# The q-Onsager algebra (Finite Groups, Vertex Operator Algebras and Combinatorics)


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# The $q$-Onsager algebra

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This article gives a summary of the finite-dimesional irreducible representations of the $q$-Onsager algebra, which are treated in detail in [3].

Fix a nonzero scalar $q \in \mathbb{C}$ which is not a root of unity. Let $\mathcal{A} = \mathcal{A}_q$ denote the associative $\mathbb{C}$-algebra with 1 defined by genarators $z, z^*$ subject to the relations

$$(\mathrm{TD}) \left\{ \begin{array}{lcl} [z,\ z^2z^* - \beta z\, z^*z + z^*z^2] & = & \delta\,[z,\ z^*], \\ [z^*,\ z^{*2}z - \beta z^*z\, z^* + z\, z^{*2}] & = & \delta\,[z^*,\ z], \end{array} \right.$$

where $\beta = q^2+q^{-2}$ and $\delta = -(q^2-q^{-2})^2$. (TD) can be regarded as a $q$-analogue of the Dolan-Grady relations and we call $\mathcal{A}$ the *$q$-Onsager algebra.* We classify the finite-dimensional irreducible representations of $\mathcal{A}$. All such representations are explicitly constructed via embeddings of $\mathcal{A}$ into the $U_q(sl_2)$-loop algebra. As an application, tridiagonal pairs of $q$-Racah type over $\mathbb{C}$ are classified in the case where $q$ is not a root of unity.

The $U_q(sl_2)$-loop algebra $\mathcal{L} = U_q(L(sl_2))$ is the associative $\mathbb{C}$-algebra with 1 generated by $e_i^+, e_i^-, k_i, k_i^{-1}$ $(i = 0, 1)$ subject to the relations

$$\begin{array}{rcl} k_0k_1 & = & k_1k_0 = 1, \\ k_ik_i^{-1} & = & k_i^{-1}k_i = 1, \\ k_ie_i^{\pm}k_i^{-1} & = & q^{\pm 2}e_i^{\pm}, \\ k_ie_j^{\pm}k_i^{-1} & = & q^{\mp 2}e_j^{\pm} \quad (i \neq j), \\ [e_i^+, e_i^-] & = & \dfrac{k_i - k_i^{-1}}{q-q^{-1}}, \\ [e_i^+, e_j^-] & = & 0 \quad (i \neq j), \end{array}$$

$$[e_i^{\pm}, (e_i^{\pm})^2e_j^{\pm} - (q^2+q^{-2})e_i^{\pm}e_j^{\pm}e_i^{\pm} + e_j^{\pm}(e_i^{\pm})^2] = 0 \quad (i \neq j).$$

Note that if we replace $k_0k_1 = k_1k_0 = 1$ in the defining relations for $\mathcal{L}$ by $k_0k_1 = k_1k_0$, then we have the quantum affine algebra $U_q(\widehat{sl}_2)$: $\mathcal{L}$ is isomorphic to the quotient algebra of $U_q(\widehat{sl}_2)$ by the two-sided ideal generated by $k_0k_1 - 1$.

**Proposition 1** *For arbitrary nonzero $s, t \in \mathbb{C}$, there exists an algebra homomorphism $\varphi_{s,t}$ from $\mathcal{A}$ to $\mathcal{L}$ that sends $z$, $z^*$ to*

$$\begin{array}{rcl} z_t(s) & = & x(s) + t\,k(s) + t^{-1}k(s)^{-1}, \\ z_t^*(s) & = & y(s) + t^{-1}k(s) + t\,k(s)^{-1}, \end{array}$$

*respectively, where*

$$\begin{aligned} x(s) &= \alpha(se_0^+ + s^{-1}e_1^- k_1) \quad \text{with } \alpha = -q^{-1}(q-q^{-1})^2, \\ y(s) &= se_0^- k_0 + s^{-1}e_1^+, \\ k(s) &= sk_0. \end{aligned}$$

*Moreover $\varphi_{s,t}$ is injective.*

We give an overview of finite-dimensional representations of $\mathcal{L}$ that we need to state our explicit construction of irreducible $\mathcal{A}$-modules via $\varphi_{s,t}$. For $a \in \mathbb{C}$ $(a \neq 0)$ and $\ell \in \mathbb{Z}$ $(\ell > 0)$, $V(\ell, a)$ denotes the *evaluation module* of $\mathcal{L}$, i.e., $V(\ell, a)$ is an $(\ell+1)$-dimensional vector space over $\mathbb{C}$ with a basis $v_0, v_1, \ldots, v_\ell$ on which $\mathcal{L}$ acts as follows:

$$\begin{aligned} k_0 v_i &= q^{2i-\ell} v_i, \\ k_1 v_i &= q^{\ell-2i} v_i, \\ e_0^+ v_i &= a\, q\, [i+1]\, v_{i+1}, \\ e_0^- v_i &= a^{-1} q^{-1} [\ell - i + 1]\, v_{i-1}, \\ e_1^+ v_i &= [\ell - i + 1]\, v_{i-1}, \\ e_1^- v_i &= [i+1]\, v_{i+1}, \end{aligned}$$

where $v_{-1} = v_{\ell+1} = 0$ and $[j] = [j]_q = (q^j - q^{-j})/(q - q^{-1})$. $V(\ell, a)$ is an irreducible $\mathcal{L}$-module. We call $v_0, v_1, \ldots, v_\ell$ a *standard basis.*

Let $\Delta$ denote the *coproduct* of $\mathcal{L}$: the algebra homomorphism from $\mathcal{L}$ to $\mathcal{L} \otimes \mathcal{L}$ defined by

$$\begin{aligned} \Delta(k_i^{\pm 1}) &= k_i^{\pm 1} \otimes k_i^{\pm 1}, \\ \Delta(e_i^+) &= k_i \otimes e_i^+ + e_i^+ \otimes 1, \\ \Delta(e_i^- k_i) &= k_i \otimes e_i^- k_i + e_i^- k_i \otimes 1. \end{aligned}$$

Given $\mathcal{L}$-modules $V_1, V_2$, the tensor product $V_1 \otimes V_2$ becomes an $\mathcal{L}$-module via $\Delta$. Given a set of evaluation modules $V(\ell_i, a_i)$ $(1 \leq i \leq n)$ of $\mathcal{L}$, the tensor product

$$V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$$

makes sense as an $\mathcal{L}$-module without being affected by the parentheses for the tensor product because of the coassociativity of $\Delta$.

With an evaluation module $V(\ell, a)$ of $\mathcal{L}$, we associate the set $S(\ell, a)$ of scalars $a\, q^{-\ell+1}$, $a\, q^{-\ell+3}$, $\cdots$, $a\, q^{\ell-1}$:

$$S(\ell, a) = \{a\, q^{2i-\ell+1} \mid 0 \leq i \leq \ell - 1\}.$$

The set $S(\ell, a)$ is called a *q-string* of length $\ell$. Two $q$-strings $S(\ell, a)$, $S(\ell', a')$ are said to be *adjacent* if $S(\ell, a) \cup S(\ell', a')$ is a longer $q$-string, i.e., $S(\ell, a) \cup S(\ell', a') = S(\ell'', a'')$ for some $\ell''$, $a''$ with $\ell'' > \max\{\ell, \ell'\}$. It can be easily checked that $S(\ell, a)$, $S(\ell', a')$ are adjacent if and only if $a^{-1}a' = q^{\pm i}$ for some

$$i \in \{|\ell - \ell'| + 2, |\ell - \ell'| + 4, \cdots, \ell + \ell'\}.$$

Two $q$-strings $S(\ell, a)$, $S(\ell', a')$ are defined to be *in general position* if they are not adjacent, i.e., if either

(i) $S(\ell, a) \cup S(\ell', a')$ is not a $q$-string,
or
(ii) $S(\ell, a) \subseteq S(\ell', a')$ or $S(\ell, a) \supseteq S(\ell', a')$.

A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings is said to be *in general position* if $S(\ell_i, a_i)$ and $S(\ell_j, a_j)$ are in general position for any $i$, $j$ ($i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$). The following fact is well-known and easy to prove. Let $\Omega$ be a finite multi-set of nonzero scalars from $\mathbb{C}$. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings in general position such that

$$\Omega = \bigcup_{i=1}^{n} S(\ell_i, a_i)$$

as multi-sets of nonzero scalars. Moreover such a multi-set of $q$-strings is uniquely determined by $\Omega$.

With a tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules $V(\ell_i, a_i)$ ($1 \leq i \leq n$), we associate the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings. The following (i), (ii), (iii) are well-known [1]:

(i) A tensor product $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as an $\mathcal{L}$ -module if and only if the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings is in geneal position.

(ii) Set $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots V(\ell'_{n'}, a'_{n'})$ and assume that $V$, $V'$ are both irreducible as an $\mathcal{L}$-module. Then $V$, $V'$ are isomorphic as $\mathcal{L}$-modules if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$, $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i$, $a_i = a'_i$ for all $i$ ($1 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \cdots, S(\ell'_n, a'_n)$.

(iii) Every nontrivial finite-dimensional irreducible $\mathcal{L}$-module of type (1,1) is isomorphic to some $V(\ell_1, a_1) \otimes \cdots V(\ell_n, a_n)$.

Two multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n, \{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ of $q$-strings are defined to be *equivalent* if there exists $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$) such that $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ and $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ coincide, i.e., $n = n'$ and $\ell_i = \ell'_i$, $a_i^{\varepsilon_i} = a'_i$ for all $i$ ($0 \leq i \leq n$) with a suitable reordering of $S(\ell'_1, a'_1), \cdots, S(\ell'_n, a'_n)$. A multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings is defined to be *strongly in general position* if any multi-set of $q$-strings equivalent to $\{S(\ell_i, a_i)\}_{i=1}^n$ is in general position, i.e., the multi-set $\{S(\ell_i, a_i^{\varepsilon_i})\}_{i=1}^n$ is in general position for any choice of $\varepsilon_i \in \{\pm 1\}$ ($1 \leq i \leq n$).

**Lemma 1** *Let $\Omega$ be a finite multi-set of nonzero scalars from $\mathbb{C}$ such that $c$ and $c^{-1}$ appear in $\Omega$ in pairs, i.e., $c$ and $c^{-1}$ have the same multiplicity in $\Omega$ for each $c \in \Omega$, where we understand that if 1 or -1 appears in $\Omega$, it has even multiplicity. Then there exists a multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of $q$-strings strongly in general position such that*

$$\Omega = \bigcup_{i=1}^{n} \left(S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})\right)$$

*as multi-sets of nonzero scalars. Such a multi-set of q-strings is uniquely determined by $\Omega$ up to equivalence.*

For an $\mathcal{L}$-module $V$, let $\rho_V$ denote the representation of $\mathcal{L}$ afforded by the $\mathcal{L}$-module $V$. Then $\rho_V \circ \varphi_{s,t}$ is a representation of $\mathcal{A}$.

**Theorem 1** *The following (i), (ii), (iii) hold.*

(i) *For an $\mathcal{L}$-module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and nonzero $s$, $t \in \mathbb{C}$, the representaiton $\rho_V \circ \varphi_{s,t}$ of $\mathcal{A}$ is irreducible if and only if*

(i.1) *the multi-set $\{S(\ell_i, a_i)\}_{i=1}^n$ of q-strings is strongly in general position,*

(i.2) *none of $-s^2$, $-t^2$ belongs to $S(\ell_i, a_i) \cup S(\ell_i, a_i^{-1})$ for any $i$ $(1 \le i \le n)$,*

(i.3) *none of the four scalars $\pm st$, $\pm st^{-1}$ equals $q^i$ for any $i \in \mathbb{Z}$ $(-d+1 \le i \le d-1)$.*

(ii) *For $\mathcal{L}$-modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ and $(s,t)$, $(s',t') \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$, set $\rho = \rho_V \circ \varphi_{s,t}$ and $\rho' = \rho_{V'} \circ \varphi_{s',t'}$. Assume that the representations $\rho$, $\rho'$ of $\mathcal{A}$ are both irreducible. Then they are isomorphic as representations of $\mathcal{A}$ if and only if the multi-sets $\{S(\ell_i, a_i)\}_{i=1}^n$, $\{S(\ell'_i, a'_i)\}_{i=1}^{n'}$ are equivalent and*

$$(s', t') \in \{\pm(s,t),\ \pm(t^{-1}, s^{-1}),\ \pm(t,s),\ \pm(s^{-1}, t^{-1})\}.$$

(iii) *Every nontrivial finite-dimensional irreducible representation of $\mathcal{A}$ is isomorphic to $\rho_V \circ \varphi_{s,t}$ for some $\mathcal{L}$-module $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ and $(s,t) \in (\mathbb{C}\backslash\{0\}) \times (\mathbb{C}\backslash\{0\})$.*

Let $A, A^* \in \mathrm{End}(V)$ be a TD-pair [2] with eigenspaces $\{V_i\}_{i=0}^d$, $\{V_i^*\}_{i=0}^d$ respectively. Then we have the split decomposition:

$$V = \bigoplus_{i=0}^{d} U_i,$$

where

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d).$$

By [2, Corollary 5.7 ], it holds that

$$\dim U_i = \dim V_i = \dim V_i^* \quad (0 \le i \le d),$$

and

$$\dim U_i = \dim U_{d-i} \qquad (0 \le i \le d).$$

Note that $\dim U_i$ is invariant under standardization of $A$, $A^*$ by affine transformations $\lambda A + \mu I$, $\lambda^* A^* + \mu^* I$. For an $\mathcal{L}$-module

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n),$$

set $A = \rho_V \circ \varphi_{s,t}(z)$, $A^* = \rho_V \circ \varphi_{s,t}(z^*)$. If the conditions (i.1), (i.2), (i.3) in Theorem 1 hold, then $A$, $A^*$ are a standardized TD-pair of $q$-Racah type. Every standardized TD-pair $A$, $A^*$ of $q$-Racah type arises in this way. The split decomposition of $V$ for $A$, $A^*$ coincides with the eigenspace decomposition of the element $k_0$ of $\mathcal{L}$ acting on $V$. Thus the generating function for $\dim U_i$

$$g(\lambda) = \sum_{i=0}^{d} (\dim U_i)\, \lambda^i$$

is given as follows.

**Proposition 2** ([2, Conjecture 13.7 ])

$$g(\lambda) \;=\; \prod_{i=1}^{n} (1 + \lambda + \lambda^2 + \cdots + \lambda^{\ell_i}).$$

A TD-pair $A, A^*$ is called a *Leonard pair* if $\dim\, U_i = 1$ for all $i$ $(0 \le i \le d)$. A standardized TD-pair $A, A^*$ of $q$-Racah type is a Leonard pair if and only if it is afforded by an evaluation module. In view of this fact, a standardized TD-pair $A$, $A^*$ of $q$-Racah type is regarded as a 'tensor product of Leonard pairs'.

# References


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