Tridiagonal pairs and the quantum affine algebra $U_q(\widehat{sl}_2)$*

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Abstract

Let $\mathbb{K}$ denote an algebraically closed field and let q denote a nonzero scalar in $\mathbb{K}$ that is not a root of unity. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$) denote a standard ordering of the eigenvalues of A (resp. A^*). We assume there exist nonzero scalars a, a^* in $\mathbb{K}$ such that $\theta_i = aq^{2i-d}$ and $\theta_i^* = a^*q^{d-2i}$ for $0 \leq i \leq d$. We display two irreducible $U_q(\widehat{sl}_2)$-module structures on V and discuss how these are related to the actions of A and A^*.

1 The quantum affine algebra $U_q(\widehat{sl}_2)$

Throughout this paper $\mathbb{K}$ will denote an algebraically closed field. We fix a nonzero scalar $q \in \mathbb{K}$ that is not a root of unity. We will use the following notation.

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \qquad n = 0, 1, \ldots \tag{1}$$

We now recall the definition of $U_q(\widehat{sl}_2)$.

Definition 1.1 [3, p. 262] The quantum affine algebra $U_q(\widehat{sl}_2)$ is the unital associative $\mathbb{K}$-algebra with generators $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ and the following relations:

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \tag{2}$$

$$K_0 K_1 = K_1 K_0, \tag{3}$$

$$K_i e_i^{\pm} K_i^{-1} = q^{\pm 2} e_i^{\pm}, \tag{4}$$

$$K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm}, \qquad i \neq j, \tag{5}$$

$$[e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{6}$$

$$[e_0^{\pm}, e_1^{\mp}] = 0, \tag{7}$$

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$$(e_i^{\pm})^3 e_j^{\pm} - [3]_q (e_i^{\pm})^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} (e_i^{\pm})^2 - e_j^{\pm} (e_i^{\pm})^3 = 0, \qquad i \neq j. \tag{8}$$

We call $e_i^{\pm}$, $K_i^{\pm 1}$, $i \in \{0, 1\}$ the *Chevalley generators* for $U_q(\widehat{sl}_2)$.

Remark 1.2 The equations (8) are called the *q-Serre relations*.

2 A presentation of $U_q(\widehat{sl}_2)$

In order to state our main result we introduce an alternate presentation of $U_q(\widehat{sl}_2)$. This presentation is given below.

Theorem 2.1 *The quantum affine algebra $U_q(\widehat{sl}_2)$ is isomorphic to the unital associative $\mathbb{K}$-algebra with generators $y_i^{\pm}$, $k_i^{\pm 1}$, $i \in \{0, 1\}$ and the following relations:*

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \tag{9}$$

$$k_0 k_1 \text{ is central}, \tag{10}$$

$$\frac{q y_i^+ k_i - q^{-1} k_i y_i^+}{q - q^{-1}} = 1, \tag{11}$$

$$\frac{q k_i y_i^- - q^{-1} y_i^- k_i}{q - q^{-1}} = 1, \tag{12}$$

$$\frac{q y_i^- y_i^+ - q^{-1} y_i^+ y_i^-}{q - q^{-1}} = 1, \tag{13}$$

$$\frac{q y_i^+ y_j^- - q^{-1} y_j^- y_i^+}{q - q^{-1}} = k_0^{-1} k_1^{-1}, \qquad i \neq j, \tag{14}$$

$$(y_i^{\pm})^3 y_j^{\pm} - [3]_q (y_i^{\pm})^2 y_j^{\pm} y_i^{\pm} + [3]_q y_i^{\pm} y_j^{\pm} (y_i^{\pm})^2 - y_j^{\pm} (y_i^{\pm})^3 = 0, \qquad i \neq j. \tag{15}$$

An isomorphism with the presentation in Definition 1.1 is given by:

$$\begin{aligned} k_i^{\pm} &\to K_i^{\pm}, \\ y_i^- &\to K_i^{-1} + e_i^-, \\ y_i^+ &\to K_i^{-1} - q(q - q^{-1})^2 K_i^{-1} e_i^+. \end{aligned}$$

The inverse of this isomorphism is given by:

$$\begin{aligned} K_i^{\pm} &\to k_i^{\pm}, \\ e_i^- &\to y_i^- - k_i^{-1}, \\ e_i^+ &\to \frac{1 - k_i y_i^+}{q(q - q^{-1})^2}. \end{aligned}$$

Proof: One readily checks that each map is a homomorphism of $\mathbb{K}$-algebras and that the maps are inverses. It follows each map is an isomorphism of $\mathbb{K}$-algebras. □

Definition 2.2 With reference to Theorem 2.1 we call $y_i^{\pm}$, $k_i^{\pm 1}$, $i \in \{0, 1\}$ the *alternate generators* of $U_q(\widehat{sl}_2)$.

3 Tridiagonal pairs

We now recall the notion of a *tridiagonal pair* [7], [12]. We will use the following terms. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $A : V \to V$ denote a linear transformation and let W denote a subspace of V. We call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that

$$W = \{v \in V \mid Av = \theta v\}.$$

We say A is *diagonalizable* whenever V is spanned by the eigenspaces of A.

Definition 3.1 *[7, Definition 1.1]* Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. By a *tridiagonal pair* on V, we mean an ordered pair A, A^* where $A : V \to V$ and $A^* : V \to V$ are linear transformations that satisfy the following four conditions.

(i) Each of A, A^* is diagonalizable.

(ii) There exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \leq i \leq d), \tag{16}$$

where $V_{-1} = 0$, $V_{d+1} = 0$.

(iii) There exists an ordering $V_0^*, V_1^*, \ldots, V_\delta^*$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \leq i \leq \delta), \tag{17}$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$.

(iv) There does not exist a subspace W of V such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

Note 3.2 According to a common notational convention, A^* denotes the conjugate transpose of A. We are not using this convention. In a tridiagonal pair A, A^* the linear transformations A and A^* are arbitrary subject to (i)–(iv) above.

Our interest in tridiagonal pairs evolved from our interest in the following special case. A tridiagonal pair for which the V_i, V_i^* all have dimension 1 is called a *Leonard pair* [11]. There is a natural correspondence between the Leonard pairs and a family of orthogonal polynomials consisting of the q-Racah polynomials [1], [6] and some related polynomials in the Askey-scheme [9], [18]. This correspondence follows from the classification of Leonard pairs [11], [18]. We remark that this classification amounts to a linear algebraic version of a theorem of D. Leonard [2], [10] concerning the q-Racah polynomials. See [8], [12], [13], [14], [15], [16], [17], [19], [20] for more information about Leonard pairs.

Given these comments on Leonard pairs, it is natural to attempt a classification of the tridiagonal pairs. At present we do not have this classification; however we do have a result that might lead to one. In order to state the result we recall a few basic facts about tridiagonal pairs. Let A, A^* denote a tridiagonal pair on V and let d, δ be as in Definition 3.1(ii), (iii).

By [7, Lemma 4.5] we have $d = \delta$; we call this common value the *diameter* of A, A^*. An ordering of the eigenspaces of A (resp. A^*) will be called *standard* whenever it satisfies (16) (resp. (17)). We comment on the uniqueness of the standard ordering. Let $V_0, V_1, \ldots, V_d$ denote a standard ordering of the eigenspaces of A. Then the ordering $V_d, V_{d-1}, \ldots, V_0$ is standard and no other ordering is standard. A similar result holds for the eigenspaces of A^*. An ordering of the eigenvalues of A (resp. A^*) will be called *standard* whenever the corresponding ordering of the eigenspaces of A (resp. A^*) is standard. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$) denote a standard ordering of the eigenvalues of A (resp. A^*). The θ_i, θ^*_i satisfy a number of equations [12, Theorem 4.3] that have been solved in closed form [12, Theorem 4.4]. In a special case of interest, there exist nonzero scalars a, a^* in $\mathbb{K}$ such that $\theta_i = aq^{2i-d}$ and $\theta^*_i = a^*q^{d-2i}$ for $0 \leq i \leq d$ [7, Example 1.7], [8].

We now state our main result.

Theorem 3.3 *Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $\theta_0, \theta_1, \ldots, \theta_d$ (resp. $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$) denote a standard ordering of the eigenvalues of A (resp. A^*). We assume there exist nonzero scalars a, a^* in $\mathbb{K}$ such that $\theta_i = aq^{2i-d}$ and $\theta^*_i = a^*q^{d-2i}$ for $0 \leq i \leq d$. Then with reference to Definition 2.1, there exists a unique $U_q(\widehat{sl}_2)$-module structure on V such that ay^-_1 acts as A and $a^*y^-_0$ acts as A^*. Moreover there exists a unique $U_q(\widehat{sl}_2)$-module structure on V such that ay^+_0 acts as A and $a^*y^+_1$ acts as A^*. Both $U_q(\widehat{sl}_2)$-module structures are irreducible.*

The proof of Theorem 3.3 appears in Sections 13, 14 below.

Remark 3.4 The finite dimensional irreducible modules for $U_q(\widehat{sl}_2)$ are described in [3]. In a future paper we hope to use [3] to obtain a classification of the tridiagonal pairs that satisfy the assumptions of Theorem 3.3. See Lemma 15.1 and Problem 16.1 below for a discussion of the issues involved.

Remark 3.5 Theorem 3.3 extends some work of Curtin and Al-Najjar [4], [5]. They give a $U_q(\widehat{sl}_2)$-action for those tridiagonal pairs that satisfy the assumptions of Theorem 3.3 and for which the dimensions of the V_i, V^*_i are all at most 2.

4 Six decompositions

In this section and the next we collect some results about tridiagonal pairs which we will use to prove Theorem 3.3.

We will use the following notation. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let d denote a nonnegative integer. By a *decomposition of V of length d*, we mean a sequence $U_0, U_1, \ldots, U_d$ consisting of nonzero subspaces of V such that

$$V = U_0 + U_1 + \cdots + U_d \qquad \text{(direct sum)}.$$

We do not assume each of $U_0, U_1, \ldots, U_d$ has dimension 1. For $0 \leq i \leq d$ we call U_i the *ith subspace* of the decomposition. For notational convenience we define $U_{-1} := 0$ and $U_{d+1} := 0$.

We will refer to the following setup.

Definition 4.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $V_0, V_1, \ldots, V_d$ (resp. $V_0^*, V_1^*, \ldots, V_d^*$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V_i^*).

With reference to Definition 4.1, we are about to define six decompositions of V. In order to keep track of these decompositions we will give each of them a name. Our naming scheme is as follows. Let Ω denote the set consisting of the four symbols $0, D, 0^*, D^*$. Each of the six decompositions will get a name $[u]$ where u is a two-element subset of Ω. We now define the six decompositions.

Lemma 4.2 *With reference to Definition 4.1, for each of the six rows in the table below, and for* $0 \leq i \leq d$, *let* U_i *denote the* i*th subspace described in that row. Then the sequence* $U_0, U_1, \ldots, U_d$ *is a decomposition of* V.

name	ith subspace of the decomposition
$[0D]$	V_i
$[0^*D^*]$	V_i^*
$[0^*D]$	$(V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$
$[0^*0]$	$(V_0^* + \cdots + V_i^*) \cap (V_0 + \cdots + V_{d-i})$
$[D^*0]$	$(V_{d-i}^* + \cdots + V_d^*) \cap (V_0 + \cdots + V_{d-i})$
$[D^*D]$	$(V_{d-i}^* + \cdots + V_d^*) \cap (V_i + \cdots + V_d)$

Proof: We consider each of the six rows of the table.
$[0D]$: Recall $V_0, V_1, \ldots, V_d$ are the eigenspaces of A and that A is diagonalizable.
$[0^*D^*]$: Recall $V_0^*, V_1^*, \ldots, V_d^*$ are the eigenspaces of A^* and that A^* is diagonalizable.
$[0^*D]$: Define $U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$ for $0 \leq i \leq d$. Then the sequence $U_0, U_1, \ldots, U_d$ is a decomposition of V by [7, Theorem 4.6].
$[0^*0]$: Apply the present Lemma, row $[0^*D]$, with V_i replaced by V_{d-i} for $0 \leq i \leq d$.
$[D^*0]$: Apply the present Lemma, row $[0^*D]$, with V_i replaced by V_{d-i} and V_i^* replaced by V_{d-i}^* for $0 \leq i \leq d$.
$[D^*D]$: Apply the present Lemma, row $[0^*D]$, with V_i^* replaced by V_{d-i}^* for $0 \leq i \leq d$. $\square$

The six decompositions from Lemma 4.2 are related to each other as follows.

Lemma 4.3 *Adopt the assumptions of Definition 4.1 and let* $U_0, U_1, \ldots, U_d$ *denote any one of the six decompositions of* V *given in Lemma 4.2. Then for* $0 \leq i \leq d$ *the sums* $U_0 + \cdots + U_i$ *and* $U_i + \cdots + U_d$ *are given as follows.*

name	$U_0 + \cdots + U_i$	$U_i + \cdots + U_d$
$[0D]$	$V_0 + \cdots + V_i$	$V_i + \cdots + V_d$
$[0^*D^*]$	$V_0^* + \cdots + V_i^*$	$V_i^* + \cdots + V_d^*$
$[0^*D]$	$V_0^* + \cdots + V_i^*$	$V_i + \cdots + V_d$
$[0^*0]$	$V_0^* + \cdots + V_i^*$	$V_0 + \cdots + V_{d-i}$
$[D^*0]$	$V_{d-i}^* + \cdots + V_d^*$	$V_0 + \cdots + V_{d-i}$
$[D^*D]$	$V_{d-i}^* + \cdots + V_d^*$	$V_i + \cdots + V_d$

Proof: We consider each of the six rows of the table.
[$0D$]: Immediate from Lemma 4.2, row [$0D$].
[0^*D^*]: Immediate from Lemma 4.2, row [0^*D^*].
[0^*D]: Let $U_0, U_1, \ldots, U_d$ denote the decomposition [0^*D]. By [7, Theorem 4.6] we find $U_0 + \cdots + U_i = V_0^* + \cdots + V_i^*$ and $U_i + \cdots + U_d = V_i + \cdots + V_d$ for $0 \leq i \leq d$.
[0^*0]: Apply the present Lemma, row [0^*D], with V_i replaced by V_{d-i} for $0 \leq i \leq d$.
[D^*0]: Apply the present Lemma, row [0^*D], with V_i replaced by V_{d-i} and V_i^* replaced by V_{d-i}^* for $0 \leq i \leq d$.
[D^*D]: Apply the present Lemma, row [0^*D], with V_i^* replaced by V_{d-i}^* for $0 \leq i \leq d$. □

We have a comment.

Lemma 4.4 *[7, Corollary 5.7, Corollary 6.6] Adopt the assumptions of Definition 4.1 and let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V given in Lemma 4.2. For $0 \leq i \leq d$ let ρ_i denote the dimenension of U_i. Then the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is independent of the decomposition. Moreover the sequence $\rho_0, \rho_1, \ldots, \rho_d$ is unimodal and symmetric; that is $\rho_i = \rho_{d-i}$ for $0 \leq i \leq d$ and $\rho_{i-1} \leq \rho_i$ for $1 \leq i \leq d/2$.*

Referring to Lemma 4.4, we call the sequence $\rho_0, \rho_1, \ldots, \rho_d$ the *shape* of the tridiagonal pair. As we indicated in Section 2, a tridiagonal pair of shape $1, 1, \ldots, 1$ is the same thing as a Leonard pair [7].

5 The action of A and A^* on the six decompositions

With reference to Definition 4.1, in this section we describe the actions of A and A^* on each of the six decompositions given in Lemma 4.2.

Lemma 5.1 *Adopt the assumptions of Definition 4.1 and let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V given in Lemma 4.2. Then for $0 \leq i \leq d$ the action of A and A^* on U_i is described as follows.*

name	action of A on U_i	action of A^* on U_i
[$0D$]	$(A - \theta_i I)U_i = 0$	$A^*U_i \subseteq U_{i-1} + U_i + U_{i+1}$
[0^*D^*]	$AU_i \subseteq U_{i-1} + U_i + U_{i+1}$	$(A^* - \theta_i^* I)U_i = 0$
[0^*D]	$(A - \theta_i I)U_i \subseteq U_{i+1}$	$(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$
[0^*0]	$(A - \theta_{d-i} I)U_i \subseteq U_{i+1}$	$(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$
[D^*0]	$(A - \theta_{d-i} I)U_i \subseteq U_{i+1}$	$(A^* - \theta_{d-i}^* I)U_i \subseteq U_{i-1}$
[D^*D]	$(A - \theta_i I)U_i \subseteq U_{i+1}$	$(A^* - \theta_{d-i}^* I)U_i \subseteq U_{i-1}$

Proof: We consider each of the six rows of the table.
[$0D$]: For $0 \leq i \leq d$ the space V_i is an eigenspace for A with eigenvalue θ_i. Therefore $(A - \theta_i I)V_i = 0$. We have $A^*V_i \subseteq V_{i-1} + V_i + V_{i+1}$ by (16).
[0^*D^*]: For $0 \leq i \leq d$ we find $AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^*$ by (17). The space V_i^* is an eigenspace for A^* with eigenvalue θ_i^*. Therefore $(A^* - \theta_i^* I)V_i^* = 0$.
[0^*D]: Let $U_0, U_1, \ldots, U_d$ denote the decomposition [0^*D]. By [7, Theorem 4.6] we find

$(A - \theta_i I)U_i \subseteq U_{i+1}$ and $(A^* - \theta^*_i I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$.
$[0^*0]$: Apply the present Lemma, row $[0^*D]$, with V_i replaced by V_{d-i} for $0 \leq i \leq d$.
$[D^*0]$: Apply the present Lemma, row $[0^*D]$, with V_i replaced by V_{d-i} and V^*_i replaced by V^*_{d-i} for $0 \leq i \leq d$.
$[D^*D]$: Apply the present Lemma, row $[0^*D]$, with V^*_i replaced by V^*_{d-i} for $0 \leq i \leq d$. □

6 The linear transformations B, B^*, K, K^*

In the previous two sections we discussed general tridiagonal pairs. For the rest of this paper we restrict our attention to the special case mentioned in Theorem 3.3. We will refer to the following setup.

Definition 6.1 Let V denote a vector space over $\mathbb{K}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V. Let $V_0, V_1, \ldots, V_d$ (resp. $V^*_0, V^*_1, \ldots, V^*_d$) denote a standard ordering of the eigenspaces of A (resp. A^*). For $0 \leq i \leq d$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) associated with V_i (resp. V^*_i). We assume there exist nonzero scalars a, a^* in $\mathbb{K}$ such that

$$\theta_i = aq^{2i-d}, \qquad \theta^*_i = a^* q^{d-2i} \qquad (0 \leq i \leq d). \tag{18}$$

Let b and b^* denote nonzero scalars in $\mathbb{K}$.

Definition 6.2 Adopt the assumptions of Definition 6.1.

(i) We let $B : V \to V$ denote the unique linear transformation such that for $0 \leq i \leq d$,

$$(V^*_0 + \cdots + V^*_i) \cap (V_0 + \cdots + V_{d-i}) \tag{19}$$

is an eigenspace of B with eigenvalue bq^{2i-d}. We remark (19) is the ith subspace of the decomposition $[0^*0]$ from Lemma 4.2.

(ii) We let $B^* : V \to V$ denote the unique linear transformation such that for $0 \leq i \leq d$,

$$(V^*_{d-i} + \cdots + V^*_d) \cap (V_i + \cdots + V_d) \tag{20}$$

is an eigenspace of B^* with eigenvalue $b^* q^{d-2i}$. We remark (20) is the ith subspace of the decomposition $[D^*D]$ from Lemma 4.2.

(iii) We let $K : V \to V$ denote the unique linear transformation such that for $0 \leq i \leq d$,

$$(V^*_0 + \cdots + V^*_i) \cap (V_i + \cdots + V_d) \tag{21}$$

is an eigenspace of K with eigenvalue q^{2i-d}. We remark (21) is the ith subspace of the decomposition $[0^*D]$ from Lemma 4.2.

(iv) We let $K^*: V \to V$ denote the unique linear transformation such that for $0 \le i \le d$,

$$(V^*_{d-i} + \cdots + V^*_d) \cap (V_0 + \cdots + V_{d-i}) \tag{22}$$

is an eigenspace of K^* with eigenvalue q^{2i-d}. We remark (22) is the ith subspace of the decomposition $[D^*0]$ from Lemma 4.2.

Remark 6.3 With reference to Definition 6.1 and Definition 6.2, the following (i), (ii) hold.

(i) If we replace $(A, A^*, V_i, V^*_i, a, a^*, B, B^*, b, b^*, K, K^*, q)$ by $(A^*, A, V^*_{d-i}, V_{d-i}, a^*, a, B^*, B, b^*, b, K^{-1}, K^{*-1}, q)$ then the requirements of Definition 6.1 and Definition 6.2 are still satisfied.

(ii) If we replace $(A, A^*, V_i, V^*_i, a, a^*, B, B^*, b, b^*, K, K^*, q)$ by $(A, A^*, V_{d-i}, V^*_{d-i}, a, a^*, B^*, B, b^*, b, K^{*-1}, K^{-1}, q^{-1})$ then the requirements of Definition 6.1 and Definition 6.2 are still satisfied.

We will use Remark 6.3 to streamline a few proofs later in the paper.

7 Some relations involving A, A^*, B, B^*

In this section we give four relations involving the tridiagonal pair A, A^* from Definition 6.1 and the elements B, B^* from Definition 6.2.

Theorem 7.1 *With reference to Definition 6.1 and Definition 6.2,*

$$\frac{qAB - q^{-1}BA}{q - q^{-1}} = abI, \tag{23}$$

$$\frac{qBA^* - q^{-1}A^*B}{q - q^{-1}} = a^*bI, \tag{24}$$

$$\frac{qA^*B^* - q^{-1}B^*A^*}{q - q^{-1}} = a^*b^*I, \tag{25}$$

$$\frac{qB^*A - q^{-1}AB^*}{q - q^{-1}} = ab^*I. \tag{26}$$

Proof: We first show (23). Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*0]$ from Lemma 4.2. We show $qAB - q^{-1}BA - ab(q - q^{-1})I$ vanishes on U_i for $0 \le i \le d$. Let i be given. By Definition 6.2(i) we find $B - bq^{2i-d}I$ vanishes on U_i so

$$(A - aq^{d-2i-2}I)(B - bq^{2i-d}I) \tag{27}$$

vanishes on U_i. From the table of Lemma 5.1, row $[0^*0]$, and using (18), we find $(A - aq^{d-2i}I)U_i \subseteq U_{i+1}$. Therefore

$$(B - bq^{2i+2-d}I)(A - aq^{d-2i}I) \tag{28}$$

vanishes on U_i. Subtracting q^{-1}times (28) from q times (27) we find $qAB - q^{-1}BA - ab(q - q^{-1})I$ vanishes on U_i. Line (23) follows. To get (25) use (23) and the involution given in Remark 6.3(i). To get (26) use (23) and the involution given in Remark 6.3(ii). To get (24) use (25) and the involution given in Remark 6.3(ii). □

8 The action of B and B^* on the six decompositions

In this section we describe how the elements B, B^* from Definition 6.2 act on the six decompositions given in Lemma 4.2.

Theorem 8.1 *Adopt the assumptions of Definition 6.1 and let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V given in Lemma 4.2. Let the maps B, B^* be as in Definition 6.2. Then for $0 \le i \le d$ the action of B and B^* on U_i is described as follows.*

name	action of B on U_i	action of B^* on U_i
$[0D]$	$(B - bq^{d-2i}I)U_i \subseteq U_{i-1}$	$(B^* - b^*q^{d-2i}I)U_i \subseteq U_{i+1}$
$[0^*D^*]$	$(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$	$(B^* - b^*q^{2i-d}I)U_i \subseteq U_{i+1}$
$[0^*D]$	$(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$	$(B^* - b^*q^{d-2i}I)U_i \subseteq U_{i+1}$
$[0^*0]$	$(B - bq^{2i-d}I)U_i = 0$	$B^*U_i \subseteq U_{i-1} + U_i + U_{i+1}$
$[D^*0]$	$(B - bq^{2i-d}I)U_i \subseteq U_{i+1}$	$(B^* - b^*q^{d-2i}I)U_i \subseteq U_{i-1}$
$[D^*D]$	$BU_i \subseteq U_{i-1} + U_i + U_{i+1}$	$(B^* - b^*q^{d-2i}I)U_i = 0$

Proof: We first give the action of B for each of the six rows in the table.
$[0D]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0D]$. From Lemma 5.1, row $[0D]$, and using (18), we find that for $0 \le i \le d$, U_i is an eigenspace for A with eigenvalue aq^{2i-d}. We show $(B - bq^{d-2i}I)U_i \subseteq U_{i-1}$ for $0 \le i \le d$. To do this, it suffices to show

$$(A - aq^{2i-2-d}I)(B - bq^{d-2i}I) \tag{29}$$

vanishes on U_i for $0 \le i \le d$. Let i be given. Observe $A - aq^{2i-d}I$ vanishes on U_i so

$$(B - bq^{d-2i+2}I)(A - aq^{2i-d}I) \tag{30}$$

vanishes on U_i. Using (23) we find

$$qAB - q^{-1}BA - ab(q - q^{-1})I \tag{31}$$

vanishes on U_i. Adding (30) to q times (31) we find (29) vanishes on U_i. We conclude $(B - bq^{d-2i}I)U_i \subseteq U_{i-1}$ for $0 \le i \le d$.
$[0^*D^*]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*D^*]$. From Lemma 5.1, row $[0^*D^*]$, and using (18), we find that for $0 \le i \le d$, U_i is an eigenspace for A^* with eigenvalue a^*q^{d-2i}. We show $(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$ for $0 \le i \le d$. To do this, it suffices to show

$$(A^* - a^*q^{d-2i+2}I)(B - bq^{2i-d}I) \tag{32}$$

vanishes on U_i for $0 \le i \le d$. Let i be given. Observe $A^* - a^*q^{d-2i}I$ vanishes on U_i so

$$(B - bq^{2i-d-2}I)(A^* - a^*q^{d-2i}I) \tag{33}$$

vanishes on U_i. Using (24) we find

$$qBA^* - q^{-1}A^*B - a^*b(q - q^{-1})I \tag{34}$$

vanishes on U_i. Subtracting (33) from q^{-1} times (34) we find (32) vanishes on U_i. We conclude $(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$.
$[0^*D]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*D]$. We show $(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned}
(B - bq^{2i-d}I)U_i &\subseteq (B - bq^{2i-d}I)(U_0 + \cdots + U_i) \\
&= (B - bq^{2i-d}I)(V_0^* + \cdots + V_i^*) \qquad \text{(by Lemma 4.3, row } [0^*D]) \\
&\subseteq V_0^* + \cdots + V_{i-1}^* \qquad \text{(by present Theorem, row } [0^*D^*]) \\
&= U_0 + \cdots + U_{i-1} \qquad \text{(by Lemma 4.3, row } [0^*D])
\end{aligned}$$

and also

$$\begin{aligned}
(B - bq^{2i-d}I)U_i &\subseteq (B - bq^{2i-d}I)(U_i + \cdots + U_d) \\
&= (B - bq^{2i-d}I)(V_i + \cdots + V_d) \qquad \text{(by Lemma 4.3, row } [0^*D]) \\
&\subseteq V_{i-1} + \cdots + V_d \qquad \text{(by present Theorem, row } [0D]) \\
&= U_{i-1} + \cdots + U_d \qquad \text{(by Lemma 4.3, row } [0^*D]).
\end{aligned}$$

Combining these observations we obtain $(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$.
$[0^*0]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*0]$. Then $(B - bq^{2i-d}I)U_i = 0$ for $0 \leq i \leq d$ by Definition 6.2(i).
$[D^*0]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[D^*0]$. We show $(B - bq^{2i-d}I)U_i \subseteq U_{i+1}$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned}
(B - bq^{2i-d}I)U_i &\subseteq (B - bq^{2i-d}I)(U_0 + \cdots + U_i) \\
&= (B - bq^{2i-d}I)(V_{d-i}^* + \cdots + V_d^*) \qquad \text{(by Lemma 4.3, row } [D^*0]) \\
&\subseteq V_{d-i-1}^* + \cdots + V_d^* \qquad \text{(by present Theorem, row } [0^*D^*]) \\
&= U_0 + \cdots + U_{i+1} \qquad \text{(by Lemma 4.3, row } [D^*0])
\end{aligned}$$

and also

$$\begin{aligned}
(B - bq^{2i-d}I)U_i &\subseteq (B - bq^{2i-d}I)(U_i + \cdots + U_d) \\
&= (B - bq^{2i-d}I)(V_0 + \cdots + V_{d-i}) \qquad \text{(by Lemma 4.3, row } [D^*0]) \\
&\subseteq V_0 + \cdots + V_{d-i-1} \qquad \text{(by present Theorem, row } [0D]) \\
&= U_{i+1} + \cdots + U_d \qquad \text{(by Lemma 4.3, row } [D^*0]).
\end{aligned}$$

Combining these observations we obtain $(B - bq^{2i-d}I)U_i \subseteq U_{i+1}$ for $0 \leq i \leq d$.
$[D^*D]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[D^*D]$. We show $BU_i \subseteq U_{i-1} + U_i + U_{i+1}$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned}
BU_i &\subseteq B(U_0 + \cdots + U_i) \\
&= B(V_{d-i}^* + \cdots + V_d^*) \qquad \text{(by Lemma 4.3, row } [D^*D]) \\
&\subseteq V_{d-i-1}^* + \cdots + V_d^* \qquad \text{(by present Theorem, row } [0^*D^*]) \\
&= U_0 + \cdots + U_{i+1} \qquad \text{(by Lemma 4.3, row } [D^*D])
\end{aligned}$$

and also

$$\begin{aligned}
BU_i &\subseteq B(U_i + \cdots + U_d) \\
&= B(V_i + \cdots + V_d) && \text{(by Lemma 4.3, row } [D^*D]) \\
&\subseteq V_{i-1} + \cdots + V_d && \text{(by present Theorem, row } [0D]) \\
&= U_{i-1} + \cdots + U_d && \text{(by Lemma 4.3, row } [D^*D]).
\end{aligned}$$

Combining these observations we find $BU_i \subseteq U_{i-1} + U_i + U_{i+1}$ for $0 \le i \le d$.
We have now given the action of B on each of the six decompositions. Using this and the involution from Remark 6.3(i), we find B^* acts on the six decompositions as claimed. □

9 The pair B, B^* is a tridiagonal pair

In this section we show that the linear transformations B, B^* from Definition 6.2 form a tridiagonal pair.

Theorem 9.1 *Adopt the assumptions of Definition 6.1 and let the maps B, B^* be as in Definition 6.2. Then the pair B, B^* is a tridiagonal pair on V. The sequence bq^{2i-d} $(0 \le i \le d)$ is a standard ordering of the eigenvalues of B and the sequence b^*q^{d-2i} $(0 \le i \le d)$ is a standard ordering of the eigenvalues of B^*.*

Proof: For the duration of this proof let $U_0, \ldots, U_d$ (resp. $U_0^*, \ldots, U_d^*$) denote the decomposition $[0^*0]$ (resp. $[D^*D]$) from Lemma 4.2. We show the pair B, B^* is a tridiagonal pair on V. To do this we show B, B^* satisfies conditions (i)–(iv) in Definition 3.1.
Proof that B, B^* satisfies Definition 3.1(i): Each of $U_0, \ldots, U_d$ is an eigenspace of B by Definition 6.2(i) and these eigenspaces span V so B is diagonalizable. Each of $U_0^*, \ldots, U_d^*$ is an eigenspace of B^* by Definition 6.2(ii) and these eigenspaces span V so B^* is diagonalizable.
Proof that B, B^* satisfies Definition 3.1(ii): From the construction $U_0, \ldots, U_d$ is an ordering of the eigenspaces of B. By Theorem 8.1, row $[0^*0]$ we find $B^*U_i \subseteq U_{i-1} + U_i + U_{i+1}$ for $0 \le i \le d$.
Proof that B, B^* satisfies Definition 3.1(iii): From the construction $U_0^*, \ldots, U_d^*$ is an ordering of the eigenspaces of B^*. By Theorem 8.1, row $[D^*D]$ we find $BU_i^* \subseteq U_{i-1}^* + U_i^* + U_{i+1}^*$ for $0 \le i \le d$.
Proof that B, B^* satisfies Definition 3.1(iv): We let W denote an irreducible (B, B^*)-submodule of V and show $W = V$. To obtain $W = V$ we will show $AW \subseteq W$ and $A^*W \subseteq W$. We first show $AW \subseteq W$. We define $\tilde{W} := \{w \in W | Aw \in W\}$ and show $\tilde{W} = W$. Using (23) we routinely find $B\tilde{W} \subseteq \tilde{W}$. Using (26) we routinely find $B^*\tilde{W} \subseteq \tilde{W}$. We claim $\tilde{W} \neq 0$. To prove the claim, We define $W_i = W \cap U_i$ for $0 \le i \le d$. From the table of Lemma 4.3, row $[0^*0]$ we find both

$$W_0 + \cdots + W_i \subseteq V_0^* + \cdots + V_i^* \qquad (0 \le i \le d), \tag{35}$$

$$W_i + \cdots + W_d \subseteq V_0 + \cdots + V_{d-i} \qquad (0 \le i \le d). \tag{36}$$

The nonzero spaces among $W_0, \ldots, W_d$ are the eigenspaces of B on W so $W = \sum_{i=0}^{d} W_i$. By this and since $W \neq 0$ we find $W_0, \ldots, W_d$ are not all 0. Define $r = \max\{i | 0 \leq i \leq d,\ W_i \neq 0\}$. We define $W_i^* = W \cap U_i^*$ for $0 \leq i \leq d$. From the table of Lemma 4.3, row $[D^*D]$ we find

$$W_0^* + \cdots + W_i^* \subseteq V_{d-i}^* + \cdots + V_d^* \qquad (0 \leq i \leq d), \tag{37}$$

$$W_i^* + \cdots + W_d^* \subseteq V_i + \cdots + V_d \qquad (0 \leq i \leq d). \tag{38}$$

The nonzero spaces among $W_0^*, \ldots, W_d^*$ are the eigenspaces of B^* on W so $W = \sum_{i=0}^{d} W_i^*$. By this and since $W \neq 0$ we find $W_0^*, \ldots, W_d^*$ are not all 0. Define $t = \min\{i | 0 \leq i \leq d,\ W_i^* \neq 0\}$. Suppose for the moment that $r + t < d$. Setting $i = r$ in (35) and using $W_0 + \cdots + W_r = W$ we find $W \subseteq V_0^* + \cdots + V_r^*$. Setting $i = t$ in (37) we find $W_t^* \subseteq V_{d-t}^* + \cdots + V_d^*$. Of course $W_t^* \subseteq W$ so

$$\begin{aligned} W_t^* &= W \cap W_t^* \\ &\subseteq (V_0^* + \cdots + V_r^*) \cap (V_{d-t}^* + \cdots + V_d^*) \\ &= 0 \end{aligned}$$

for a contradiction. Therefore $r + t \geq d$. Setting $i = r$ in (36) we find $W_r \subseteq V_0 + \cdots + V_{d-r}$. Setting $i = t$ in (38) and using $W_t^* + \cdots + W_d^* = W$ we find $W \subseteq V_t + \cdots + V_d$. Of course $W_r \subseteq W$ so

$$\begin{aligned} W_r &= W_r \cap W \\ &\subseteq (V_0 + \cdots + V_{d-r}) \cap (V_t + \cdots + V_d). \end{aligned}$$

By this and since $r + t \geq d$ we find $r + t = d$ and then $W_r \subseteq V_{d-r}$. Recall V_{d-r} is an eigenspace for A so $AW_r \subseteq W_r$. Therefore $AW_r \subseteq W$ so $W_r \subseteq \tilde{W}$. Consequently $\tilde{W} \neq 0$ as desired. We have shown $\tilde{W}$ is nonzero and invariant under each of B, B^*. Therefore $\tilde{W} = W$ since W is irreducible as a (B, B^*)-module. We have now shown $AW \subseteq W$. Using this and the involution in Remark 6.3(i) we find $A^*W \subseteq W$. Applying Definition 3.1(iv) to A, A^* we find $W = V$.

We have now shown the pair B, B^* satisfies conditions (i)–(iv) of Definition 3.1. Therefore B, B^* is a tridiagonal pair on V. From the construction $U_0, \ldots, U_d$ is a standard ordering of the eigenspaces of B. For $0 \leq i \leq d$ the scalar bq^{2i-d} is the eigenvalue of B associated with U_i. Therefore the sequence bq^{2i-d} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues of B. From the construction $U_0^*, \ldots, U_d^*$ is a standard ordering of the eigenspaces of B^*. For $0 \leq i \leq d$ the scalar b^*q^{d-2i} is the eigenvalue of B^* associated with U_i^*. Therefore the sequence b^*q^{d-2i} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues of B^*. □

10 Some relations involving A, A^*, B, B^*, K, K^*

In this section we give some relations involving the tridiagonal pair A, A^* from Definition 6.1, the tridiagonal pair B, B^* from Definition 6.2, and the elements K, K^* from Definition 6.2.

Theorem 10.1 *With reference to Definition 6.1 and Definition 6.2,*

$$\frac{qK^{-1}A - q^{-1}AK^{-1}}{q-q^{-1}} = aI, \tag{39}$$

$$\frac{qBK^{-1} - q^{-1}K^{-1}B}{q-q^{-1}} = bI, \tag{40}$$

$$\frac{qKA^* - q^{-1}A^*K}{q-q^{-1}} = a^*I, \tag{41}$$

$$\frac{qB^*K - q^{-1}KB^*}{q-q^{-1}} = b^*I. \tag{42}$$

Proof: We first show (39), (40). Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*D]$ from Lemma 4.2. Concerning (39), we show $qK^{-1}A - q^{-1}AK^{-1} - a(q-q^{-1})I$ vanishes on U_i for $0 \leq i \leq d$. Let i be given. Observe $K - q^{2i-d}I$ vanishes on U_i by Definition 6.2 so $K^{-1} - q^{d-2i}I$ vanishes on U_i; from this we find

$$(A - aq^{2i-d+2}I)(K^{-1} - q^{d-2i}I) \tag{43}$$

vanishes on U_i. From the table of Lemma 5.1, row $[0^*D]$, and using (18), we find $(A - aq^{2i-d}I)U_i \subseteq U_{i+1}$. Therefore

$$(K^{-1} - q^{d-2i-2}I)(A - aq^{2i-d}I) \tag{44}$$

vanishes on U_i. Subtracting q^{-1}times (43) from q times (44) we find $qK^{-1}A - q^{-1}AK^{-1} - a(q-q^{-1})I$ vanishes on U_i. Line (39) follows. Concerning (40), we show $qBK^{-1} - q^{-1}K^{-1}B - b(q-q^{-1})I$ vanishes on U_i for $0 \leq i \leq d$. Let i be given. We mentioned earlier that $K^{-1} - q^{d-2i}I$ vanishes on U_i so

$$(B - bq^{2i-d-2}I)(K^{-1} - q^{d-2i}I) \tag{45}$$

vanishes on U_i. From the table of Lemma 8.1, row $[0^*D]$, we find $(B - bq^{2i-d}I)U_i \subseteq U_{i-1}$. Therefore

$$(K^{-1} - q^{d-2i+2}I)(B - bq^{2i-d}I) \tag{46}$$

vanishes on U_i. Subtracting q^{-1}times (46) from q times (45) we find $qBK^{-1} - q^{-1}K^{-1}B - b(q-q^{-1})I$ vanishes on U_i. Line (40) follows. To obtain (41), (42) apply (39), (40) and the involution given in Remark 6.3(i). □

Theorem 10.2 *With reference to Definition 6.1 and Definition 6.2,*

$$\frac{qAK^* - q^{-1}K^*A}{q-q^{-1}} = aI. \tag{47}$$

$$\frac{qK^{*-1}B - q^{-1}BK^{*-1}}{q-q^{-1}} = bI, \tag{48}$$

$$\frac{qA^*K^{*-1} - q^{-1}K^{*-1}A^*}{q-q^{-1}} = a^*I, \tag{49}$$

$$\frac{qK^*B^* - q^{-1}B^*K^*}{q-q^{-1}} = b^*I, \tag{50}$$

Proof: Use Theorem 10.1 and the involution given in Remark 6.3(ii). □

11 The actions of K, K^{-1}, K^*, K^{*-1} on the six decompositions

In this section we describe how the elements K, K^{-1}, K^*, K^{*-1} from Definition 6.2 act on the six decompositions from Lemma 4.2. We begin with K and K^{-1}.

Theorem 11.1 *Adopt the assumptions of Definition 6.1 and let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V given in Lemma 4.2. Let the map K be as in Definition 6.2. Then for $0 \le i \le d$ the action of K and K^{-1} on U_i is described as follows.*

name	action of K on U_i	action of K^{-1} on U_i
$[0D]$	$(K - q^{2i-d}I)U_i \subseteq U_{i+1} + \cdots + U_d$	$(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i+1}$
$[0^*D^*]$	$(K - q^{2i-d}I)U_i \subseteq U_{i-1}$	$(K^{-1} - q^{d-2i}I)U_i \subseteq U_0 + \cdots + U_{i-1}$
$[0^*D]$	$(K - q^{2i-d}I)U_i = 0$	$(K^{-1} - q^{d-2i}I)U_i = 0$
$[0^*0]$	$(K - q^{2i-d}I)U_i \subseteq U_0 + \cdots + U_{i-1}$	$(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i-1}$
$[D^*0]$	$KU_i \subseteq U_0 + \cdots + U_{i+1}$	$K^{-1}U_i \subseteq U_{i-1} + \cdots + U_d$
$[D^*D]$	$(K - q^{2i-d}I)U_i \subseteq U_{i+1}$	$(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i+1} + \cdots + U_d$

Proof: We consider each of the six rows of the table.
$[0D]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0D]$. From Lemma 5.1, row $[0D]$, and using (18), we find that for $0 \le i \le d$, U_i is an eigenspace for A with eigenvalue aq^{2i-d}. We show $(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i+1}$ for $0 \le i \le d$. To do this, it suffices to show

$$(A - aq^{2i+2-d}I)(K^{-1} - q^{d-2i}I) \tag{51}$$

vanishes on U_i for $0 \le i \le d$. Let i be given. Observe $A - aq^{2i-d}I$ vanishes on U_i so

$$(K^{-1} - q^{d-2i-2}I)(A - aq^{2i-d}I) \tag{52}$$

vanishes on U_i. Using (39) we find

$$qK^{-1}A - q^{-1}AK^{-1} - a(q - q^{-1})I \tag{53}$$

vanishes on U_i. Subtracting (52) from q^{-1} times (53) we find (51) vanishes on U_i. We conclude $(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i+1}$ for $0 \le i \le d$. From this we find $(K - q^{2i-d}I)U_i \subseteq U_{i+1} + \cdots + U_d$ for $0 \le i \le d$.
$[0^*D^*]$: Use the present Theorem, row $[0D]$ and the involution given in Remark 6.3(i).
$[0^*D]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*D]$. From Definition 6.2 we find $(K - q^{2i-d}I)U_i = 0$ for $0 \le i \le d$. It follows $(K^{-1} - q^{d-2i}I)U_i = 0$ for $0 \le i \le d$.
$[0^*0]$: Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*0]$. From Definition 6.2 we find that for $0 \le i \le d$, U_i is an eigenspace for B with eigenvalue bq^{2i-d}. We show $(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i-1}$ for $0 \le i \le d$. To do this, it suffices to show

$$(B - bq^{2i-2-d}I)(K^{-1} - q^{d-2i}I) \tag{54}$$

vanishes on U_i for $0 \leq i \leq d$. Let i be given. Observe $B - bq^{2i-d}I$ vanishes on U_i so

$$(K^{-1} - q^{d-2i+2}I)(B - bq^{2i-d}I) \tag{55}$$

vanishes on U_i. Using (40) we find

$$qBK^{-1} - q^{-1}K^{-1}B - b(q - q^{-1})I \tag{56}$$

vanishes on U_i. Adding (55) to q times (56) we find (54) vanishes on U_i. We conclude $(K^{-1} - q^{d-2i}I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. It follows $(K - q^{2i-d}I)U_i \subseteq U_0 + \cdots + U_{i-1}$ for $0 \leq i \leq d$.
[D^*0]: Let $U_0, U_1, \ldots, U_d$ denote the decomposition [D^*0]. We show $KU_i \subseteq U_0 + \cdots + U_{i+1}$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned} KU_i &\subseteq K(U_0 + \cdots + U_i) \\ &= K(V^*_{d-i} + \cdots + V^*_d) && \text{(by Lemma 4.3, row } [D^*0]) \\ &\subseteq V^*_{d-i-1} + \cdots + V^*_d && \text{(by present Theorem, row } [0^*D^*]) \\ &= U_0 + \cdots + U_{i+1} && \text{(by Lemma 4.3, row } [D^*0]). \end{aligned}$$

Next we show $K^{-1}U_i \subseteq U_{i-1} + \cdots + U_d$ for $0 \leq i \leq d$. Let i be given. We have

$$\begin{aligned} K^{-1}U_i &\subseteq K^{-1}(U_i + \cdots + U_d) \\ &= K^{-1}(V_0 + \cdots + V_{d-i}) && \text{(by Lemma 4.3, row } [D^*0]) \\ &\subseteq V_0 + \cdots + V_{d-i+1} && \text{(by present Theorem, row } [0D]) \\ &= U_{i-1} + \cdots + U_d && \text{(by Lemma 4.3, row } [D^*0]). \end{aligned}$$

[D^*D]: Use the present Theorem, row [0^*0] and the involution given in Remark 6.3(i). □

We now describe the action of K^* and K^{*-1} on each of the six decompositions from Lemma 4.2.

Theorem 11.2 *Adopt the assumptions of Definition 6.1 and let $U_0, U_1, \ldots, U_d$ denote any one of the six decompositions of V given in Lemma 4.2. Let the map K^* be as in Definition 6.2. Then for $0 \leq i \leq d$ the action of K^* and K^{*-1} on U_i is described as follows.*

name	action of K^* on U_i	action of K^{*-1} on U_i
[$0D$]	$(K^* - q^{d-2i}I)U_i \subseteq U_{i-1}$	$(K^{*-1} - q^{2i-d}I)U_i \subseteq U_0 + \cdots + U_{i-1}$
[0^*D^*]	$(K^* - q^{d-2i}I)U_i \subseteq U_{i+1} + \cdots + U_d$	$(K^{*-1} - q^{2i-d}I)U_i \subseteq U_{i+1}$
[0^*D]	$K^*U_i \subseteq U_{i-1} + \cdots + U_d$	$K^{*-1}U_i \subseteq U_0 + \cdots + U_{i+1}$
[0^*0]	$(K^* - q^{2i-d}I)U_i \subseteq U_{i+1} + \cdots + U_d$	$(K^{*-1} - q^{d-2i}I)U_i \subseteq U_{i+1}$
[D^*0]	$(K^* - q^{2i-d}I)U_i = 0$	$(K^{*-1} - q^{d-2i}I)U_i = 0$
[D^*D]	$(K^* - q^{2i-d}I)U_i \subseteq U_{i-1}$	$(K^{*-1} - q^{d-2i}I)U_i \subseteq U_0 + \cdots + U_{i-1}$

Proof: Use Theorem 11.1 and the involution given in Remark 6.3(ii). □

12 The q-Serre relations

In this section we give two relations involving the tridiagonal pair A, A^* from Definition 6.1, and two relations involving the tridiagonal pair B, B^* from Definition 6.2.

Theorem 12.1 *With reference to Definition 6.1 and Definition 6.2,*

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \qquad (57)$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0, \qquad (58)$$
$$B^3B^* - [3]_q B^2B^*B + [3]_q BB^*B^2 - B^*B^3 = 0, \qquad (59)$$
$$B^{*3}B - [3]_q B^{*2}BB^* + [3]_q B^*BB^{*2} - BB^{*3} = 0. \qquad (60)$$

Proof: We first show (57). let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0D]$ from Lemma 4.2. By Lemma 5.1, row $[0D]$, and using (18), we find that for $0 \leq i \leq d$ the space U_i is an eigenspace for A with eigenvalue aq^{2i-d}. Abbreviate $\Psi = A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3$. We show $\Psi = 0$. To do this we show $\Psi U_i = 0$ for $0 \leq i \leq d$. Let i be given and pick $v \in U_i$. Observe $A^*v \in U_{i-1} + U_i + U_{i+1}$ by Lemma 5.1, row $[0D]$. Observe $(A - aq^{2i-2-d}I)U_{i-1} = 0$, $(A - aq^{2i-d}I)U_i = 0$, and $(A - aq^{2i+2-d}I)U_{i+1} = 0$. By these comments

$$(A - aq^{2i-2-d}I)(A - aq^{2i-d}I)(A - aq^{2i+2-d}I)A^*v = 0.$$

We may now argue

$$\begin{aligned}
\Psi v &= (A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3)v \\
&= (A^3A^* - [3]_q A^2A^*aq^{2i-d} + [3]_q AA^*a^2q^{4i-2d} - A^*a^3q^{6i-3d})v \\
&= (A - aq^{2i-2-d}I)(A - aq^{2i-d}I)(A - aq^{2i+2-d}I)A^*v \\
&= 0.
\end{aligned}$$

We have now shown $\Psi U_i = 0$ for $0 \leq i \leq d$. We conclude $\Psi = 0$ and (57) follows. To get (58) use (57) and the involution in Remark 6.3(i). To get (59), (60) apply (57), (58) to the tridiagonal pair B, B^*. $\square$

13 Two modules for $U_q(\widehat{sl}_2)$

In this section we prove the existence part of Theorem 3.3. We begin with two theorems.

Theorem 13.1 *Adopt the assumptions of Definition 6.1. Let B, B^*, K be as in Definition 6.2. Then V is an irreducible $U_q(\widehat{sl}_2)$-module on which the alternate generators act as follows.*

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	$b^{*-1}B^*$	$b^{-1}B$	$a^{*-1}A^*$	$a^{-1}A$	K	K^{-1}	K^{-1}	K

Proof: To see that the above action on V gives a $U_q(\widehat{sl}_2)$-module, compare the equations in Theorem 7.1, Theorem 10.1, and Theorem 12.1 with the defining relations for $U_q(\widehat{sl}_2)$ given in Theorem 2.1. The $U_q(\widehat{sl}_2)$-module V is irreducible by Definition 3.1(iv). □

Theorem 13.2 *Adopt the assumptions of Definition 6.1. Let B, B^*, K^* be as in Definition 6.2. Then V is an irreducible $U_q(\widehat{sl}_2)$-module on which the alternate generators act as follows.*

generator	y_0^+	y_1^+	y_0^-	y_1^-	k_0	k_1	k_0^{-1}	k_1^{-1}
action on V	$a^{-1}A$	$a^{*-1}A^*$	$b^{*-1}B^*$	$b^{-1}B$	K^*	K^{*-1}	K^{*-1}	K^*

Proof: To see that the above action on V gives a $U_q(\widehat{sl}_2)$-module, compare the equations in Theorem 7.1, Theorem 10.2, and Theorem 12.1 with the defining relations for $U_q(\widehat{sl}_2)$ given in Theorem 2.1. The $U_q(\widehat{sl}_2)$-module V is irreducible by Definition 3.1(iv). □

It is now a simple matter to prove the existence part of Theorem 3.3.

Proof of Theorem 3.3 (existence): By Theorem 13.1 there exists an irreducible $U_q(\widehat{sl}_2)$-module structure on V such that ay_1^- acts as A and $a^*y_0^-$ acts A^*. By Theorem 13.2 there exists an irreducible $U_q(\widehat{sl}_2)$-module structure on V such that ay_0^+ acts as A and $a^*y_1^+$ acts as A^*. □

14 Uniqueness

In this section we prove the uniqueness part of Theorem 3.3.

We begin with a comment concerning finite dimensional irreducible $U_q(\widehat{sl}_2)$-modules.

Lemma 14.1 *Let V denote a finite dimensional irreducible $U_q(\widehat{sl}_2)$-module. Then there exist nonzero scalars $\varepsilon_0, \varepsilon_1$ in $\mathbb{K}$ and there exists a decomposition $U_0, U_1, \ldots, U_d$ of V such that both*

$$(k_0 - \varepsilon_0 q^{2i-d} I)U_i = 0, \qquad (k_1 - \varepsilon_1 q^{d-2i} I)U_i = 0 \qquad (0 \le i \le d). \tag{61}$$

The sequence $\varepsilon_0, \varepsilon_1; U_0, U_1, \ldots, U_d$ is unique. Moreover for $0 \le i \le d$ we have

$$(\varepsilon_0 y_0^+ - q^{d-2i} I)U_i \subseteq U_{i+1}, \qquad (\varepsilon_1 y_1^- - q^{2i-d} I)U_i \subseteq U_{i+1}, \tag{62}$$

$$(\varepsilon_0 y_0^- - q^{d-2i} I)U_i \subseteq U_{i-1}, \qquad (\varepsilon_1 y_1^+ - q^{2i-d} I)U_i \subseteq U_{i-1}. \tag{63}$$

Proof: By the construction V has finite positive dimension. Since k_0k_1 is central in $U_q(\widehat{sl}_2)$ and since $\mathbb{K}$ is algebraically closed, there exists $\alpha \in \mathbb{K}$ such that $(k_0k_1 - \alpha I)V = 0$. Observe $\alpha \neq 0$ since each of k_0, k_1 is invertible on V. For $\theta \in \mathbb{K}$ we define $V(\theta) = \{v \in V | k_0 v = \theta v\}$. We observe $V(\theta) \neq 0$ if and only if θ is an eigenvalue of k_0 on V, and in this case $V(\theta)$ is the corresponding eigenspace. For nonzero $\theta \in \mathbb{K}$ we find using (11), (12) that

$$(y_0^+ - \theta^{-1} I)V(\theta) \subseteq V(q^2\theta), \qquad (y_1^- - \theta\alpha^{-1} I)V(\theta) \subseteq V(q^2\theta), \tag{64}$$

$$(y_0^- - \theta^{-1} I)V(\theta) \subseteq V(q^{-2}\theta), \qquad (y_1^+ - \theta\alpha^{-1} I)V(\theta) \subseteq V(q^{-2}\theta). \tag{65}$$

Since $\mathbb{K}$ is algebraically closed and since V has finite positive dimension, there exists $\theta \in \mathbb{K}$ such that $V(\theta) \neq 0$. We observe $\theta \neq 0$ since k_0 is invertible on V. Since q is not a root of unity the scalars $\theta, q^{-2}\theta, q^{-4}\theta, \ldots$ are mutually distinct. These scalars cannot all be eigenvalues of k_0 on V; consequently there exists a nonzero $\eta \in \mathbb{K}$ such that $V(\eta) \neq 0$ and $V(q^{-2}\eta) = 0$. Similarly the scalars $\eta, q^2\eta, q^4\eta, \ldots$ are mutually distinct so they are not all eigenvalues of k_0 on V; consequently there exists a nonnegative integer d such that $V(q^{2i}\eta)$ is nonzero for $0 \leq i \leq d$ and zero for $i = d+1$. We abbreviate $U_i = V(q^{2i}\eta)$ for $0 \leq i \leq d$. From the construction

$$(k_0 - q^{2i}\eta I)U_i = 0, \qquad (k_1 - \alpha q^{-2i}\eta^{-1} I)U_i = 0 \qquad (0 \leq i \leq d). \tag{66}$$

Define $\varepsilon_0, \varepsilon_1$ so that $\eta = \varepsilon_0 q^{-d}$ and $\varepsilon_0\varepsilon_1 = \alpha$. Observe $\varepsilon_0, \varepsilon_1$ are nonzero. Eliminating η, α in (66) using the preceeding equations we obtain (61). From (64), (65) and our above comments we obtain (62), (63), where $U_{-1} = 0$ and $U_{d+1} = 0$. We claim $V = \sum_{i=0}^{d} U_i$. From (61)–(63) we find $\sum_{i=0}^{d} U_i$ is invariant under each of the alternate generators for $U_q(\widehat{sl}_2)$. Also $\sum_{i=0}^{d} U_i$ is nonzero since each of $U_0, \ldots, U_d$ is nonzero. We conclude $V = \sum_{i=0}^{d} U_i$ since V is irreducible as a $U_q(\widehat{sl}_2)$-module. The sum $\sum_{i=0}^{d} U_i$ is direct since each of $U_0, \ldots, U_d$ is an eigenspace for k_0 and the corresponding eigenvalues are mutually distinct. We now see $U_0, \ldots, U_d$ is a decomposition of V. It is clear that the sequence $\varepsilon_0, \varepsilon_1; U_0, U_1, \ldots, U_d$ is unique. □

Remark 14.2 We will not use this fact, but it turns out that the scalars $\varepsilon_0, \varepsilon_1$ from Lemma 14.1 are both in $\{1, -1\}$. See for example [3, Proposition 3.2]. That proof assumes $\mathbb{K} = \mathbb{C}$ but the assumption is unnecessary.

Definition 14.3 Referring to Lemma 14.1, we call the sequence $U_0, U_1, \ldots, U_d$ the *weight space decomposition* of V. We call the ordered pair $(\varepsilon_0, \varepsilon_1)$ the *type* of V.

Example 14.4 *Adopt the assumptions of Definition 6.1. For the $U_q(\widehat{sl}_2)$-module structure on V given in Theorem 13.1 (resp. Theorem 13.2), the weight space decomposition coincides with the decomposition $[0^*D]$ (resp. $[D^*0]$) from Lemma 4.2. Both module structures have type $(1,1)$.*

Proof: We first consider the $U_q(\widehat{sl}_2)$-module structure from Theorem 13.1. Let $U_0, U_1, \ldots, U_d$ denote the decomposition $[0^*D]$. By Definition 6.2(iii) we find $(K - q^{2i-d}I)U_i = 0$ for $0 \leq i \leq d$. By Theorem 13.1 we find k_0, k_1 act on V as K, K^{-1} respectively. Therefore $(k_0 - q^{2i-d}I)U_i = 0$ and $(k_1 - q^{d-2i}I)U_i = 0$ for $0 \leq i \leq d$. Define $\varepsilon_0 = 1$, $\varepsilon_1 = 1$ and observe these values satisfy (61). By Definition 14.3, V has weight space decompostion $U_0, U_1, \ldots, U_d$ and type $(1,1)$. We have now proved our assertions concerning the $U_q(\widehat{sl}_2)$-module structure from Theorem 13.1. The proof for the $U_q(\widehat{sl}_2)$-module structure from Theorem 13.2 is similar and omitted. □

Proof of Theorem 3.3(uniqueness): For $0 \leq i \leq d$ let V_i (resp. V_i^*) denote the eigenspace of A (resp. A^*) associated with θ_i (resp. θ_i^*). We assume a $U_q(\widehat{sl}_2)$-module structure on V

such that ay_1^- acts as A and $a^*y_0^-$ acts as A^*. We show the alternate generators for $U_q(\widehat{sl}_2)$ act on V according to the table of Theorem 13.1. Observe the $U_q(\widehat{sl}_2)$-module structure is irreducible in view of Definition 3.1(iv). Let $(\varepsilon_0, \varepsilon_1)$ denote the type of the $U_q(\widehat{sl}_2)$-module structure. We claim $(\varepsilon_0, \varepsilon_1) = (1, 1)$. To see this, consider the weight space decomposition $U_0, U_1, \ldots, U_d$ from Lemma 14.1. By (62) and since ay_1^- acts on V as A we find

$$(\varepsilon_1 A - aq^{2i-d} I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d). \tag{67}$$

Similarly

$$(\varepsilon_0 A^* - a^* q^{d-2i} I)U_i \subseteq U_{i-1} \qquad (0 \leq i \leq d). \tag{68}$$

From (67) we find that for $0 \leq i \leq d$ the scalar $\varepsilon_1^{-1} aq^{2i-d}$ is an eigenvalue of A and the dimension of the corresponding eigenspace has the same dimension as U_i. Apparently the sequence $\varepsilon_1^{-1} aq^{2i-d}$ $(0 \leq i \leq d)$ is an ordering the eigenvalues of A. Recall $\theta_i = aq^{2i-d}$ for $0 \leq i \leq d$. Therefore the sequence $\varepsilon_1^{-1} aq^{2i-d}$ $(0 \leq i \leq d)$ is a permutation of the sequence aq^{2i-d} $(0 \leq i \leq d)$. Since q is not a root of unity we must have $\varepsilon_1 = 1$. By a similar argument we find $\varepsilon_0 = 1$. Setting $(\varepsilon_0, \varepsilon_1) = (1, 1)$ in (67), (68) we find $(A - \theta_i I)U_i \subseteq U_{i+1}$ and $(A^* - \theta_i^* I)U_i \subseteq U_{i-1}$ for $0 \leq i \leq d$. By this and [7, Theorem 4.6] we find $U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d)$ for $0 \leq i \leq d$. In other words $U_0, \ldots, U_d$ is the decomposition $[0^*D]$ from Lemma 4.2. By Definition 6.2(iii) we have $(K - q^{2i-d} I)U_i = 0$ for $0 \leq i \leq d$. Comparing this with (61) and recalling $(\varepsilon_0, \varepsilon_1) = (1, 1)$ we find k_0, k_1 act on V as K, K^{-1} respectively. Apparently k_0^{-1}, k_1^{-1} act on V as K^{-1}, K respectively. We show by_1^+ acts on V as B. Define $W = \{v \in V | (by_1^+ - B)v = 0\}$. We show $W = V$. To do this we show $W \neq 0$, $AW \subseteq W$, $A^*W \subseteq W$. Observe $(B - bq^{-d} I)U_0 = 0$ by Theorem 8.1, row $[0^*D]$. Observe $(y_1^+ - q^{-d} I)U_0 = 0$ by (63). By these comments $by_1^+ - B$ vanishes on U_0. Therefore $U_0 \subseteq W$ so $W \neq 0$. By (13) (with $i = 1$), by (23), and since ay_1^-, A agree on V, we find $(by_1^+ - B)A$, $q^2 A(by_1^+ - B)$ agree on V. Using this we find $AW \subseteq W$. By (14) (with $i = 1$), by (24), and since $a^*y_0^-, A^*$ agree on V, we find $(by_1^+ - B)A^*$, $q^{-2} A^*(by_1^+ - B)$ agree on V. Using this we find $A^*W \subseteq W$. We have now shown $W \neq 0$, $AW \subseteq W$, $A^*W \subseteq W$. Now $W = V$ in view of Definition 3.1(iv). We conclude $(by_1^+ - B)V = 0$ so by_1^+ acts on V as B. By a similar argument we find $b^*y_0^+$ acts on V as B^*. We have now shown $y_i^\pm$, $k_i^{\pm 1}$, $i \in \{0, 1\}$ act on V according to the table of Theorem 13.1. It follows the given $U_q(\widehat{sl}_2)$-module structure is unique. By a similar argument we obtain the uniqueness of the irreducible $U_q(\widehat{sl}_2)$-module structure on V such that ay_0^+ acts as A and $a^*y_1^+$ acts as A^*. □

15 Comments

We have a comment on Theorem 3.3.

Lemma 15.1 *Let a, a^* denote nonzero scalars in $\mathbb{K}$. Let A, A^* denote elements in $U_q(\widehat{sl}_2)$ which satisfy*

$$A = ay_1^-, \qquad A^* = a^* y_0^- \tag{69}$$

or

$$A = ay_0^+, \qquad\qquad A^* = a^*y_1^+. \tag{70}$$

Let V denote a finite dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1,1)$. Assume V is irreducible as an (A, A^*)-module. Then the pair A, A^* acts on V as a tridiagonal pair. Denoting the diameter of this pair by d, the sequence aq^{2i-d} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues for A on V and the sequence a^*q^{d-2i} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues for A^* on V.

Proof: First assume (69). By (15) and (69) we find both

$$A^3A^* - [3]_q A^2A^*A + [3]_q AA^*A^2 - A^*A^3 = 0, \tag{71}$$
$$A^{*3}A - [3]_q A^{*2}AA^* + [3]_q A^*AA^{*2} - AA^{*3} = 0. \tag{72}$$

Let $U_0, U_1, \ldots, U_d$ denote the weight space decomposition of V from Definition 14.3. Setting $(\varepsilon_0, \varepsilon_1) = (1, 1)$ in Lemma 14.1 and using (69) we find both

$$(A - aq^{2i-d}I)U_i \subseteq U_{i+1} \qquad (0 \leq i \leq d), \tag{73}$$
$$(A^* - a^*q^{d-2i}I)U_i \subseteq U_{i-1} \qquad (0 \leq i \leq d). \tag{74}$$

We draw several conclusions from these lines. From (73) (resp. (74)) the action of A (resp. A^*) on V is diagonalizable. Also for $0 \leq i \leq d$ the scalar aq^{2i-d} (resp. a^*q^{d-2i}) is an eigenvalue for this action and the corresponding eigenspace has the same dimension as U_i. In particular the scalars aq^{2i-d} $(0 \leq i \leq d)$ (resp. a^*q^{d-2i} $(0 \leq i \leq d)$) are the eigenvalues of A (resp. A^*) on V. We are assuming V is irreducible as an (A, A^*)-module. This means there does not exist a subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$. We show A, A^* acts on V as a tridiagonal pair. To do this we apply [7, Example 1.7]. In order to apply this example we must show neither of A, A^* is nilpotent on V. We mentioned above that each of A, A^* is diagonalizable on V. Neither of A, A^* is zero on V so neither of A, A^* is nilpotent on V. Now by [7, Example 1.7] we find A, A^* act on V as a tridiagonal pair. The diameter of this pair is d since each of A, A^* has $d+1$ distinct eigenvalues. By [12, Lemma 4.8] there exists a standard ordering of the eigenvalues of A (resp. A^*) on V of the form αq^{2i-d} $(0 \leq i \leq d)$ (resp. $\alpha^* q^{d-2i}$ $(0 \leq i \leq d)$), where α (resp. α^*) is an appropriate nonzero scalar in $\mathbb{K}$. Combining this with our above remarks we find $\alpha = a$ and $\alpha^* = a^*$. Therefore the sequence aq^{2i-d} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues for A on V and the sequence a^*q^{d-2i} $(0 \leq i \leq d)$ is a standard ordering of the eigenvalues for A^* on V. We have now proved the result for case (69). For the case (70) the proof is similar and omitted. $\square$

16 Suggestions for further research

In this section we give some open problems. The first problem is motivated by Lemma 15.1.

Problem 16.1 Let a, a^* denote nonzero scalars in $\mathbb{K}$ and let A, A^* denote the elements of $U_q(\widehat{sl}_2)$ given in (69) or (70). Let V denote a finite dimensional irreducible $U_q(\widehat{sl}_2)$-module of type $(1,1)$. Find a necessary and sufficient condition for V to be irreducible as an (A, A^*)-module.

In order to state the next problem we recall a few terms. Let V denote a vector space over $\mathbb{K}$ with finite positive dimension. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of all linear transformations from V to V. By an *antiautomorphism* of $\mathrm{End}(V)$ we mean a $\mathbb{K}$-linear bijection $\dagger : \mathrm{End}(V) \to \mathrm{End}(V)$ such that $(XY)^\dagger = Y^\dagger X^\dagger$ for all $X, Y \in \mathrm{End}(V)$.

Problem 16.2 Let A, A^* denote a tridiagonal pair on V. Show there exists an antiautomorphism $\dagger$ of $\mathrm{End}(V)$ such that $A^\dagger = A$ and $A^{*\dagger} = A^*$. We remark that $\dagger$ exists if A, A^* is a Leonard pair [19, Theorem 7.1].

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