big(e^{2\pi i\sum_j(r_j+is_j)\phi_j}z_1, e^{2\pi i\sum_j(p_{1j}+iq_{1j})\phi_j}w_1, \ldots, e^{2\pi i\sum_j(p_{lj}+iq_{lj})\phi_j}w_l\big) = \beta(z_1,w)$$

for all $\phi_j \in \mathbf{R}$. So, letting $\beta(z_1,w) = \sum A_{\mu\nu} z_1^\mu w^\nu$ be the Laurent expansion of β and assuming $A_{\mu\nu} \neq 0$, we obtain that

$$\mu\sum_{j=1}^{l}(r_j+is_j)\phi_j + \nu_1\sum_{j=1}^{l}(p_{1j}+iq_{1j})\phi_j + \cdots + \nu_l\sum_{j=1}^{l}(p_{lj}+iq_{lj})\phi_j = 0$$

for all $\phi_j \in \mathbf{R}$, which says that

$$M \cdot {}^t(\mu, \nu_1, \ldots, \nu_l) \;=\; 0. \tag{3.15}$$

Next consider the component function F_{k+b} $(1 \le b \le l)$ of F. Then, since $F_{k+b}(z_1, Az', w) = F_{k+b}(z, w)$ for all $A \in SU(k-1)$, F_{k+b} has the form $F_{k+b}(z, w) = F_{k+b}(z_1, w)$. Let $F_{k+b}(z_1, w) = \sum A_{\mu\nu} z_1^{\mu} w^{\nu}$ be the Laurent expansion of F_{k+b} and assume that $A_{\mu\nu} \neq 0$. By repeating the same argument as above, we then have

$$(3.16) \qquad M \cdot {}^t(\mu, \nu_1, \dots, \nu_b - 1, \dots, \nu_l) \; = \; 0.$$

Notice that $l \le \operatorname{rank} M \le l+1$ by (3.2). Thus we have two cases to consider.

Case (a). $\operatorname{rank} M = l+1$: According to (3.14), (3.15) or (3.16), we have $\mu - 1 = \nu_1 = \cdots = \nu_l = 0$, $\mu = \nu_1 = \cdots = \nu_l = 0$, or $\mu = \nu_1 = \cdots = \nu_b - 1 = \cdots = \nu_l = 0$, respectively, and hence, F is of the form $F(z, w) = (\alpha z_1, \beta z', \gamma_1 w_1, \dots, \gamma_l w_l)$ with some non-zero constants α, β, γ_j. Thus $Z_D(\Lambda)$ is a trivial group.

Case (b). $\operatorname{rank} M = l$: Since $\{\mathbf{q}_1, \dots, \mathbf{q}_l\}$ is linearly independent, we have

$$\operatorname{rank} \begin{pmatrix} s_1 & q_{11} & \cdots & q_{l1} \\ \vdots & \vdots & & \vdots \\ s_l & q_{1l} & \cdots & q_{ll} \end{pmatrix} = l.$$

Therefore, by (3.14) we have $\nu_j = -\lambda_j(\mu - 1)$, $1 \le j \le s$, and $\nu_{s+1} = \cdots = \nu_l = 0$. Since $D \cap \{z = 0\} \neq \emptyset$, we see that $\mu \ge 0$. Here, if $\mu = 0$, then $\lambda_j = \nu_j \in \mathbf{Z}$ for all $1 \le j \le s$. Consequently, $D \cap \{w_1 \cdots w_s = 0\} \neq \emptyset$ by our assumption (3.11). Hence $\nu_{j_0} \ge 0$ for some $1 \le j_0 \le s$, which contradicts the fact $\nu_{j_0} = \lambda_{j_0} < 0$. As a result, we have seen that $\mu \ge 1$. Taking this into account, we put

$$(3.17) \qquad n_1 \; = \; \min\{n \in \mathbf{N}\,; -\lambda_j n \in \mathbf{Z},\, 1 \le j \le s\}, \;\; m_j \; = \; -\lambda_j n_1, \;\; 1 \le j \le s.$$

Then $n_1, m_j \in \mathbf{N}$ and

$$(\mu - 1, \nu_1, \dots, \nu_s, \nu_{s+1}, \dots, \nu_l) = m(n_1, m_1, \dots, m_s, 0, \dots, 0), \;\; m = 0, 1, 2, \dots .$$

Thus F_1 has the form $F_1(z, w) = \alpha(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}) z_1$. Here, noting that

$$|z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}| = \big(|z_1||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s}\big)^{n_1},$$

we see that α is a nowhere vanishing holomorphic function on the unit disc Δ.

For the mapping F', we have by (3.15) that $\mu \mathbf{s} + \nu_1 \mathbf{q}_1 + \cdots + \nu_l \mathbf{q}_l = \mathbf{0}$, from which it follows that $\nu_j = -\lambda_j \mu$, $1 \le j \le s$, and $\nu_{s+1} = \cdots = \nu_l = 0$. Since $D \cap \{z = 0\} \neq \emptyset$, we have $\mu \ge 0$, and so

$$(\mu, \nu_1, \dots, \nu_s, \nu_{s+1}, \dots, \nu_l) = m(n_1, m_1, \dots, m_s, 0, \dots, 0), \;\; m = 0, 1, 2, \dots .$$

Therefore F' can be expressed as $F'(z,w) = \beta(z_1^{n_1}w_1^{m_1}\cdots w_s^{m_s})z'$, where β is a nowhere vanishing holomorphic function on Δ.

For the function F_{k+b} ($1 \le b \le l$), we see by (3.16) that

$$\mu\mathbf{s} + \nu_1\mathbf{q}_1 + \cdots + (\nu_b - 1)\mathbf{q}_b + \cdots + \nu_l\mathbf{q}_l \;=\; \mathbf{0},$$

and so

$$\begin{cases} \nu_b - 1 = -\lambda_b\mu, \\ \nu_j = -\lambda_j\mu, \;\; 1 \le j \le s,\, j \ne b, \\ \nu_j = 0, \;\; s+1 \le j \le l \end{cases} \quad \text{or} \quad \begin{cases} \nu_b = 1, \\ \nu_j = -\lambda_j\mu, \;\; 1 \le j \le s, \\ \nu_j = 0, \;\; s+1 \le j \le l,\, j \ne b \end{cases}$$

according to $1 \le b \le s$ or $s+1 \le b \le l$. Hence, by using the integers n_1, m_j defined in (3.17), F_{k+b} can be written in the form $F_{k+b}(z,w) = \gamma_b(z_1^{n_1}w_1^{m_1}\cdots w_s^{m_s})w_b$, where γ_b is a nowhere vanishing holomorphic function on Δ.

Summarizing our result obtained so far, we have shown that $C_D(\Lambda)$ consists of all elements $F \in \mathrm{Aut}(D)$ having the form

$$(3.18) \qquad F(z,w) = (\alpha(u)z_1, \beta(u)z', \gamma_1(u)w_1, \ldots, \gamma_l(u)w_l), \;\; u = z_1^{n_1}w_1^{m_1}\cdots w_s^{m_s},$$

where α, β and γ_j are nowhere vanishing holomorphic functions on Δ.

Finally we assert that $Z_D(\Lambda)$ is, in fact, an abelian group. To this end, we set

$$(3.19) \qquad \hat{D} = \big\{(z,w) \in \mathbf{C}^n \,;\, \|z\||w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s} < 1\big\}.$$

Then, since $n_1, m_j \in \mathbf{N}$ for all $1 \le j \le s$, the group $C_D(\Lambda)$ can be regarded as a subgroup of $\mathrm{Aut}(\hat{D})$ by (3.18). We now verify our assertion only in the case where $l = 2$ and $s = 1$, since the verification in the general case is almost identical. Since $\|z\|^{n_1}|w_1|^{m_1} = (\|z\||w_1|^{-\lambda_1})^{n_1}$ in this case, we have

$$\hat{D} = \big\{(z,w) \in \mathbf{C}^k \times \mathbf{C}^2 \,;\, \|z\|^{n_1}|w_1|^{m_1} < 1\big\}$$

and we know that each $F \in C_D(\Lambda)$ has the form

$$F(z,w) = (\alpha(u)z_1, \beta(u)z', \gamma_1(u)w_1, \gamma_2(u)w_2), \;\; u = z_1^{n_1}w_1^{m_1},$$

with nowhere vanishing holomorphic functions α, β and γ_j on Δ.

We first claim that

$$(3.20) \qquad \alpha(u)^{n_1}\gamma_1(u)^{m_1} = \alpha(0)^{n_1}\gamma_1(0)^{m_1} \;\text{ on } \Delta, \text{ and } \; |\alpha(0)^{n_1}\gamma_1(0)^{m_1}| = 1.$$

To prove our claim, consider the holomorphic function $f(u) := \alpha(u)^{n_1}\gamma_1(u)^{m_1}u$ on Δ and take an arbitrary point $u \in \Delta$. Then there is a point $(z_1, 0', w_1, 0) \in \hat{D}$ with

$u = z_1^{n_1} w_1^{m_1}$. Since $F(z_1, 0', w_1, 0) = (\alpha(u)z_1, 0', \gamma_1(u)w_1, 0)$ belongs to $\hat{D}$, we have $|f(u)| < 1$, and hence $f(\Delta) \subset \Delta$. Moreover, put

$$M_c = \left\{(z_1, 0', w_1, 0) \in \hat{D} \,;\, z_1^{n_1} w_1^{m_1} = c\right\} \text{ for each } c \in \Delta.$$

Then it is not difficult to see that $F(M_c) \subset M_{f(c)}$. This, combined with the fact $F^{-1} \in C_D(\Lambda)$, yields at once that f is an automorphism of Δ with $f(0) = 0$. Thus we have $f(u) = Au$ with $|A| = 1$, proving the claim (3.20).

Thanks to (3.20), if we set $A = \alpha(0)^{n_1}\gamma_1(0)^{m_1}$, then F^{-1} is given by

$$F^{-1}(z, w) = (\alpha(A^{-1}u)^{-1}z_1, \beta(A^{-1}u)^{-1}z', \gamma_1(A^{-1}u)^{-1}w_1, \gamma_2(A^{-1}u)^{-1}w_2).$$

Thus, repeating the same computations as in the proof of [1, Theorem], one can show that $Z_D(\Lambda)$ is abelian. Hence we have proved Sublemma 3.7. $\square$

Since $Z_{\Omega^*}(\Gamma)$ is now isomorphic to $Z_D(\Lambda)$, we have a contradiction by Sublemmas 3.6 and 3.7, which completes the proof of Lemma 3.5 for $k \geq 3$.

Proof of Lemma 3.5 for $k = 2$. In this case, we consider the subgroup

$$\Gamma = \left\{\left(\begin{pmatrix} 1 & 0 \\ 0 & e^{4\pi i\theta}\end{pmatrix}, e^{2\pi i(\theta_1 + i\phi_1)}, \ldots, e^{2\pi i(\theta_l + i\phi_l)}\right) ; \theta, \theta_j, \phi_j \in \mathbf{R},\, 1 \leq j \leq l\right\}$$

of $U(k) \times (\mathbf{C}^*)^l \subset \mathrm{Aut}(\Omega^*)$ and put $\Lambda = \Phi(\Gamma)$. Since we may assume that $\Phi_s(A) = A$ for $A \in SU(2)$, it follows from (3.4) that

$$\begin{aligned}\Lambda = \Big\{\big(&e^{2\pi i[\{(a-1)\theta + \sum_j b_j\theta_j\} + \sum_j (r_j + is_j)\phi_j]}, e^{2\pi i[\{(a+1)\theta + \sum_j b_j\theta_j\} + \sum_j (r_j + is_j)\phi_j]}, \\ &e^{2\pi i\{(c_1\theta + \sum_j d_{1j}\theta_j) + \sum_j (p_{1j} + iq_{1j})\phi_j\}}, \ldots, \\ &e^{2\pi i\{(c_l\theta + \sum_j d_{lj}\theta_j) + \sum_j (p_{lj} + iq_{lj})\phi_j\}}\big) ; \theta, \theta_j, \phi_j \in \mathbf{R},\, 1 \leq j \leq l\Big\}.\end{aligned}$$

By the same reasoning as in the proof of Sublemma 3.6, we have the following:

Sublemma 3.8. *The group $Z_{\Omega^*}(\Gamma)$ is a non-abelian group.*

We shall complete the proof of Lemma 3.5 for $k = 2$ by showing the following:

Sublemma 3.9. *The group $Z_D(\Lambda)$ is an abelian group.*

PROOF. Let $F = (F_1, \dots, F_n) \in C_D(\Lambda)$ and let $F_1(z, w) = \sum A_{\mu\nu} z^\mu w^\nu$ be the Laurent expansion of F_1. Then, for any non-zero coefficient $A_{\mu\nu}$, we have

$$\begin{aligned}(\mu_1 - 1)&\left[\left\{(a-1)\theta + \sum_{j=1}^{l} b_j\theta_j\right\} + \sum_{j=1}^{l}(r_j + is_j)\phi_j\right] \\ &+ \mu_2\left[\left\{(a+1)\theta + \sum_{j=1}^{l} b_j\theta_j\right\} + \sum_{j=1}^{l}(r_j + is_j)\phi_j\right] \\ &\quad + \nu_1\left\{\left(c_1\theta + \sum_{j=1}^{l} d_{1j}\theta_j\right) + \sum_{j=1}^{l}(p_{1j} + iq_{1j})\phi_j\right\} + \cdots \\ &\qquad + \nu_l\left\{\left(c_l\theta + \sum_{j=1}^{l} d_{lj}\theta_j\right) + \sum_{j=1}^{l}(p_{lj} + iq_{lj})\phi_j\right\} = 0\end{aligned}$$

for all $\theta, \theta_j, \phi_j \in \mathbf{R}$, $1 \le j \le l$. Hence

$$\begin{pmatrix} a & c_1 & \dots & c_l \\ b_1 & d_{11} & \dots & d_{l1} \\ \vdots & \vdots & & \vdots \\ b_l & d_{1l} & \dots & d_{ll} \\ r_1 & p_{11} & \dots & p_{l1} \\ \vdots & \vdots & & \vdots \\ r_l & p_{1l} & \dots & p_{ll} \\ s_1 & q_{11} & \dots & q_{l1} \\ \vdots & \vdots & & \vdots \\ s_l & q_{1l} & \dots & q_{ll} \end{pmatrix} \begin{pmatrix} \mu_1 + \mu_2 - 1 \\ \nu_1 \\ \vdots \\ \nu_l \end{pmatrix} = \begin{pmatrix} \mu_1 - \mu_2 - 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Put

$$M = \begin{pmatrix} a & c_1 & \dots & c_l \\ b_1 & d_{11} & \dots & d_{l1} \\ \vdots & \vdots & & \vdots \\ b_l & d_{1l} & \dots & d_{ll} \end{pmatrix} \quad \text{and} \quad M^{-1} = \begin{pmatrix} \alpha_1 & * \\ \alpha_2 & * \\ \vdots & \vdots \\ \alpha_{l+1} & * \end{pmatrix}.$$

(Note that $M = {}^tM_1 \in GL(l+1, \mathbf{Z})$ by (3.2).) Then

$$\alpha_1 = \det(d_{ij})\big/\det M \in \mathbf{Z} \setminus \{0\}, \quad \alpha_j \in \mathbf{Z}, \; 2 \le j \le l+1, \text{ and}$$

$$\begin{pmatrix} \mu_1 + \mu_2 - 1 \\ \nu_1 \\ \vdots \\ \nu_l \end{pmatrix} = \begin{pmatrix} \alpha_1(\mu_1 - \mu_2 - 1) \\ \alpha_2(\mu_1 - \mu_2 - 1) \\ \vdots \\ \alpha_{l+1}(\mu_1 - \mu_2 - 1) \end{pmatrix}. \tag{3.21}$$

On the other hand, since $(\mu_1 + \mu_2 - 1)\mathbf{s} + \nu_1\mathbf{q}_1 + \cdots + \nu_l\mathbf{q}_l = \mathbf{0}$ and $\{\mathbf{q}_1, \dots, \mathbf{q}_l\}$ is linearly independent, we have

$$\nu_j = -\lambda_j(\mu_1 + \mu_2 - 1), \; 1 \le j \le s, \text{ and } \nu_j = 0, \; s+1 \le j \le l. \tag{3.22}$$

Let $F_2(z,w) = \sum A_{\mu\nu} z^\mu w^\nu$ be the Laurent expansion of F_2 and assume that $A_{\mu\nu} \neq 0$. Then, in exactly the same way as in the case of F_1, we have

$$\begin{pmatrix} \mu_1+\mu_2-1 \\ \nu_1 \\ \vdots \\ \nu_l \end{pmatrix} = \begin{pmatrix} \alpha_1(\mu_1-\mu_2+1) \\ \alpha_2(\mu_1-\mu_2+1) \\ \vdots \\ \alpha_{l+1}(\mu_1-\mu_2+1) \end{pmatrix}, \text{ and} \tag{3.23}$$

$$\nu_j = -\lambda_j(\mu_1+\mu_2-1),\ 1 \le j \le s, \ \text{ and } \ \nu_j = 0,\ s+1 \le j \le l. \tag{3.24}$$

Finally, denoting by $F_{2+b}(z,w) = \sum A_{\mu\nu} z^\mu w^\nu$ the Laurent expansion of F_{2+b} for $1 \le b \le l$ and assuming $A_{\mu\nu} \neq 0$, we obtain that

$$\begin{pmatrix} \mu_1+\mu_2 \\ \nu_1 \\ \vdots \\ \nu_b - 1 \\ \vdots \\ \nu_l \end{pmatrix} = \begin{pmatrix} \alpha_1(\mu_1-\mu_2) \\ \alpha_2(\mu_1-\mu_2) \\ \vdots \\ \alpha_{l+1}(\mu_1-\mu_2) \end{pmatrix}. \tag{3.25}$$

Thus

$$\begin{cases} \nu_b - 1 = -\lambda_b(\mu_1+\mu_2), \\ \nu_j = -\lambda_j(\mu_1+\mu_2),\ 1 \le j \le s,\, j \neq b, \\ \nu_j = 0,\ s+1 \le j \le l \end{cases} \quad \text{or} \quad \begin{cases} \nu_j = -\lambda_j(\mu_1+\mu_2),\ 1 \le j \le s, \\ \nu_b = 1, \\ \nu_j = 0,\ s+1 \le j \le l,\, j \neq b \end{cases}$$

according to $1 \le b \le s$ or $s+1 \le b \le l$.

Our next task is to find more concrete description of F. We proceed with a case by case analysis.

Case (a). $\alpha_1 = 1$: For F_1, we have $\mu_2 = 0$ by (3.21) and so $\nu_j = -\lambda_j(\mu_1 - 1)$, $1 \le j \le s$, and $\nu_j = 0$, $s+1 \le j \le l$, by (3.22). On the other hand, since $D \cap \{z = 0\} \neq \emptyset$, we see $\mu_1 \ge 0$. If $\mu_1 = 0$, then $\lambda_j = \nu_j \in \mathbf{Z}$, $1 \le j \le s$, and hence $D \cap \{w_1 \cdots w_s = 0\} \neq \emptyset$ by our assumption (3.11). Thus $\nu_{j_0} \ge 0$ for some $1 \le j_0 \le s$, contradicting the fact $\nu_{j_0} = \lambda_{j_0} < 0$. Therefore we have $\mu_1 \ge 1$. Put

$$n_1 = \min\{n \in \mathbf{N}\,; -\lambda_j n \in \mathbf{Z},\ 1 \le j \le s\}, \quad m_j = -\lambda_j n_1,\ 1 \le j \le s.$$

Then $n_1, m_j \in \mathbf{N}$ and

$$(\mu_1 - 1, \mu_2, \nu_1, \dots, \nu_s, \nu_{s+1}, \dots, \nu_l) = m(n_1, 0, m_1, \dots, m_s, 0, \dots, 0)$$

for $m = 0, 1, 2, \dots$. So F_1 has to be of the form $F_1(z,w) = \alpha(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}) z_1$.

As to F_2, we have $\mu_2 = 1$ by (3.23), so that $\nu_j = -\lambda_j\mu_1$, $1 \le j \le s$, and $\nu_j = 0$, $s+1 \le j \le l$, by (3.24). Since $\mu_1 \ge 0$, we have

$$(\mu_1, \mu_2 - 1, \nu_1, \dots, \nu_s, \nu_{s+1}, \dots, \nu_l) = m(n_1, 0, m_1, \dots, m_s, 0, \dots, 0)$$

for $m = 0, 1, 2, \dots$. Hence, F_2 has the form $F_2(z, w) = \beta(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}) z_2$.

Finally, consider the function F_{2+b} $(1 \le b \le l)$. Then we have $\mu_2 = 0$ by (3.25) and hence

$$\begin{cases} \nu_b - 1 = -\lambda_b\mu_1, \\ \nu_j = -\lambda_j\mu_1, \ \ 1 \le j \le s,\ j \ne b, \\ \nu_j = 0, \ \ s+1 \le j \le l \end{cases} \quad \text{or} \quad \begin{cases} \nu_j = -\lambda_j\mu_1, \ \ 1 \le j \le s, \\ \nu_b = 1, \\ \nu_j = 0, \ \ s+1 \le j \le l,\ j \ne b \end{cases}$$

according to $1 \le b \le s$ or $s+1 \le b \le l$. Since $\mu_1 \ge 0$, we conclude that F_{2+b} can be expressed as $F_{2+b}(z, w) = \gamma_b(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}) w_b$.

Eventually, we have shown that, if $\alpha_1 = 1$, $C_D(\Lambda)$ consists of all elements $F \in \mathrm{Aut}(D)$ written in the form

$$F(z, w) = (\alpha(u)z_1, \beta(u)z_2, \gamma_1(u)w_1, \dots, \gamma_l(u)w_l), \ \ u = z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s},$$

where α, β and γ_j are nowhere vanishing holomorphic functions on the unit disc Δ. In partiqular, we may regard $C_D(\Lambda)$ as a subgroup of $\mathrm{Aut}(\hat{D})$, where $\hat{D}$ is the domain defined in (3.19). Therefore, by repeating the same argument as in the proof of Sublemma 3.7, one can verify that $Z_D(\Lambda)$ is an abelian group.

Case (b). $\alpha_1 = -1$: In this case, put

$$n_2 \ = \ \min\{n \in \mathbf{N}\,; -\lambda_j n \in \mathbf{Z}, \ \ 1 \le j \le s\}, \quad m_j \ = \ -\lambda_j n_2, \ 1 \le j \le s.$$

Then, just as in the Case (a), we can see that each $F \in C_D(\Lambda)$ has the form

$$F(z, w) = (\alpha(u)z_1, \beta(u)z_2, \gamma_1(u)w_1, \dots, \gamma_l(u)w_l), \ \ u = z_2^{n_2} w_1^{m_1} \cdots w_s^{m_s},$$

where α, β and γ_j are nowhere vanishing holomorphic functions on Δ. Hence, $Z_D(\Lambda)$ has to be an abelian group.

Case (c). $\alpha_1 \ne \pm 1$: Notice that $(\alpha_1 - 1)/(\alpha_1 + 1)$ and $\alpha_1/(\alpha_1 + 1)$ are positive rational mumbers, in this case. Taking this into account, we put

$$\begin{aligned} n_1 \ &= \ \min\left\{ n \in \mathbf{N}\,; \frac{\alpha_1 - 1}{\alpha_1 + 1} n \in \mathbf{Z}, \ \frac{-2\alpha_1\lambda_j}{\alpha_1 + 1} n \in \mathbf{Z}, \ 1 \le j \le s \right\}, \\ n_2 \ &= \ \frac{\alpha_1 - 1}{\alpha_1 + 1} n_1, \quad m_j \ = \ \frac{-2\alpha_1\lambda_j}{\alpha_1 + 1} n_1, \ 1 \le j \le s. \end{aligned}$$

Also, noting that

$$|z_1|^{n_1}|z_2|^{n_2}|w_1|^{m_1}\cdots|w_s|^{m_s} \le \left(\|z\||w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s}\right)^{2\alpha_1 n_1/(\alpha_1+1)} < 1$$

for every point $(z,w)\in D$, we put $R = \sup\left\{\,|z_1^{n_1}z_2^{n_2}w_1^{m_1}\cdots w_s^{m_s}|\,;(z,w)\in D\right\}$. Then, by repeating the same argument as above, it can be verified that every element $F\in C_D(\Lambda)$ has the form

$$F(z,w) = (\alpha(u)z_1, \beta(u)z_2, \gamma_1(u)w_1, \dots, \gamma_l(u)w_l), \quad u = z_1^{n_1}z_2^{n_2}w_1^{m_1}\cdots w_s^{m_s},$$

where α,β and γ_j are nowhere vanishing holomorphic functions on the open disc $\{u\in\mathbf{C}\,;|u|<R\}$. From this we conclude that $Z_D(\Lambda)$ is an abelian group.

Therefore, we have shown that $Z_D(\Lambda)$ is an abelian group in any cases, as desired. □

Eventually, by Sublemmas 3.8 and 3.9 we have proved Lemma 3.5 for $k=2$. Hence we complete the proof of Theorem in the case $k\ge 2$.

3.2. Proof of Theorem in the case $k=1$. The method of our proof for $k=1$ is almost identical to that for $k\ge 2$. Therefore we shall give only an outline of the proof, and the detail is left to the reader.

We now proceed to define the following subgroup Γ of $\mathrm{Aut}(\Omega^*)$ and consider the image $\Lambda := \Phi(\Gamma)$ of it under the isomorphism Φ:

$$\begin{aligned}\Gamma &= \left\{(1, e^{-2\pi\phi_1},\dots,e^{-2\pi\phi_l})\,;\phi_j\in\mathbf{R}\right\},\\ \Lambda &= \left\{\left(e^{2\pi i\sum_j(r_j+is_j)\phi_j}, e^{2\pi i\sum_j(p_{1j}+iq_{1j})\phi_j},\dots,e^{2\pi i\sum_j(p_{lj}+iq_{lj})\phi_j}\right)\,;\phi_j\in\mathbf{R}\right\}.\end{aligned}$$

Now take a point $p_0=(z_0,w_1^0,\dots,w_l^0)\in\partial D\cap(\mathbf{C}^*)^n$ and let $(z,w)\in\Lambda\cdot p_0$. Then there exists an element $\phi\in\mathbf{R}^l$ such that

$$(|z|,|w_1|,\dots,|w_l|) = \left(e^{-2\pi\mathbf{s}\cdot\phi}|z_0|, e^{-2\pi\mathbf{q}_1\cdot\phi}|w_1^0|,\dots,e^{-2\pi\mathbf{q}_l\cdot\phi}|w_l^0|\right). \tag{3.26}$$

The following lemma can be proved just as in the case of Sublemma 3.6:

LEMMA 3.10. *The group $Z_{\Omega^*}(\Gamma)$ is a non-abelian group.*

As in Subsection 3.1, we divide the proof of Theorem for $k=1$ into two cases.

CASE I. $\{\mathbf{q}_1,\dots,\mathbf{q}_l\}$ is linearly dependent in $\mathbf{R}^l$.

By (3.2) we may assume that $\{\mathbf{q}_1,\dots,\mathbf{q}_{l-1}\}$ is linearly independent in $\mathbf{R}^l$. Then $\{\mathbf{s},\mathbf{q}_1,\dots,\mathbf{q}_{l-1}\}$ is linearly independent and $\mathbf{q}_l$ can be written uniquely as $\mathbf{q}_l =$

$\lambda_1\mathbf{q}_1 + \cdots + \lambda_{l-1}\mathbf{q}_{l-1}$, $\lambda_j \in \mathbf{R}$. Thus $|w_1|^{-\lambda_1}\cdots|w_{l-1}|^{-\lambda_{l-1}}|w_l|$ is constant on the orbit $\Lambda\cdot p_0$ for each point $p_0 \in \partial D \cap (\mathbf{C}^*)^n$ by (3.26).

Case (I-1). $\mathbf{q}_l = \mathbf{0}$: In this case, after a change of coordinates by an algebraic automorphism, D may coincide with one of the following three domains:

$$\text{(B.1)}\ \{|z|<1\}\times X_{a,b},\quad \text{(B.2)}\ \{0<|z|<1\}\times X_{a,b},\quad \text{(B.3)}\ \{r<|z|<1\}\times X_{a,b},$$

where a, b are non-negative integers with $a+b=l$ and $0<r<1$.

LEMMA 3.11. (1) *The case* (B.1) *occurs only when* $a=0$ *and hence* D *is biholomorphically equivalent to* Ω^*.
(2) *The cases* (B.2) *and* (B.3) *do not occur.*

PROOF. (1) If $a=0$, then $b=l$ and $D=\Omega^*$. Assume that $a>0$. Then we will arrive at a contradiction saying that $\mathcal{G}(\Omega^*)$ and $\mathcal{G}(D)$ are not isomorphic. Indeed, this can be achieved by employing the same computations as in the proof of Lemma 3.2.

(2) Assume that either of the cases (B.2) or (B.3) occurs. Then we can find a subgroup G of $GL(l,\mathbf{Z})$ such that

$$\mathcal{G}(D) = \left\{ \begin{pmatrix} 1 & 0 \\ N & A \end{pmatrix} ; N \in \mathbf{Z}^l,\ A \in G \right\}.$$

Now, as in the proof of Lemma 3.2, consider the group isomorphism $\tilde{\Phi}\colon \mathcal{G}(\Omega^*) \to \mathcal{G}(D)$ given by the matrix $L = \left(\begin{smallmatrix} P & Q \\ R & S \end{smallmatrix}\right) \in GL(n,\mathbf{Z})$ with $L^{-1} = \left(\begin{smallmatrix} P^* & Q^* \\ R^* & S^* \end{smallmatrix}\right)$. Then $\tilde{\Phi}^{-1}\left(\left(\begin{smallmatrix} 1 & 0 \\ N & E_l \end{smallmatrix}\right)\right) \in \mathcal{G}(\Omega^*)$ for all $N \in \mathbf{Z}^l$. Thus $Q^*NP=0$, $Q^*NQ=0$ and $S^*NP=0$. In particular, $Q=0$ by Lemma 2.4 and hence $\tilde{\Phi}^{-1}\left(\left(\begin{smallmatrix} 1 & 0 \\ N & E_l \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & E_l \end{smallmatrix}\right)$ for all $N \in \mathbf{Z}^l$. Clearly this is absurd, thereby the proof of (2) is completed. □

Case (I-2). $\mathbf{q}_l \neq \mathbf{0}$: We may assume that $\lambda_1\cdots\lambda_s \neq 0, \lambda_{s+1} = \cdots = \lambda_{l-1} = 0$ for some s. Thus, up to an algebraic automorphism, we have three cases:

(C.1) $D = X_{a,b} \times \left\{|w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s}|w_l| < 1,\ w_j, w_l \in X^\sharp\right\}$
with $\lambda_1, \ldots, \lambda_s < 0$,

(C.2) $D = X_{a,b} \times \left\{|w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s}|w_l| > 1,\ w_j \in X^\sharp,\ w_l \in \mathbf{C}^*\right\}$
with $\lambda_1, \ldots, \lambda_s > 0$,

(C.3) $D = X_{a,b} \times \left\{r < |w_1|^{-\lambda_1}\cdots|w_s|^{-\lambda_s}|w_l| < 1,\ w_j,\ w_l \in \mathbf{C}^*\right\}$
with $\lambda_1, \ldots, \lambda_s < 0$,

where a, b are non-negative integers with $a+b=l-s$ and $0<r<1$.

The proof of the following lemma is identical to that of Lemma 3.3:

LEMMA 3.12. *In the cases* (C.1) *through* (C.3)*, all* λ_j*'s are rational numbers.*

LEMMA 3.13. *We have* $D \cap \{w_1 \cdots w_s w_l = 0\} \neq \emptyset$*. In particular, the case* (C.3) *does not occur.*

PROOF. Assuming that $D \cap \{w_1 \cdots w_s w_l = 0\} = \emptyset$, we shall derive a contradiction.

In the case (C.1), we put

$$m_l = \min\{n \in \mathbf{N}\,; -\lambda_j n \in \mathbf{Z},\ 1 \leq j \leq s\}, \quad m_j = -\lambda_j m_l,\ 1 \leq j \leq s. \tag{3.27}$$

Then $m_l, m_j \in \mathbf{N}$ and $\gcd(m_1, \ldots, m_s, m_l) = 1$. Moreover, noting that

$$\big(|w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s}|w_l|\big)^{m_l} = |w_1|^{m_1} \cdots |w_s|^{m_s}|w_l|^{m_l},$$

we have $D = X_{a,b} \times \big\{0 < |w_1|^{m_1} \cdots |w_s|^{m_s}|w_l|^{m_l} < 1\big\}$. On the other hand, since $\gcd(m_1, \ldots, m_s, m_l) = 1$, there exists a matrix $(a_{ij}) \in GL(s+1, \mathbf{Z})$ such that $(a_{(s+1)1}, \ldots, a_{(s+1)(s+1)}) = (m_1, \ldots, m_s, m_l)$. Using this, we define the algebraic automorphism $\varphi(z, w) = (z^*, w^*)$ of $X_{a,b} \times (\mathbf{C}^*)^{s+1}$ by

$$z^* = z, \quad w_j^* = w_1^{a_{j1}} \cdots w_s^{a_{js}} w_l^{a_{j(s+1)}}, \ 1 \leq j \leq s,$$
$$w_l^* = w_1^{m_1} \cdots w_s^{m_s} w_l^{m_l}, \text{ and } w_j^* = w_j \ \text{ for } j \neq l.$$

Then $\varphi(D) = X_{a,b+s} \times \{0 < |w_l^*| < 1\}$. But this is impossible by Lemma 3.11.

In the case (C.2), D is algebraically equivalent to

$$D = X_{a,b} \times \big\{0 < |w_1|^{\lambda_1} \cdots |w_s|^{\lambda_s}|w_l| < 1\big\},$$

which is a domain of type (C.1). Thus this is also impossible.

In the case (C.3), the same algebraic isomorphism φ introduced above gives a biholomorphic mapping from D to the domain $\varphi(D) = X_{a,b+s} \times \{r < |w_l^*| < 1\}$ of type (B.3). Hence the case (C.3) does not occur, thereby the lemma is proved. □

By virtue of Lemma 3.13, it is enough to consider the following two cases:

Case (I-2-1). There exist integers $i_0, j_0 \in \{1, \ldots, s, l\}$ such that

$$i_0 \neq j_0, \ D \cap \{w_{i_0} = 0\} \neq \emptyset, \ D \cap \{w_{j_0} = 0\} \neq \emptyset;$$

Case (I-2-2). There exists an integer $i_0 \in \{1, \ldots, s, l\}$ such that

$$D \cap \{w_{i_0} = 0\} \neq \emptyset, \ D \cap \{w_j = 0\} = \emptyset \text{ for every } j \in \{1, \ldots, s, l\} \setminus \{i_0\}.$$

LEMMA 3.14. *If D is a domain of type* (C.1)*, then Case* (I-2-1) *does not occur.*

PROOF. Assume this does, and take an arbitrary element $F \in C_D(\Lambda)$. Then, it is possible to show, by the Laurent series argument as in the previous subsection, that F has the form

$$F(z, w) = (\alpha(u)z, \gamma_1(u)w_1, \dots, \gamma_l(u)w_l), \quad u = w_1^{m_1} \cdots w_s^{m_s} w_l^{m_l}, \tag{3.28}$$

where α, γ_j are nowhere vanishing holomorphic functions on Δ and m_j, m_l are the same integers appearing in (3.27). Therefore, $Z_D(\Lambda)$ must be abelian. Since $Z_D(\Lambda)$ is isomorphic to $Z_{\Omega^*}(\Gamma)$ and since $Z_{\Omega^*}(\Gamma)$ is non-abelian by Lemma 3.10, this is a contradiction. Thus we have proved the lemma. □

LEMMA 3.15. *Let D be a domain of type* (C.1) *and assume that Case* (I-2-2) *occurs. Then D is biholomorphically equivalent to Ω^*.*

PROOF. First we assert that $m_{i_0} = 1$. Indeed, assume that $m_{i_0} \geq 2$. Then, by using the fact that $\gcd(m_1, \dots, m_s, m_l) = 1$, we can verify that each $F \in C_D(\Lambda)$ has the same form as in (3.28). Thus $Z_D(\Lambda)$ has to be abelian, contradicting the assertion in Lemma 3.10.

When $m_{i_0} = 1$, let us define the algebraic automorphism $\varphi(z, w) = (z^*, w^*)$ by

$$z^* = z,\ w_{i_0}^* = w_1^{m_1} \cdots (w_{i_0-1})^{m_{i_0-1}} w_{i_0} (w_{i_0+1})^{m_{i_0+1}} \cdots w_l^{m_l},\ w_j^* = w_j,\ j \neq j_0.$$

Then D is biholomorphically equivalent to $\varphi(D) = X_{a,b+s} \times \{|w_{i_0}^*| < 1\}$. Consequently, our assertion follows from Lemma 3.11. □

The domain of type (C.2) can be transformed into some domain of type (C.1) by an algebraic automorphism. Thus, the investigation of such a domain is reduced to that of type (C.1). Therefore, we have proved Theorem for $k = 1$ in Case I.

CASE II. $\{\mathbf{q}_1, \dots, \mathbf{q}_l\}$ is linearly independent in $\mathbf{R}^l$.

In this case, $\mathbf{s}$ can be expressed as $\mathbf{s} = \lambda_1 \mathbf{q}_1 + \cdots + \lambda_l \mathbf{q}_l$, $\lambda_j \in \mathbf{R}$.

Case (II-1). $\mathbf{s} = \mathbf{0}$: By passing to an algebraic image if necessary, D can be described as $D = W \times X_{a,b}$, where $0 \leq a, b \in \mathbf{Z}$, $a + b = l$, and W is one of the three domains

$$\{|z| < 1\}, \quad \{0 < |z| < 1\} \quad \text{and} \quad \{r < |z| < 1\}\ (0 < r < 1)$$

in $\mathbf{C}$. Hence, this case is reduced to Case (I-1) and D is biholomorphically equivalent to Ω^*.

Case (II-2). $\mathbf{s} \neq \mathbf{0}$: We may assume that $\lambda_1 \cdots \lambda_s \neq 0$, $\lambda_{s+1} = \cdots = \lambda_l = 0$ for some s. Then, it is easily seen that $|z||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s}$ is a constant on the orbit $\Lambda \cdot p_0$ for each point $p_0 \in \partial D \cap (\mathbf{C}^*)^n$. So, by considering again suitable algebraic automorophisms if necessary, we have the following three cases:

$$\begin{aligned}
&\text{(D.1)} \quad D = \left\{|z||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} < 1,\ z, w_j \in X^\sharp\right\} \times X_{a,b} \\
&\qquad\qquad \text{with } \lambda_1, \ldots, \lambda_s < 0, \\
&\text{(D.2)} \quad D = \left\{|z||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} > 1,\ z \in \mathbf{C}^*,\ w_j \in X^\sharp\right\} \times X_{a,b} \\
&\qquad\qquad \text{with } \lambda_1, \ldots, \lambda_s > 0, \\
&\text{(D.3)} \quad D = \left\{r < |z||w_1|^{-\lambda_1} \cdots |w_s|^{-\lambda_s} < 1,\ z,\ w_j \in \mathbf{C}^*\right\} \times X_{a,b} \\
&\qquad\qquad \text{with } \lambda_1, \ldots, \lambda_s < 0,\ 0 < r < 1.
\end{aligned}$$

Again, repeating the same argument as before, we can show the following:

LEMMA 3.16. *In the cases* (D.1) *through* (D.3), *all λ_j's are rational numbers.*

LEMMA 3.17. *We have $D \cap \{zw_1 \cdots w_s = 0\} \neq \emptyset$. In particular, the case* (D.3) *does not occur.*

PROOF. Assuming that $D \cap \{zw_1 \cdots w_s = 0\} = \emptyset$, we will arrive at a contradiction as follows. Consider first the case (D.1) and put

$$N = \min\{n \in \mathbf{N}\,; -\lambda_j n \in \mathbf{Z},\ 1 \leq j \leq s\}, \quad m_j = -\lambda_j N,\ 1 \leq j \leq s. \tag{3.29}$$

Then $N, m_j \in \mathbf{N}$ and $\gcd(N, m_1, \ldots, m_s) = 1$. Moreover, D can be expressed as

$$D = \left\{0 < |z|^N |w_1|^{m_1} \cdots |w_s|^{m_s} < 1\right\} \times X_{a,b}.$$

Here, since $\gcd(N, m_1, \cdots, m_s) = 1$, by the same reasoning as in the proof of Lemma 3.13, there exists an algebraic automorphism φ of $(\mathbf{C}^*)^{s+1} \times X_{a,b}$ that transforms D into the domain $\{0 < |z^*| < 1\} \times (\mathbf{C}^*)^s \times X_{a,b}$. However, this is impossible by Lemma 3.11.

In the case (D.2), D is obviously algebraically equivalent to the domain

$$\left\{0 < |z||w_1|^{\lambda_1} \cdots |w_s|^{\lambda_s} < 1\right\} \times X_{a,b}.$$

But, since every $\lambda_j > 0$, this is also impossible by the above case (D.1).

In the case (D.3), by the same algebraic automorphism φ considered in the case (D.1), D is transformed into the domain $\{r < |z^*| < 1\} \times (\mathbf{C}^*)^s \times X_{a,b}$. Again this is impossible by Lemma 3.11. Therefore we have proved the lemma. □

By Lemma 3.17, we have the following four cases to consider:

Case (II-2-1). There exist integers $i_0, j_0 \in \{1, \dots, s\}$ such that

$$i_0 \neq j_0,\ D \cap \{w_{i_0} = 0\} \neq \emptyset,\ D \cap \{w_{j_0} = 0\} \neq \emptyset.$$

Case (II-2-2). $D \cap \{z = 0\} \neq \emptyset$, $D \cap \{w_{i_0} = 0\} \neq \emptyset$ for some $i_0 \in \{1, \dots, s\}$.

Case (II-2-3). $D \cap \{z = 0\} \neq \emptyset$, $D \cap \{w_j = 0\} = \emptyset$ for every $1 \leq j \leq s$.

Case (II-2-4). $D \cap \{z = 0\} = \emptyset$ and, for some $i_0 \in \{1, \dots, s\}$, one has

$$D \cap \{w_{i_0} = 0\} \neq \emptyset,\ D \cap \{w_j = 0\} = \emptyset \text{ for every } j \in \{1, \cdots, s\} \setminus \{i_0\}.$$

LEMMA 3.18. *If D is a domain of type* (D.1)*, then Cases* (II-2-1) *and* (II-2-2) *do not occur.*

PROOF. Assume that one of these cases occurs and choose an element $F = (F_1, \dots, F_n) \in C_D(\Lambda)$ arbitrarily. Let $F_j(z, w) = \sum A_{\mu\nu} z^\mu w^\nu$ be the Laurent expansion of F_j $(1 \leq j \leq n)$ and let $A_{\mu\nu}$ be a non-zero coefficient of it. Then, by the conditions in Cases (II-2-1) or (II-2-2) one may obtain a bunch of relations between the exponents μ, $\nu = (\nu_1, \dots, \nu_l)$ and λ_j as in the proof of Sublemma 3.9 that forces F to be of the form

$$F(z, w) = (\alpha(u)z, \gamma_1(u)w_1, \dots, \gamma_l(u)w_l), \ \ u = z^N w_1^{m_1} \cdots w_s^{m_s},$$

where α, γ_j are nowhere vanishing holomorphic functions on Δ and N, m_j are the integers defined in (3.29). Thus, $Z_D(\Lambda)$ has to be abelian. This is a contradiction as mentioned before. Hence Lemma 3.18 is proved. □

LEMMA 3.19. *Let D be a domain of type* (D.1) *and assume that either Case* (II-2-3) *or Case* (II-2-4) *occurs. Then D is biholomorphically equivalent to Ω^*.*

PROOF. Let us consider Case (II-2-3). First of all, by making use of the fact $\gcd(N, m_1, \dots, m_s) = 1$, one can show that $N = 1$. Then, there exists an algebraic automorphism φ of $\mathbf{C} \times (\mathbf{C}^*)^s \times X_{a,b}$ that induces a biholomorphic mapping of D onto $\varphi(D) = \{|z^*| < 1\} \times (\mathbf{C}^*)^s \times X_{a,b}$. Thus D is biholomorphically equivalent to Ω^* by Lemma 3.11.

Next, consider Case (II-2-4). If $m_{i_0} \geq 2$, then we may obtain a contradiction just as in the proof of Lemma 3.15. Moreover, in the case when $m_{i_0} = 1$, D is algebraically equivalent to the domain $\{|w_{i_0}^*| < 1\} \times (\mathbf{C}^*)^s \times X_{a,b}$. Accordingly, D is biholomorphically equivalent to Ω^* by Lemma 3.11, as desired. □

Finally, notice that the domain of type (D.2) is algebraically equivalent to a domain of type (D.1). Therefore we have proved Theorem for $k = 1$ in Case II.

Summing up the results obtained above, we conclude that D is biholomorphically equivalent to Ω^*. Hence we complete the proof of Theorem in the case $k = 1$.

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