Diffeomorphisms between Siegel domains of the first kind preserving the holomorphic automorphism groups and applications

メタデータ	言語: eng
	出版者:
	公開日: 2017-12-05
	キーワード (Ja):
	キーワード (En):
	作成者:
	メールアドレス:
	所属:
URL	https://doi.org/10.24517/00010603

A. KODAMA AND S. SHIMIZU
KODAI MATH. J.
36 (2013), 299–312

DIFFEOMORPHISMS BETWEEN SIEGEL DOMAINS OF THE FIRST KIND PRESERVING THE HOLOMORPHIC AUTOMORPHISM GROUPS AND APPLICATIONS

Dedicated to the memory of Professor Shoshichi Kobayashi

AKIO KODAMA AND SATORU SHIMIZU

Abstract

This is a continuation of our previous paper [4]. In the class of hyperbolic manifolds in the sense of S. Kobayashi [3], we obtained in [4] an intrinsic characterization of bounded symmetric domains in $\mathbf{C}^n$ from the viewpoint of the holomorphic automorphism group. In connection with this, we give in this paper a structure theorem on diffeomorphisms between Siegel domains of the first kind that preserve the holomorphic automorphism groups. As an application, we obtain a well-known fact [2] that two Siegel domains of the first kind are biholomorphically equivalent if and only if they are linearly equivalent.

1. Introduction

Let M be a connected complex manifold and $\mathrm{Aut}(M)$ the group of all biholomorphic automorphisms of M equipped with the compact-open topology. Then one of the fundamental problems in complex geometric analysis is to determine the complex analytic structure of M by its holomorphic automorphism group $\mathrm{Aut}(M)$. Of course, this is impossible without any further assumptions on M, since there exist many domains in $\mathbf{C}^n$ that are not biholomorphically equivalent although their holomorphic automorphism groups are isomorphic as topological groups. However, in the class of hyperbolic manifolds in the sense of Kobayashi [3], we obtained the following characterization of bounded symmetric domains by their holomorphic automorphism groups (cf. [4; Corollary 2]): *Let M be a connected hyperbolic manifold of dimension n and let D be a bounded symmetric domain in $\mathbf{C}^n$. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(D)$ *as*

2000 *Mathematics Subject Classification.* Primary 32M05; Secondary 32Q28.

Key words and phrases. Holomorphic automorphism groups, Siegel domains of the first kind, Holomorphic equivalence problem.

The authors are partially supported by the Grant-in-Aid for Scientific Research (C) No. 24540166 and (C) No. 22540167, the Ministry of Education, Science, Sports and Culture, Japan.

Received July 25, 2012; revised December 11, 2012.

topological groups. Then M is biholomorphically equivalent to D. Here it would be natural to ask what happens when the domain D is a homogeneous bounded (not necessarily symmetric) domains in $\mathbf{C}^n$. In this paper, we study exclusively Siegel domains of the first kind in conection with this question and establish the following: (For the definition of Siegel domains of the first kind, see Section 1.)

THEOREM 1. *Let T_Ω and $T_{\Omega'}$ be the Siegel domains of the first kind in $\mathbf{C}^n$ associated to convex cones Ω and Ω' in $\mathbf{R}^n$, respectively. Let $G(T_\Omega)$ and $G(T_{\Omega'})$ be the identity components of* $\mathrm{Aut}(T_\Omega)$ *and* $\mathrm{Aut}(T_{\Omega'})$, *respectively. Assume that there exists a real analytic diffeomorphism $F : T_\Omega \to T_{\Omega'}$ with $FG(T_\Omega)F^{-1} = G(T_{\Omega'})$. Then, after replacing F by a suitable diffeomorphism of the form $\sigma \cdot F$, $\sigma \in G(T_{\Omega'})$, if necessary, F can be written in the form*

$$F(z) = Px + U(y) + \sqrt{-1}V(y), \quad z = x + \sqrt{-1}y \in T_\Omega,$$

where $P \in GL(n, \mathbf{R})$ and $U : \Omega \to \mathbf{R}^n$ is a real analytic mapping, and $V : \Omega \to \Omega'$ is a real analytic diffeomorphism which satisfies the following:

$$(*) \qquad V(hy) = PhP^{-1}V(y) \quad \text{for all } h \in G(\Omega),\ y \in \Omega,$$

where $G(\Omega)$ is the identity component of the linear automorphism group $GL(\Omega)$ of the convex cone Ω in $\mathbf{R}^n$.

As an immediate consequence of this theorem, we obtain the following well-known fact (cf. [2], [5]):

COROLLARY. *Let T_Ω and $T_{\Omega'}$ be Siegel domains of the first kind in $\mathbf{C}^n$. Then T_Ω is biholomorphically equivalent to $T_{\Omega'}$ if and only if there exists a diffeomorphism $V : \Omega \to \Omega'$ having the form $V(y) = Py$, $y \in \Omega$, with $P \in GL(n, \mathbf{R})$.*

THEOREM 2. *In the same notations as in Theorem* 1, *assume that there exists a topological group isomorphism $\Phi : G(T_\Omega) \to G(T_{\Omega'})$ and assume further that one of two Siegel domains T_Ω and $T_{\Omega'}$ is homogeneous. Then the other is also homogeneous and Φ induces a real analytic diffeomorphism $F : T_\Omega \to T_{\Omega'}$ such that $F(g \cdot z) = \Phi(g) \cdot F(z)$ for all $g \in G(T_\Omega)$ and all $z \in T_\Omega$. In particular, after replacing Φ by a suitable topological group isomorphism of the form $\sigma\Phi(\cdot)\sigma^{-1}$, $\sigma \in G(T_{\Omega'})$, if necessary, F can be expressed just as in Theorem* 1. *Moreover, the restriction of Φ to the identity component $A(T_\Omega)$ of the affine automorphism group of T_Ω can be explicitly given by the following:*

$$(\#) \qquad F(g \cdot z) = PhP^{-1}F(z) + Pa + (I_n - PhP^{-1})\mathfrak{a}_o, \quad z \in T_\Omega,$$

for every element $g = (h, a) \in G(\Omega) \times \mathbf{R}^n = A(T_\Omega)$, where $P \in GL(n, \mathbf{R})$ and I_n is the identity matrix of degree n, and $\mathfrak{a}_o$ is an element of $\mathbf{R}^n$.

In Section 1, we recall basic concepts on Siegel domains of the first kind in $\mathbf{C}^n$ and collect some preliminary facts on them. After that, our Theorems and Corollary will be proven in Section 2. In the final Section 3, we study the linear equivalence problem for homogeneous convex cones in $\mathbf{R}^n$ in connection with our subject in this paper. In particular, we show that two homogeneous Siegel domains of the first kind in $\mathbf{C}^n$ with $n \leq 3$ are biholomorphically equivalent if and only if the identity components of their holomorphic automorphism groups are isomorphic as topological groups.

1. Preliminaries

For later purpose, in this section we shall recall some definitions and known facts on Siegel domains introduced by Pyatetskii-Shapiro [6].

Throughout this paper, we use the following notation: Let $z = (z_1, \dots, z_n)$ be a point of $\mathbf{C}^n$. Then, writing $z_j = x_j + \sqrt{-1}y_j$ with $x_j, y_j \in \mathbf{R}$ for $1 \leq j \leq n$, we set $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Thus the point $z \in \mathbf{C}^n$ can be expressed as $z = x + \sqrt{-1}y$ with $x, y \in \mathbf{R}^n$.

Let Ω be a convex domain in $\mathbf{R}^n$. Then Ω is called a *convex cone* if Ω contains no entire straight line and $\lambda y \in \Omega$ whenever $y \in \Omega$ and $\lambda > 0$. Let Ω be a convex cone in $\mathbf{R}^n$. Then the domain T_Ω in $\mathbf{C}^n$ defined by

$$T_\Omega = \{x + \sqrt{-1}y \in \mathbf{C}^n \mid y \in \Omega\}$$

is called a *Siegel domain of the first kind associated to the convex cone* Ω. It is known [6] that every Siegel domain T_Ω is biholomorphically equivalent to a bounded domain in $\mathbf{C}^n$. Hence, its holomorphic automorphism group $\mathrm{Aut}(T_\Omega)$ has the structure of a Lie group with respect to the compact-open topology by a well-known theorem of H. Cartan and the Lie algebra $\mathfrak{g}(T_\Omega)$ of all complete holomorphic vector fields on T_Ω is canonically identified with the Lie algebra of $\mathrm{Aut}(T_\Omega)$. By the definition of T_Ω, the one-parameter groups $\{\phi_t^a\}_{t\in\mathbf{R}}$, $\{\psi_t\}_{t\in\mathbf{R}}$ defined by

$$\phi_t^a : z \mapsto z + ta, \quad \psi_t : z \mapsto (\exp t)z$$

are contained in $\mathrm{Aut}(T_\Omega)$, where a is an arbitrary element of $\mathbf{R}^n$. Hence, the following holomorphic vector fields on T_Ω are contained in $\mathfrak{g}(T_\Omega)$:

$$\partial_j := \frac{\partial}{\partial z_j} \quad (1 \leq j \leq n) \quad \text{and} \quad \partial := \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}.$$

More precisely, we have the following:

Structure Theorem (Kaup-Matsushima-Ochiai [2]). *Let T_Ω be the Siegel domain of the first kind in $\mathbf{C}^n$ associated to a convex cone Ω in $\mathbf{R}^n$ and let $\mathfrak{g}(T_\Omega)$ be the Lie algebra consisting of all complete holomorphic vector fields on T_Ω. Then we have*:

(1) $\mathfrak{g}(T_\Omega) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$, *where each $\mathfrak{g}_\lambda$ consists of all vector fields in $\mathfrak{g}(T_\Omega)$ whose components are homogeneous polynomials in $z_1, \dots, z_n$ of degree $\lambda + 1$ with real coefficients. In particular, $\mathfrak{g}_{-1}$, $\mathfrak{g}_0$, respectively, consist of vector fields of the form $\sum_{j=1}^n a_j\partial_j$, $\sum_{k=1}^n(\sum_{j=1}^n a_{kj}z_j)\partial_k$ with $a_j, a_{kj} \in \mathbf{R}$ for all j, k.*

(2) $\mathfrak{g}_\lambda$ *is the eigenspace of* ad ∂ *for the eigenvalue* λ *and* $[\mathfrak{g}_\lambda, \mathfrak{g}_\sigma] \subset \mathfrak{g}_{\lambda+\sigma}$ *for all* λ, σ.

(3) *Let* $\mathfrak{a}(T_\Omega) = \mathfrak{g}_{-1} + \mathfrak{g}_0$. *Then* $\mathfrak{a}(T_\Omega)$ *is the subalgebra of* $\mathfrak{g}(T_\Omega)$ *corresponding to the subgroup* $\mathrm{Aff}(T_\Omega)$ *of* $\mathrm{Aut}(T_\Omega)$ *consisting of all affine transformations of* $\mathbf{C}^n$ *leaving* T_Ω *invariant. And,* $\mathfrak{g}_0$ *is the Lie algebra of all* $X \in \mathfrak{g}(T_\Omega)$ *which are tangent to the real submanifold* $\sqrt{-1}\Omega = \{x + \sqrt{-1}y \in T_\Omega \mid x = 0\}$; *and hence* $\mathfrak{g}_0$ *is identified with the Lie algebra of the group* $GL(\Omega)$ *consisting of all nonsingular linear transformations of* $\mathbf{R}^n$ *leaving* Ω *invariant.*

In this paper, we denote by $A(T_\Omega)$ and $G(\Omega)$ the identity components of $\mathrm{Aff}(T_\Omega)$ and $GL(\Omega)$, respectively. Then $A(T_\Omega)$ may be regarded as the Lie group $G(\Omega) \times \mathbf{R}^n$ with the multiplication given by

$$(h_1, a_1)(h_2, a_2) = (h_1 h_2, h_1 a_2 + a_1) \tag{1.1}$$

for $(h_j, a_j) \in G(\Omega) \times \mathbf{R}^n$, $j = 1, 2$, and the action of $A(T_\Omega) = G(\Omega) \times \mathbf{R}^n$ on T_Ω is given by

$$g \cdot z = hx + a + \sqrt{-1}hy, \quad z = x + \sqrt{-1}y \in T_\Omega, \tag{1.2}$$

for all $g = (h, a) \in G(\Omega) \times \mathbf{R}^n = A(T_\Omega)$.

Now let us consider a real semi-simple Lie algebra $\mathfrak{s}$ and its subalgebra $\mathfrak{t}$. Then $\mathfrak{t}$ is said to be *triangular* if there exists a basis of $\mathfrak{s}$ with respect to which ad X is represented by an upper triangular matrix for every $X \in \mathfrak{t}$. Clearly, any triangular subalgebra is solvable. Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{t}$ a solvable subalgebra of $\mathfrak{g}$. Then $\mathfrak{t}$ is said to be *maximal triangular* if it contains the radical $\mathfrak{r}$ of $\mathfrak{g}$ and if $\mathfrak{t}/\mathfrak{r}$ is a maximal triangular subalgebra of the semi-simple Lie algebra $\mathfrak{g}/\mathfrak{r}$. We finish this section by recalling the following:

LEMMA 1 (cf. [5; Lemmas 6 and 7]). *Let* $\mathfrak{g}$ *be a real Lie algebra and* $\mathfrak{g}(T_\Omega)$ *the Lie algebra consisting of all complete holomorphic vector fields on a Siegel domain* T_Ω *of the first kind in* $\mathbf{C}^n$. *Then we have*:

(1) *Any two maximal triangular subalgebras of* $\mathfrak{g}$ *are conjugate under an inner automorphism of* $\mathfrak{g}$.

(2) *There exists a maximal triangular subalgebra* $\mathfrak{t}$ *of* $\mathfrak{g}(T_\Omega)$ *such that* $\partial \in \mathfrak{t}$ *and* $\mathfrak{t} = \mathfrak{g}_{-1} + \mathfrak{t} \cap \mathfrak{g}_0$.

2. Proofs of Theorems and Corollary

We start with the following lemma, which will play an important role in the proof of Theorem 1.

LEMMA 2. *Let* T_Ω *be a Siegel domain of the first kind in* $\mathbf{C}^n$ *and let* $\mathfrak{a}(T_\Omega) = \mathfrak{g}_{-1} + \mathfrak{g}_0$ *be the Lie algebra of* $\mathrm{Aff}(T_\Omega)$, *as in the Structure Theorem. Let* $\mathfrak{a}$ *be any abelian ideal of* $\mathfrak{a}(T_\Omega)$. *Then* $\mathfrak{a}$ *is contained in the ideal* $\mathfrak{g}_{-1}$ *of* $\mathfrak{a}(T_\Omega)$. *In particular,* $\mathfrak{g}_{-1}$ *is the unique maximal abelian ideal of* $\mathfrak{a}(T_\Omega)$.

Proof. Our proof is almost identical to that of [8; Lemma 4]. Although there are overlaps with that, we carry out the proof in detail for the sake of completeness and self-containedness.

We first observe that every element X of $\mathfrak{a}(T_\Omega)$ can be written in the form

$$X = \sum_{j=1}^{n} f_j(z)\partial_j, \tag{2.1}$$

where every $f_j(z)$ is a polynomial of degree at most one, and that

$$[\partial_i, X] = \sum_{j=1}^{n} (\partial_i f_j(z))\partial_j \quad \text{for } 1 \le i \le n. \tag{2.2}$$

Now, assuming to the contrary that $\mathfrak{a} \not\subset \mathfrak{g}_{-1}$, we would like to derive a contradiction. To this end, note that

$$[\mathfrak{g}_{-1}, \mathfrak{a}] \subset \mathfrak{g}_{-1} \cap \mathfrak{a}, \tag{2.3}$$

since $\mathfrak{g}_{-1}$ and $\mathfrak{a}$ are ideals of $\mathfrak{a}(T_\Omega)$. By setting $\ell = \dim(\mathfrak{g}_{-1} \cap \mathfrak{a})$, the proof is now divided into three cases as follows:

Case (a). $\ell = 0$: By (2.3) we have $[\partial_i, \mathfrak{a}] = \{0\}$ for every $1 \le i \le n$. Thus, taking any element X of $\mathfrak{a}$ and writing X in the form (2.1), we have

$$\partial_i f_j(z) = 0 \quad \text{for } 1 \le i \le n.$$

This combined with the fact $\mathfrak{g}(T_\Omega) \cap \sqrt{-1}\mathfrak{g}(T_\Omega) = \{0\}$ yields that every $f_j(z)$ is a real constant. Accordingly, $X \in \mathfrak{g}_{-1}$ and $\mathfrak{a} \subset \mathfrak{g}_{-1}$, a contradiction.

Case (b). $0 < \ell < n$: Set $k = n - \ell$. Then, by a change of linear coordinates, if necessary, we may assume that

$$\mathfrak{g}_{-1} \cap \mathfrak{a} = \mathbf{R}\{\partial_{k+1}, \dots, \partial_n\},$$

where the set on the right-hand side is the subspace of $\mathfrak{a}(T_\Omega)$ spanned by ∂_j, $k+1 \le j \le n$, over $\mathbf{R}$. Thus

$$[\partial_i, \mathfrak{a}] \subset \mathbf{R}\{\partial_{k+1}, \dots, \partial_n\} \quad \text{for } 1 \le i \le n \tag{2.4}$$

by (2.3). Also, since $\mathfrak{a}$ is abelian, we have

$$[\partial_i, \mathfrak{a}] = \{0\} \quad \text{for } k+1 \le i \le n. \tag{2.5}$$

Take any element X of $\mathfrak{a}$ and write X in the form (2.1). It then follows from (2.4) and (2.5) that

$$\partial_i f_j(z) = 0 \quad \text{for } 1 \le i \le n,\ 1 \le j \le k; \quad \text{and}$$
$$\partial_i f_j(z) = 0 \quad \text{for } k+1 \le i \le n,\ 1 \le j \le n.$$

Hence, every function $f_j(z)$ for $1 \le j \le k$ reduces to a constant $\alpha_j \in \mathbf{C}$ and every function $f_j(z)$ for $k+1 \le j \le n$ is independent on $z_{k+1}, \dots, z_n$; accordingly, X has the form

$$X = \sum_{j=1}^{k} \alpha_j \partial_j + \sum_{j=k+1}^{n} f_j(z_1, \dots, z_k)\partial_j.$$

Moreover, the Structure Theorem says that $\alpha_j \in \mathbf{R}$ for every $1 \le j \le k$, and so

$$X' := \sum_{j=k+1}^{n} f_j(z_1, \dots, z_k)\partial_j \in \mathfrak{a}(T_\Omega).$$

Therefore, applying [7; Sect. 3, Lemma 6] to the complete holomorphic vector field X' on T_Ω, we obtain that every $f_j(z_1, \dots, z_k)$ also reduces to a real constant. Thus $X \in \mathfrak{g}_{-1}$ and $\mathfrak{a} \subset \mathfrak{g}_{-1}$, a contradiction.

Case (c). $\ell = n$: Since $\mathfrak{g}_{-1} \subset \mathfrak{a}$ in this case and since $\mathfrak{a}$ is abelian, we have $[\mathfrak{g}_{-1}, \mathfrak{a}] = \{0\}$. Thus, repeating the same argument as in the case (a), one can see that $\mathfrak{a} \subset \mathfrak{g}_{-1}$; a contradiction. Therefore the proof of the lemma is completed. □

Remark. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{a}(T_\Omega)$ containing $\mathfrak{g}_{-1}$ and let $\mathfrak{a}$ be an abelian ideal of $\mathfrak{h}$. Then, a glance at the proof of Lemma 2 tells us that $\mathfrak{a} \subset \mathfrak{g}_{-1}$. Therefore, $\mathfrak{g}_{-1}$ is also the unique maximal abelian ideal of $\mathfrak{h}$.

Proof of Theorem 1. For the sake of simplicity, we set $D = T_\Omega$ and $D' = T_{\Omega'}$ for the given Siegel domains T_Ω and $T_{\Omega'}$ in $\mathbf{C}^n$ and denote by $z = (z_1, \dots, z_n)$, $z' = (z'_1, \dots, z'_n)$ the coordinate systems of the ambiant spaces of D, D' respectively. Also we denote the objects relative to D' by the corresponding symbols of D with primes. So, for instance, ∂' stands for the vector field $\partial' = \sum_{j=1}^{n} z'_j \partial'_j$ on D'.

Now, assume that there exists a real analytic diffeomorphism $F : D \to D'$ satisfying the condition $FG(T_\Omega)F^{-1} = G(T_{\Omega'})$. Then the differential F_* of F induces a Lie algebra isomorphism $F_* : \mathfrak{g}(D) \to \mathfrak{g}(D')$. Let $\mathfrak{t}$ and $\mathfrak{t}'$, respectively, be the maximal triangular subalgebras of $\mathfrak{g}(D)$ and $\mathfrak{g}(D')$ satisfying the conditions in (2) of Lemma 1. Then, by the assertion (1) of Lemma 1 there exists an element $\sigma \in G(D')$ such that $\sigma_*(F_*\mathfrak{t}) = \mathfrak{t}'$. Therefore, replacing F by $\sigma \cdot F$ if necessary, we may assume that $F_*\mathfrak{t} = \mathfrak{t}'$. Then $F_*\mathfrak{g}_{-1}$ is now a maximal abelian ideal of $\mathfrak{t}'$; and consequently, $F_*\mathfrak{g}_{-1} = \mathfrak{g}'_{-1}$ by Lemma 2.

Express the diffeomorphism $F : D \to D'$ as

$$F(z) = U(x, y) + \sqrt{-1}V(x, y), \quad z = x + \sqrt{-1}y \in D,$$

where $U : D \to \mathbf{R}^n$ and $V : D \to \Omega'$ are real analytic mappings defined on D. Since $F_*\mathfrak{g}_{-1} = \mathfrak{g}'_{-1}$ as was shown above, there exists an element $P \in GL(n, \mathbf{R})$

such that

$$F(z+a) = F(z) + Pa \quad \text{for all } z \in D,\ a \in \mathbf{R}^n$$

(think of a as column vectors), or equivalently

$$U(x+a, y) = U(x, y) + Pa, \quad V(x+a, y) = V(x, y)$$

for all $x, a \in \mathbf{R}^n$ and all $y \in \Omega$. Obviously, these equalities imply that

(2.6) $U(x, y)$ can be written in the form $U(x, y) = Px + U(y)$, where $U : \Omega \to \mathbf{R}^n$ is a real analytic mapping;

and

(2.7) $V(x, y)$ is independent on the variable x; and hence, it induces a real analytic diffeomorphism, say again, $V : \Omega \to \Omega'$.

Next we wish to prove that this $V : \Omega \to \Omega'$ satisfies the condition $(*)$ in the theorem. To this end, put $\hat{\Omega} = P^{-1}(\Omega')$, the image cone of Ω' under the non-singular linear transformation of $\mathbf{R}^n$ induced by $P^{-1} \in GL(n, \mathbf{R})$, and consider the diffeomorphism $\hat{V} : \Omega \to \hat{\Omega}$ defined by $\hat{V}(y) = P^{-1}V(y)$ for $y \in \Omega$. Once we have shown that $\hat{V}(hy) = h\hat{V}(y)$ for all $h \in G(\Omega)$ and all $y \in \Omega$, it is clear that $V : \Omega \to \Omega'$ satisfies the condition $(*)$.

Thus, taking $P^{-1}F$ instead of F, if necessary, we may assume that $P = I_n$, the identity matrix of degree n. Accordingly, we have

$$F_*(\partial_i) = \partial'_i \quad \text{for every } 1 \le i \le n.$$

Set $\tilde{\partial} = F_*(\partial)$. Then, since $\partial \in \mathfrak{t}$ and $F_*(\mathfrak{t}) = \mathfrak{t}' \subset \mathfrak{a}(D')$, $\tilde{\partial}$ can be written in the form $\tilde{\partial} = a_o + X_o$, where $a_o \in \mathfrak{g}'_{-1}$ and $X_o \in \mathfrak{g}'_0$. Moreover, since $[\partial_i, \partial] = \partial_i$ for every $1 \le i \le n$, a simple computation shows that $X_o = \partial'$; so that $\tilde{\partial} = a_o + \partial'$.

Here let us define the vector subspaces $\hat{\mathfrak{g}}'_0$, $\tilde{\mathfrak{g}}'_0$ of $\mathfrak{g}(D')$ by setting

$$\hat{\mathfrak{g}}'_0 = \{Y \in \mathfrak{g}(D') \mid [\tilde{\partial}, Y] = 0\}, \quad \tilde{\mathfrak{g}}'_0 = \{[a_o, X] + X \mid X \in \mathfrak{g}'_0\}.$$

Then it can be seen that $F_*\mathfrak{g}_0 = \hat{\mathfrak{g}}'_0$, $\tilde{\mathfrak{g}}'_0 \subset \hat{\mathfrak{g}}'_0$ and $\tilde{\mathfrak{g}}'_0$ is linearly isomorphic to $\mathfrak{g}'_0$ via the natural correspondence. Thus

$$\dim \mathfrak{g}_0 = \dim F_*(\mathfrak{g}_0) = \dim \hat{\mathfrak{g}}'_0 \ge \dim \tilde{\mathfrak{g}}'_0 = \dim \mathfrak{g}'_0.$$

Interchanging the role of F and F^{-1} in the argument above, we obtain the reverse inequality: $\dim \mathfrak{g}'_0 \ge \dim \mathfrak{g}_0$. Therefore

$$F_*\mathfrak{g}_0 = \hat{\mathfrak{g}}'_0 = \tilde{\mathfrak{g}}'_0 \quad \text{and} \quad F_*(Y) = [a_o, X] + X \quad \text{for every element } Y \in \mathfrak{g}_0,$$

where X is an element of $\mathfrak{g}'_0$ uniquely determined by the given Y. More precisely, by considering the one-parameter groups generated by the complete holomorphic vector fields $Y \in \mathfrak{g}_0$, $X \in \mathfrak{g}'_0$ and $[a_o, X] \in \mathfrak{g}'_{-1}$, respectively, it is easily checked that $X = V_*(Y)$ under the canonical identification $\mathfrak{g}_0$ (resp. $\mathfrak{g}'_0$) with the Lie algebra of $GL(\Omega)$ (resp. $GL(\Omega')$). As a result, we have shown that the induced isomorphism $F_* : \mathfrak{g}_0 \to \tilde{\mathfrak{g}}'_0$ is given by

$$F_*(Y) = [a_o, V_*(Y)] + V_*(Y) \quad \text{for } Y \in \mathfrak{g}_0$$

and, in particular, $F_*\mathfrak{a}(T_\Omega) = \mathfrak{a}(T_{\Omega'})$. On the other hand, we here assert that

$$(2.8) \qquad V_*(Y) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{kj}z_j'\right)\partial_k' \quad \text{for every } Y = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} a_{kj}z_j\right)\partial_k \in \mathfrak{g}_0.$$

Indeed, since $V_*(Y) \in \mathfrak{g}_0'$, it can be expressed as

$$V_*(Y) = \sum_{k=1}^{n}\left(\sum_{j=1}^{n} b_{kj}z_j'\right)\partial_k' \quad \text{with } b_{kj} \in \mathbf{R} \text{ for all } k, j.$$

Since $[\partial_\ell, Y] = \sum_{k=1}^{n} a_{k\ell}\partial_k \quad (1 \leq \ell \leq n)$ and $[a_o, V_*(Y)] \in \mathfrak{g}_{-1}'$, it then follows that

$$\sum_{k=1}^{n} a_{k\ell}\partial_k' = F_*([\partial_\ell, Y]) = [\partial_\ell', V_*(Y)] = \sum_{k=1}^{n} b_{k\ell}\partial_k'$$

for every $1 \leq \ell \leq n$; consequently, $a_{k\ell} = b_{k\ell}$ for all k, ℓ, as asserted. Clearly the assertion (2.8) guarantees us that $V(y)$ satisfies the condition $(*)$ in Theorem 1; thereby completing the proof of Theorem 1. □

Proof of Corollary. Assume that there exists a diffeomorphism $V : \Omega \to \Omega'$ of the form $V(y) = Py$, $y \in \Omega$, where $P \in GL(n, \mathbf{R})$. Then it is clear that the non-singular linear transformation $L(z) := Pz$ of $\mathbf{C}^n$ gives a biholomorphic equivalence between T_Ω and $T_{\Omega'}$.

Conversely, assume that there exists a biholomorphic mapping $F : T_\Omega \to T_{\Omega'}$. Of course, this F satisfies the condition $FG(T_\Omega)F^{-1} = G(T_{\Omega'})$ required in Theorem 1. Hence, by the same reasoning as in the proof of Theorem 1, we may assume that $F(z)$ has the form described in Theorem 1. Then, since F is holomorphic, it follows from the Cauchy-Riemann equations that $V(y)$ has to be of the form $V(y) = Py + y_o$, where y_o is a constant vector in $\mathbf{R}^n$. Moreover, noting the fact that the dilations $\delta_r(y) = ry$ $(r > 0)$ in $\mathbf{R}^n$ belong to $G(\Omega)$, we conclude by $(*)$ in Theorem 1 that $y_o = 0$; and hence, $V(y) = Py$ on Ω. This completes the proof of Corollary. □

Proof of Theorem 2. We retain the notation in the proof of Theorem 1. So D and D' denote the Siegel domains T_Ω and $T_{\Omega'}$, respectively. Also we set $G = G(D)$ and $G' = G(D')$, for simplicity.

Now, assume that there exists a topological group isomorphism $\Phi : G \to G'$ as in Theorem 2. Then, since both the groups G and G' are Lie groups, Φ is necessarily a Lie group isomorphism. We denote by $d\Phi : \mathfrak{g}(D) \to \mathfrak{g}(D')$ the Lie algebra isomorphism induced by Φ.

Choose two points $p \in D$ and $p' \in D'$ arbitrarily and denote by K and K' the isotropy subgroups of G and G' at p and p', respectively. Without loss of generality, we may assume that D is homogeneous; so that D can be represented as the coset space $D = G/K$ and K is a maximal compact subgroup of G. On the other hand, K' is a compact subgroup of G'. Under these situations, we first show that D' is also homogeneous. For this purpose, choose a maximal compact subgroup $\hat{K}$ of G' containing K'. Then, since any two maximal compact subgroups of a connected Lie group are always conjugate under an inner automorphism, we can find an element $\sigma \in G'$ such that $\hat{K} = \sigma\Phi(K)\sigma^{-1}$. Moreover, notice that the orbit $G' \cdot p' = G'/K'$ of G' passing through p' is a real analytic submanifold of D'. Thus

$$2n \geq \dim G'/K' \geq \dim G'/\hat{K} = \dim G/K = 2n,$$

from which we have $\dim G'/K' = 2n$ and $K' = \hat{K}$; hence, the orbit $G' \cdot p' = G'/K'$ is open in D'. Of course, the same is true for any point $q' \in D'$ with $q' \neq p'$. Consequently, the connectivity of D' implies that $D' = G'/K'$, that is, D' is also homogeneous, as desired.

Replacing Φ by the isomorphism $\sigma\Phi(\cdot)\sigma^{-1}$ if necessary, we may now assume that $K' = \Phi(K)$. Hence we can define a real analytic diffeomorphism $F : D = G/K \to G'/K' = D'$ by setting

$$F(gK) = \Phi(g)K' \quad \text{for all } g \in G.$$

Clearly this F satisfies the condition

$$F(g \cdot z) = \Phi(g) \cdot F(z) \quad \text{for all } g \in G,\ z \in D;$$

accordingly, we have $FG(D)F^{-1} = G(D')$ and $F_* = d\Phi$. Therefore, by the same reasoning as in the proof of Theorem 1, we may assume that

(2.9) $$F_*\mathfrak{a}(D) = \mathfrak{a}(D'), \quad F_*(\mathfrak{t}) = \mathfrak{t}';$$

and

(2.10) F has the form $F(z) = Px + U(y) + \sqrt{-1}V(y)$ as in Theorem 1.

Under these assumptions, our next task is to show that there exists the relationship (#) between Φ and F. We first verify this for every element h of the subgroup $G(\Omega)$ of $A(D)$, that is, for $g = (h, 0) \in G(\Omega) \times \mathbf{R}^n = A(D)$. To this end, notice that $\Phi(A(D)) = A(D')$ by (2.9). Thus we can write

$$\Phi((h,0)) = (\chi(h), \psi(h)) \quad \text{for every } h \in G(\Omega),$$

where $\chi : G(\Omega) \to G(\Omega')$ and $\psi : G(\Omega) \to \mathbf{R}^n$ are real analytic mappings. From the relation $F((h,0) \cdot z) = \Phi((h,0)) \cdot F(z)$ and the action rule of $A(D)$ (resp. $A(D')$) on D (resp. D') given in (1.2), it then follows that

$$Phx + U(hy) = \chi(h)(Px + U(y)) + \psi(h), \quad V(hy) = \chi(h)V(y).$$

These combined with (*) in Theorem 1 yield at once that

$$\chi(h) = PhP^{-1}, \quad U(hy) = PhP^{-1}U(y) + \psi(h) \quad \text{for all } h \in G(\Omega),\ y \in \Omega.$$

We now wish to prove that there exists a unique element $\mathfrak{a}_o \in \mathbf{R}^n$ such that

(2.11) $$\psi(h) = (I_n - PhP^{-1})\mathfrak{a}_o \quad \text{for all } h \in G(\Omega).$$

To prove this, recall the multiplication rule (1.1). Then

$$(2.12) \qquad \psi(h_1h_2) = Ph_1P^{-1}\psi(h_2) + \psi(h_1) \quad \text{for all } h_1, h_2 \in G(\Omega).$$

Take an arbitrary element $r \in \mathbf{R}$ with $0 < r < 1$ and consider the dilation $\delta_r(y) = ry$ in $\mathbf{R}^n$. Since δ_r belongs to the center of $G(\Omega)$, it then follows from (2.12) that

$$\psi(h) = (1-r)^{-1}(I_n - PhP^{-1})\psi(\delta_r) \quad \text{for all } h \in G(\Omega).$$

In particular, we have

$$\psi((\delta_r)^k) = (1-r)^{-1}(1-r^k)\psi(\delta_r) \quad \text{for all } k = 1, 2, \ldots.$$

Since $r^k \to 0$ as $k \to \infty$, this implies the existence of an element $\mathfrak{a}_o \in \mathbf{R}^n$ such that $(1-r)^{-1}\psi(\delta_r) = \mathfrak{a}_o$. Obviously, such an element $\mathfrak{a}_o$ is independent on the choice of $0 < r < 1$ and satisfies (2.11). Summarizing the above, we obtain the following:

$$(2.13) \qquad F((h,0)\cdot z) = PhP^{-1}F(z) + (I_n - PhP^{-1})\mathfrak{a}_o \quad \text{for } h \in G(\Omega),\ z \in D.$$

Finally, take an arbitrary element $g = (h, a) \in A(D)$. Then, since $g \cdot z = (h,0)\cdot z + a$, $F(z+a) = F(z) + Pa$ on D, our relationship (#) is now an immediate consequence of (2.13). Therefore the proof of Theorem 2 is completed. □

3. Some comments

In connection with our fundamental question in this paper, we shall study the linear equivalence problem for homogeneous convex cones in $\mathbf{R}^n$.

Throughout this section, we always assume that T_Ω and $T_{\Omega'}$ are the homogeneous Siegel domains of the first kind associated to convex cones Ω and Ω' in $\mathbf{R}^n$, respectively. Therefore the linear automorphism groups $G(\Omega)$ and $G(\Omega')$ act transitively on Ω and on Ω' (cf. [2]).

Now assume that there exists a topological group isomorphism $\Phi : G(T_\Omega) \to G(T_{\Omega'})$. It then follows from Theorems 1 and 2 that there exists a real analytic diffeomorphism $V : \Omega \to \Omega'$ satisfying the condition (*): $V(hy) = PhP^{-1}V(y)$ for all $h \in G(\Omega)$ and all $y \in \Omega$. Once it is shown that this $V : \Omega \to \Omega'$ is the restriction to Ω of some linear transformation of $\mathbf{R}^n$ or there exists a non-singular linear transformation $\ell : \mathbf{R}^n \to \mathbf{R}^n$ such that $\ell(\Omega) = P^{-1}(\Omega')$, T_Ω is necessarily biholomorphic to $T_{\Omega'}$. Taking this into account, we would like to clear up the following:

Question. Under the assumption that there exists a real anlytic diffeomorphism $V : \Omega \to \Omega'$ satisfying the condition (*), does there exist a non-singular linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ such that $L(\Omega) = \Omega'$?

Although this cannot be achieved in full generality at this moment, we first give the following Fact 1. And, after that, we will come back to this question.

In order to state Fact 1, we need a preparation. Let Λ be any homogeneous convex cone in $\mathbf{R}^n$. Then it is known that there is a $G(\Lambda)$-invariant Riemannian metric g_Λ on Λ that is naturally constructed by means of its characteristic function φ_Λ (cf. [9]). This metric is called the *canonical metric* on Λ and the characteristic function φ_Λ has the property:

$$\varphi_\Lambda(hy) = \varphi_\Lambda(y)/\det(h) \quad \text{for all } h \in G(\Lambda),\ y \in \Lambda. \tag{3.1}$$

In these notations, we have the following:

FACT 1. *The diffeomorphism $V : \Omega \to \Omega'$ is an isometry with respect to the canonical metrics g_Ω and $g_{\Omega'}$.*

Indeed, by (3.1) and (*) we have

$$\varphi_{\Omega'}(V(hy))/\varphi_\Omega(hy) = \varphi_{\Omega'}(V(y))/\varphi_\Omega(y) \quad \text{for all } h \in G(\Omega),\ y \in \Omega.$$

Hence the homogeniety of Ω tells us that $\varphi_{\Omega'}(V(y))$ coincides with $\varphi_\Omega(y)$, up to a positive constant factor. From this, we can check easily that $V : (\Omega, g_\Omega) \to (\Omega', g_{\Omega'})$ is, in fact, an isometry, as desired. □

Let us return to the question above. Since Ω and Ω' are homogeneous convex cones in $\mathbf{R}^n$, there exist maximal triangular subgroups T and T' of $G(\Omega)$ and of $G(\Omega')$ that act simply transitively on Ω and on Ω', respectively (cf. [9]). Let $\mathfrak{t}$ be the Lie algebra of T and choose a basis $\{X_1, \dots, X_n\}$ for $\mathfrak{t}$. Without loss of generality, we may assume that every element of T can be represented as an upper triangular matrix with respect to the coordinate system $(y_1, \dots, y_n)$ of the ambiant space $\mathbf{R}^n$ of Ω. Thus the complete vector field X_j on Ω has the form

$$X_j = \sum_{k=1}^{n} p_{jk}(y) \frac{\partial}{\partial y_k} \quad \text{for } 1 \le j \le n, \tag{3.2}$$

where $p_{jk}(y)$ are homogeneous polynomials in $y_k, y_{k+1}, \dots, y_n$ of degree one with real coefficients for $1 \le j \le n$, $1 \le k \le n$.

Consider now the function $P(y)$ on $\mathbf{R}^n$ and the subset $\mathscr{A}$ of $\mathbf{R}^n$ defined by

$$P(y) = \det \begin{pmatrix} p_{11}(y) & p_{12}(y) & \cdots & p_{1n}(y) \\ p_{21}(y) & p_{22}(y) & \cdots & p_{2n}(y) \\ \vdots & \vdots & & \vdots \\ p_{n1}(y) & p_{n2}(y) & \cdots & p_{nn}(y) \end{pmatrix} \quad \text{for } y \in \mathbf{R}^n \tag{3.3}$$

and

$$\mathscr{A} = \{ y \in \mathbf{R}^n \mid P(y) = 0 \}.$$

Note that, for any point $y_o \in \mathbf{R}^n$, the orbit $T \cdot y_o$ of T passing through y_o is a real analytic submanifold of $\mathbf{R}^n$ of dimension $\le n$. Then $\mathscr{A}$ is nothing but the set consisting of all points $y_o \in \mathbf{R}^n$ with $\dim T \cdot y_o < n$. In particular, $\mathscr{A}$ and its complement $\mathscr{X} := \mathbf{R}^n \backslash \mathscr{A}$ in $\mathbf{R}^n$ are invariant under the T-action on $\mathbf{R}^n$. Moreover, since $P(y)$ is a real homogeneous polynomial of degree n by (3.2) and (3.3), $\mathscr{A}$ is a real algebraic subset of $\mathbf{R}^n$ invariant under the symmetry $\sigma_0 : y \mapsto -y$ in $\mathbf{R}^n$ at the origin o. Notice that $P(y)$ can be expressed as $P(y) = y_n Q(y)$, where $Q(y)$ is a real homogeneous polynomial of degree $n-1$. Thus, the coordinate hyperplane $\{y_n = 0\}$ is contained in $\mathscr{A}$.

Now, as in the proof of Theorem 1, we put $\hat{\Omega} = P^{-1}(\Omega')$ and consider the diffeomorphism $\hat{V} : \Omega \to \hat{\Omega}$ defined by $\hat{V}(y) = P^{-1}V(y)$ for $y \in \Omega$. It then follows from the assertion (*) that

$$\hat{V}(hy) = h\hat{V}(y) \quad \text{for all } h \in T,\ y \in \Omega. \tag{3.4}$$

Therefore, T acts simply transitively on the convex cone $\hat{\Omega}$ as well as on Ω. Under this situation, our question above can be restated as follows:

Does there exist a non-singular linear transformation
$L : \mathbf{R}^n \to \mathbf{R}^n$ satisfying $L(\Omega) = \hat{\Omega}$?

To investigate this, it should be remarked that both the cones Ω and $\hat{\Omega}$ are connected components of the complement $\mathscr{X}$ of $\mathscr{A}$ in $\mathbf{R}^n$ and that T acts transitively on each connected component of $\mathscr{X}$. Taking this into account, we define a real algebraic subgroup A of $GL(n, \mathbf{R})$ by

$$A = \{h \in GL(n, \mathbf{R}) \mid h(\mathscr{A}) = \mathscr{A}\}.$$

Then A acts on the open subset $\mathscr{X}$ of $\mathbf{R}^n$ as a Lie transformation group and the identity component A^o of A leaves each connected component of $\mathscr{X}$ invariant. Therefore, T is an algebraic subgroup of A^o and hence A^o also acts transitively on each connected component of $\mathscr{X}$. Observe here that the quotient group $\hat{A} := A/A^o$ acts naturally on the set $\mathfrak{X}$ consisting of all connected components of $\mathscr{X}$. Thus, if this $\hat{A}$-action on $\mathfrak{X}$ is transitive, one may conclude that Ω is linearly equivalent to $\hat{\Omega}$. It is not clear whether this is true or not, in general. However, at least in the case where $n \leq 3$, we can give an affirmative answer to the question as follows:

FACT 2. *Let Ω and $\hat{\Omega}$ be homogeneous convex cones in $\mathbf{R}^n$ with $n \leq 3$ and assume that there exists a real analytic diffeomorphism $\hat{V} : \Omega \to \hat{\Omega}$ satisfying the condition* (3.4). *Then there exists a non-singular linear transformation $L : \mathbf{R}^n \to \mathbf{R}^n$ such that $L(\Omega) = \hat{\Omega}$.*

Since this is clear when $n \leq 2$, we consider the case of $n = 3$. Recall that the real algebraic set $\mathscr{A}$ is invariant under the symmetry $\sigma_0(y) = -y$ in $\mathbf{R}^3$ and the coordinate hyperplane $\{y_3 = 0\}$ is contained in $\mathscr{A}$. Thus, in order to show Fact 2, we may assume that both the cones Ω and $\hat{\Omega}$ are contained in the half space $\{y_3 > 0\}$. Now, put

$$P = \{(y_1, y_2, y_3) \in \Omega \mid y_3 = 1\} \quad \text{and} \quad \hat{P} = \{(y_1, y_2, y_3) \in \hat{\Omega} \mid y_3 = 1\},$$

the cross-sections of Ω and $\hat{\Omega}$ by the hyperplane $\{y_3 = 1\}$, and consider the subgroup T_1 of T consisting of all elements h having the form

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ 0 & h_{22} & h_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

Then P and $\hat{P}$ are convex domains in the hyperplane $\{y_3 = 1\}$ and clearly they contain no entire straight line. Moreover, T_1 is a two-dimensional Lie group acting simply transitively on P as well as on $\hat{P}$ as an affine transformation group. On the other hand, it is known (cf. [9]) that any two-dimensional affine homogeneous convex domain not completely containing any straight line is affinely equivalent to either

$$\mathbf{R}_+^2 := \{(x, y) \in \mathbf{R}^2 \mid x > 0, y > 0\} \quad \text{or} \quad \mathfrak{C} := \{(x, y) \in \mathbf{R}^2 \mid y > x^2\}$$

and further any maximal triangular group acting simply transitively on $\mathbf{R}_+^2$ is abelian, while that on $\mathfrak{C}$ is not. Consequently, P and $\hat{P}$ have to be affinely equivalent, that is, there exists an affine transformation $\eta : \{y_3 = 1\} \to \{y_3 = 1\}$ such that $\eta(P) = \hat{P}$. Making use of this η, we can now define a mapping $L : \Omega \to \hat{\Omega}$ by the following:

$$L(y) = y_3 \cdot \eta(y_1/y_3, y_2/y_3, 1) \quad \text{for } y = (y_1, y_2, y_3) \in \Omega.$$

Then it is an easy matter to see that L gives a linear equivalence between Ω and $\hat{\Omega}$; thereby completing the proof of Fact 2. □

As an immediate consequence of Fact 2, we obtain the following:

FACT 3. *Let T_Ω and $T_{\Omega'}$ be the homogeneous Siegel domains of the first kind associated to convex cones Ω and Ω' in $\mathbf{R}^n$ with $n \leq 3$. Then T_Ω is biholomorphically equivalent to $T_{\Omega'}$ if and only if there exists a topological group isomorphism $\Phi : G(T_\Omega) \to G(T_{\Omega'})$.*

Indeed, if there exists a topological group isomorphism $\Phi : G(T_\Omega) \to G(T_{\Omega'})$, then by the proof of Theorem 2 there exists a real analytic diffeomorphism $\hat{V} : \Omega \to \hat{\Omega}$ satisfying the condition (3.4). Hence Ω and Ω' are linearly equivalent by Fact 2; and accordingly, T_Ω and $T_{\Omega'}$ are biholomorphically equivalent. The converse is obvious; completing the proof of Fact 3. □

Finally it should be remarked that Fact 3 also comes from our previous result stated in the introduction. In fact, it is well-known that every homogeneous bounded domain in $\mathbf{C}^n$ with $n \leq 3$ is symmetric by a result of E. Cartan [1]; and hence, our previous result [4; Corollary 2] also implies Fact 3.

References

[1] É. Cartan, Sur les domaines bornés homogènes de l'espace de n variables complexes, Abh. Math. Sem. Univ. Hamburg **11** (1935), 116–162.

[2] W. Kaup, Y. Matsushima and T. Ochiai, On the automorphisms and equivalences of generalized Siegel domains, Amer. J. Math. **92** (1970), 475–497.

[3] S. Kobayashi, Hyperbolic complex spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[4] A. Kodama and S. Shimizu, Addendum to our characterization of the unit polydisc, Kodai Math. J. **33** (2010), 182–191.

[5] Y. Matsushima, On tube domains, Symmetric Spaces (W. Boothby and G. Weiss, eds.), Marcel Dekker, New York, 1972, 255–270.

[6] I. I. Pyatetskii-Shapiro, Geometry of classical domains and theory of automorphic functions, Fizmatgiz, Moscow, 1961, Gordon and Breach, New York, 1969 (English transl.).

[7] S. Shimizu, Automorphisms of tube domains, Geometry and analysis on complex manifolds, Festschrift for Professor S. Kobayashi's 60th Birthday (T. Mabuchi, J. Noguchi and T. Ochiai, eds.), World Scientific, Singapore, New Jersey, London, Hong Kong, 1994, 198–241.

[8] S. Shimizu, Automorphism and equivalence of tube domains with bounded base, Math. Ann. **315** (1999), 295–320.

[9] È. B. Vinberg, The theory of convex homogeneous cones, Trans. Moscow Math. Soc. **12** (1963), 340–403.

Akio Kodama
Faculty of Mathematics and Physics
Institute of Science and Engineering
Kanazawa University
Kanazawa, 920-1192
Japan
E-mail: kodama@se.kanazawa-u.ac.jp

Satoru Shimizu
Mathematical Institute
Tohoku University
Sendai, 980-8578
Japan
E-mail: shimizu@math.tohoku.ac.jp