On the number of discrete eigenvalues of a discrete Schrödinger operator with a finitely supported potential

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Abstract

On the d-dimensional lattice $\mathbb{Z}^d$ and the r-regular tree T^r, an exact expression for the number of discrete eigenvalues of a discrete Laplacian with a finitely supported potential is described in terms of the support and the intensities of the potential on each case. In particular, the number of eigenvalues less than the infimum of the essential spectrum is bounded by the number of negative intensities.

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1 Introduction and Main Results

Let $\mathcal{G}$ be the d-dimensional lattice $\mathbb{Z}^d$ or the r-regular tree T^r $(r \geq 3)$; we denote the vertex set of $\mathcal{G}$ by the same symbol $\mathcal{G}$. Note that the r-regular tree T^r is often called the Bethe lattice, which is an infinite connected cycle-free graph where each vertex has r neighbours and that T^2 actually coincides with $\mathbb{Z}^1$.

Discrete Schrödinger operator $\mathcal{H}$ on graph $\mathcal{G}$ is defined in the Hilbert space $\ell^2(\mathcal{G})$ of square summable functions $\psi(x)$ of discrete variable $x \in \mathcal{G}$ by means of

$$\mathcal{H}\psi(x) \equiv -\Delta\psi(x) + V(x)\psi(x),$$

where discrete Laplace operator Δ is defined by the following

$$\Delta\psi(x) = \sum_{|y-x|=1} [\psi(y) - \psi(x)].$$

Here, $|y-x|$ stands for the graph distance between two vertices y and x of $\mathcal{G}$. We suppose in this paper that potential $V(x)$ is a real function different from zero only in finite number n of points $\{x_1, \dots, x_n\} \equiv \mathrm{supp} V$, $V(x_i) \neq 0$. In this case $\mathcal{H}$ is a self-adjoint operator and its spectrum consists of essential spectrum part of an interval $\mathcal{I}$ ($\mathcal{I} = [0, 4d]$ for $\mathbb{Z}^d$, $\mathcal{I} = [r - 2\sqrt{r-1}, r + 2\sqrt{r-1}]$ for T^r) and finite number of discrete eigenvalues outside $\mathcal{I}$. The main purpose of this paper is to obtain complete description of the discrete eigenvalues of $\mathcal{H}$.

For convenience instead of the Laplace operator Δ we consider the transition operator P on $\ell^2(\mathcal{G})$ defined by

$$Pf(x) = \frac{1}{\deg x} \sum_{|y-x|=1} f(y),$$

where $\deg x = |\{y \in \mathcal{G}; |y-x| = 1\}|$. Let us denote

$$V(x) = \sum_{i=1}^{n} \alpha_i \delta_i(x),$$

where $\alpha_i \in \mathbb{R} \setminus \{0\}$ is the intensity of V at x_i. The function $\delta_i(x) = \delta_{x_i,x}$ denotes the Kronecker's delta with $x_i \in \mathcal{G}$. We always assume that $x_i \neq x_j$ $(i \neq j)$.

In this paper, we study the spectrum of the perturbed operator $L = -P + V$ on $\ell^2(\mathcal{G})$. We denote the spectrum and essential spectrum of $-P$ by $\sigma(-P)$ and $\sigma_e(-P)$, respectively. Using the spectrum $\mathcal{I}$ of $-\Delta$ or by direct computations, we can easily see that $\sigma(-P) = \sigma_e(-P) = [-\lambda_0, \lambda_0]$ with $\lambda_0 = 1$ if $\mathcal{G} = \mathbb{Z}^d$ and $\lambda_0 = 2\sqrt{r-1}/r$ if $\mathcal{G} = T^r$. Since $\sigma_e(-P)$ is stable under finite rank perturbation, we will discuss the discrete eigenvalues of L. Now, let us denote by $N_\pm(L)$ the numbers of the discrete eigenvalues of L, taking into account the multiplicity, which are greater than λ_0 and smaller than $-\lambda_0$, respectively.

Recently, one of the authors discussed the eigenvalues on T^r with a certain class of potentials in [3]. Two of the authors gave explicit expressions for $N_\pm(L)$ on $\mathbb{Z}^1$ in [5]. H. Isozaki and H. Morioka proved that L has no eigenvalue in $(-1, 1)$ on $\mathbb{Z}^d$ in [7]. If $n = 1$, F. Hiroshima, et. al. in [6] also discussed the eigenvalues of L on $\mathbb{Z}^d$.

The main purpose of this paper is to describe $N_\pm(L)$ for any n, d or r, whose concrete forms are given in Theorems 1.1, 1.2 and 1.3. For further exposition, we introduce the following notations. For a given potential V, put $m_+ = |\{j;\, \alpha_j > 0\}|$, $m_- = |\{j;\, \alpha_j < 0\}|$, and an $n \times n$-matrix

$$M = A^{-1} + (g_{i,j})_{i,j=1}^n,$$

where A is the diagonal matrix with its diagonal entries, $\alpha_1, \alpha_2, \dots, \alpha_n$, and $g_{i,j}$ are stated in the following three theorems. Furthermore, let J be the $n \times n$ all-one matrix, $Q = I - \frac{1}{n}J$ for the unit matrix I, $\mathbb{T}^d = [0, 2\pi]^d$ be the d-dimensional torus, and $x \cdot y$ be the standard inner product of two vectors x and y in $\mathbb{R}^d$. Denote by $p(S)$ the number of positive eigenvalues of a real symmetric matrix S.

Theorem 1.1. *Let $\mathcal{G}$ be $\mathbb{Z}^1$ or $\mathbb{Z}^2$. Then, we have that $N_-(L) = p(QMQ) + 1 - m_+$ with*

$$g_{i,j} = \frac{-1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos((x_i - x_j) \cdot \theta) - 1}{\frac{1}{d}\sum_{k=1}^d \cos\theta_k - 1}\, d\theta.$$

In particular, $N_-(L) \le m_-$, and $N_-(L) = m_-$ if and only if $QMQ\lfloor_{\ker Q^\perp}$ is positive definite.

Theorem 1.2. *Let $\mathcal{G}$ be $\mathbb{Z}^d$ for $d \ge 3$. Then, we have that $N_-(L) = p(M) - m_+$ with*

$$g_{i,j} = \frac{-1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos((x_i - x_j) \cdot \theta)}{\frac{1}{d}\sum_{k=1}^d \cos\theta_k - 1}\, d\theta.$$

In particular, $N_-(L) \le m_-$, and $N_-(L) = m_-$ if and only if M is positive definite.

Theorem 1.3. *Let $\mathcal{G}$ be T^r for $r \ge 3$. Then, we have that $N_-(L) = p(M) - m_+$ with*

$$g_{i,j} = \frac{r\sqrt{r-1}}{r-2} \left(\frac{1}{\sqrt{r-1}} \right)^{|x_i - x_j|}.$$

In particular, $N_-(L) \leq m_-$, and $N_-(L) = m_-$ if and only if M is positive definite.

Moreover, Theorem 2.5 in Section 2 gives us explicit expressions for the eigenfunctions of L. Here our interests concentrate only on $N_-(L)$ in the above, but we can immediately translate them into those for $N_+(L)$. Indeed, we can easily see that $U^*(-P+V)U = -(-P-V)$ using the unitary operator U on $\ell^2(\mathcal{G})$ defined by $Uf(x) = (-1)^{|x-y|}f(x)$ with a fixed vertex $y \in \mathcal{G}$.

Finally in this section, we emphasize that L is a discrete analogue of the point-interaction Hamiltonian H on $\mathbb{R}^d$, which is discussed in [1, 2, 9, 10] and in a lot of literature; H is a Schrödinger operator with pseudo-potentials $\sum_{i=1}^n \alpha_i \delta(x - x_i)$ on $\mathbb{R}^d$. Here, δ is the Dirac's delta function.

The paper is organized as follows: In Section 2, we discuss the Green function of L, the auxiliary matrix $\Gamma(\lambda)$ and its eigenvalues to give proofs of our main theorems. In Section 3, we actually prove Theorems 1.1, 1.2 and 1.3. In Section 4, we give a discrete analogue of Albeverio and Nizhnik's algorithm [2] to obtain $N_-(L)$ for $d = 1$. In Section 5, we illustrate several examples and some discussions.

2 Matrix $\Gamma(\lambda)$ and its eigenvalues

First we begin by recalling some results about Green function; the Green function $G(x, y; z)$ of an operator T on $\ell^2(\mathcal{G})$ is the kernel of the resolvent operator $(T - z)^{-1}$ for z in the resolvent set $\rho(T)$ defined by

$$(T - z)^{-1} f(x) = \sum_{y \in \mathcal{G}} G(x, y; z) f(y)$$

for all $f \in \ell^2(\mathcal{G})$. Direct computations can give us the following two expressions.

Theorem 2.1. *Let $\mathcal{G} = \mathbb{Z}^d$, $G_0(x, y; z)$ be the Green function of $-P$ and $z \in \rho(-P) = \mathbb{C} \setminus [-1, 1]$. Then, we have that*

$$\begin{aligned} G_0(x, y; z) &= \frac{-1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e^{-\sqrt{-1}(x-y)\cdot\theta}}{\frac{1}{d}\sum_{i=1}^d \cos\theta_i + z}\, d\theta \\ &= \frac{-1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos((x-y)\cdot\theta)}{\frac{1}{d}\sum_{i=1}^d \cos\theta_i + z}\, d\theta. \end{aligned}$$

THEOREM 2.2. *Let $\mathcal{G} = T^r$, $G_0(x,y;z)$ be the Green function of $-P$, $z \in \rho(-P) = \mathbb{C} \setminus [-2\sqrt{r-1}/r, 2\sqrt{r-1}/r]$, and $p = \cos^{-1}(-z\,r/2\sqrt{r-1})$ with $-\pi \le \operatorname{Re} p < \pi$ and $\operatorname{Im} p > 0$. Then, we have that*

$$G_0(x,y;z) = \frac{r}{g^{-1} - g}\, g^{|x-y|} \quad \textit{with} \quad g = \frac{e^{ip}}{\sqrt{r-1}}.$$

It is well-known that the integral $G_0(x,y;-1)$ in Theorem 2.1 diverges for $d = 1$, 2 and converges for $d \ge 3$.

REMARK 2.3. Let $d = 1$. Since $g_{i,j} = \lim_{\lambda \to -1-0}[G_0(x_i, x_j; \lambda) - G_0(0,0;\lambda)]$ (see Session 3) and $G_0(x,y;z) = \sqrt{-1}e^{\sqrt{-1}p|x-y|}/\sin p$ for $p = \cos^{-1}(-z)$ with $-\pi \le \operatorname{Re} p < \pi$ and $\operatorname{Im} p > 0$ (cf. [5, Theorem 3.1]), it follows that $g_{i,j} = -|x_i - x_j|$. Therefore, $M = A^{-1} - (|x_i - x_j|)_{i,j=1}^n$. We have no such a closed form of M for $d \ge 2$.

Next, we discuss the Green function $G(x,y;z)$ of L. Let

$$\Gamma(z) = A^{-1} + (G_0(x_i, x_j; z))_{i,j=1}^n$$

for $z \in \rho(L)$. We can obtain the following theorem employing the same proof as in Theorem 3.1 in [5].

THEOREM 2.4. *Let $G(x,y;z)$ be the Green function of L, $z \in \rho(L)$, $v_x = (G_0(x_1,x;z), G_0(x_2,x;z), \ldots, G_0(x_n,x;z))$ and ${}^t v_y$ be the transpose of v_y. We have that*

$$G(x,y;z) = G_0(x,y;z) - v_x \Gamma(z)^{-1}\, {}^t v_y.$$

In the rest of this section, we discuss the auxiliary matrix $\Gamma(z)$, which plays an important role in Section 3, and its eigenvalues.

THEOREM 2.5. *Let $\mathcal{G}$ be $\mathbb{Z}^d$ or T^r for $r \ge 3$. The following hold; (i) A real number λ in $\mathbb{R} \setminus \sigma(-P)$ is an eigenvalue of L if and only if $\det \Gamma(\lambda) = 0$; (ii) If $\Gamma(\lambda)c = 0$ for some nonzero vector $c = {}^t(c_1, c_2, \ldots, c_n)$, then the function $\psi(x) = \sum_{i=1}^n c_i\, G_0(x, x_i; \lambda)$ is an eigenfunction of L associated with the eigenvalue λ. In particular, the multiplicity of λ as an eigenvalue of L is equal to $\dim\ker\Gamma(\lambda)$.*

Proof. It is well-known that a singular point of Green function as a function of z is corresponding to an eigenvalue z of L. Since G_0 is holomorphic on $\rho(-P) = \mathbb{C} \setminus \sigma(-P)$, the singular points of G are corresponding to the zeros of $\det\Gamma(z)$ on $\rho(-P)$. Since L is self-adjoint, we obtain (i). The second assertion can be proved by direct computations. □

Consider $\Gamma(\lambda)$ as a matrix-valued function of $\lambda \in \mathbb{R} \setminus \sigma(-P)$. Let us show the positive definiteness of the derivative $d\Gamma(\lambda)/d\lambda$. Consider the case where $\mathcal{G} = \mathbb{Z}^d$.

Proposition 2.6. *Let $\mathcal{G}$ be $\mathbb{Z}^d$. Then, the derivative $d\Gamma(\lambda)/d\lambda$ is a positive definite matrix for all $\lambda \in \mathbb{R} \setminus [-1, 1]$.*

Proof. Let $\theta \in \mathbb{T}^d$, $u = {}^t(e^{\sqrt{-1}x_1\cdot\theta}, \ldots, e^{\sqrt{-1}x_n\cdot\theta})$ and define a matrix $K(\theta)$ by

$$K(\theta) = \left(e^{-\sqrt{-1}(x_i - x_j)\cdot\theta}\right)_{i,j=1}^n = \overline{u}\,{}^t u.$$

Since

$$\Gamma(\lambda) = A^{-1} + \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{-1}{\frac{1}{d}\sum_{i=1}^n \cos\theta_i + \lambda} K(\theta)\, d\theta,$$

we have

$$\frac{d\Gamma(\lambda)}{d\lambda} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{1}{(\frac{1}{d}\sum_{i=1}^n \cos\theta_i + \lambda)^2} K(\theta)\, d\theta.$$

Let v be a nonzero vector in $\mathbb{C}^n$. Since ${}^t u\, v \neq 0$ a.e. θ, we have that

$$\langle v, K(\theta)v\rangle = {}^t\overline{v}\, K(\theta)\, v = {}^t\overline{v}\,\overline{u}\ {}^t u\, v = |{}^t u\, v|^2 > 0$$

a.e. θ. Here, $\langle v, w\rangle$ is the standard inner product of $\mathbb{C}^n$. Since

$$\left\langle v, \frac{d\Gamma(\lambda)}{d\lambda} v \right\rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\langle v, K(\theta)v\rangle}{(\frac{1}{d}\sum_{i=1}^n \cos\theta_i + \lambda)^2}\, d\theta \ > \ 0,$$

the derivative $d\Gamma(\lambda)/d\lambda$ is positive definite. $\square$

In the case where $\mathcal{G} = T^r$, we can prove the positive definiteness of $d\Gamma(\lambda)/d\lambda$ with the aid of the following proposition.

Proposition 2.7. *Let $d_{i,j} \geq 0$ satisfying that there exist $v_i \in \mathbb{R}^n \setminus \{0\}$ such that $\|v_i\| + \|v_j\| \geq d_{i,j} \geq 0$ for any $i, j = 1, 2, \ldots, n$. Assume that f is a real valued function such that there exist $a > 0$ and $b > 0$ with $f(d_{i,j}) \geq ae^{-bd_{i,j}}$. Then, the matrix $A = (f(d_{i,j}))_{i,j=1}^n$ is positive definite.*

Proof. Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \setminus \{0\}$. Since

$$\frac{1}{f(d_{i,j})} \leq \frac{e^{b\,d_{i,j}}}{a} \leq \frac{e^{b(\|x_i\|+\|x_j\|)}}{a} \leq \frac{e^{2b\|x_i\|} + e^{2b\|x_j\|}}{2a},$$

we have that

$$0 < \left(\sum_{i=1}^n e^{-\lambda\frac{e^{2b\|x_i\|}}{2a}} v_i\right)^2 = \sum_{i,j=1}^n e^{-\lambda\frac{e^{2b\|x_i\|}+e^{2b\|x_j\|}}{2a}} v_i v_j \leq \sum_{i,j=1}^n e^{-\frac{\lambda}{f(d_{i,j})}} v_i v_j$$

for λ on $[0, \infty)$. Thus,

$$0 < \int_0^\infty \sum_{i,j=1}^n e^{-\frac{\lambda}{f(d_{i,j})}} v_i v_j \, d\lambda = \sum_{i,j} v_i \left[-f(d_{i,j}) e^{-\frac{\lambda}{f(d_{i,j})}} \right]_0^\infty v_j = \langle v, Av \rangle.$$

Therefore, A is positive definite. $\square$

PROPOSITION 2.8. *Let $\mathcal{G}$ be T^r for $r \geq 3$. Then, the derivative $d\Gamma(\lambda)/d\lambda$ is a positive definite matrix for all $\lambda \in \rho(-P) \cap (-\infty, 0) = (-\infty, -2\sqrt{r-1}/r)$.*

Proof. Let $p = \cos^{-1}(-\lambda r / 2\sqrt{r-1})$ and $g = e^{ip}/\sqrt{r-1}$ be the same ones as in Theorem 2.2. The equality $i \cos^{-1} t = -\cosh^{-1} t$ $(t > 1)$ implies that

$$g(\lambda) = -\frac{1}{\sqrt{\lambda - 1}} e^{-\cosh^{-1}(-\lambda r / 2\sqrt{r-1})}$$

is monotonously increasing for $\lambda < -2\sqrt{r-1}/r$. Thus, $dg/d\lambda > 0$ for $\lambda < -2\sqrt{r-1}/r$.

Let $f(x) = [(1 + g^2) + x(1 - g^2)]g^x$. Then, we have that

$$\begin{aligned} \frac{d\Gamma(\lambda)}{d\lambda} &= \frac{d}{d\lambda} (G_0(x_i, x_j; \lambda))_{i,j=1}^n = \frac{dg}{d\lambda} \frac{d}{dg} \left(\frac{r}{g^{-1} - g} g^{|x_i - x_j|} \right)_{i,j=1}^n \\ &= \frac{dg}{d\lambda} \frac{r}{(1 - g^2)^2} \left(f(|x_i - x_j|) \right)_{i,j=1}^n . \end{aligned}$$

Put $a = 1$ and $e^{-b} = g$. Since $0 < g < 1$, we have $(1 + g^2) + x(1 - g^2) \geq 1 + g^2 \geq 1$, and thus we have $f(x) \geq ae^{-bx}$ for $x > 0$, in particular, $f(|x_i - x_j|) \geq ae^{-b|x_i - x_j|}$. Fix a vertex y of T^r. Since $|x_i - y| + |y - x_j| \geq |x_i - x_j| \geq 0$, Proposition 2.7 implies the positivity of $(f(|x_i - x_j|))_{i,j=1}^n$. Therefore, $d\Gamma(\lambda)/d\lambda$ is positive definite. $\square$

As a consequence of these positive definiteness of $d\Gamma(\lambda)/d\lambda$, we obtain several properties of eigenvalues, $\mu_1(\lambda), \ldots, \mu_n(\lambda)$, of $\Gamma(\lambda)$. From now on, we always assume that $\lambda < -1$ if $d = 1$ or $d = 2$, that $\lambda \leq -1$ if $d \geq 3$, and that $\lambda \leq -2\sqrt{r-1}/r$ if $\mathcal{G} = T^r$ $(r \geq 3)$.

PROPOSITION 2.9. *The following hold.*

(i) All of $\mu_i(\lambda)$ are continuous.

(ii) All of $\mu_i(\lambda)$ are monotonously increasing.

(iii) Each $\mu_i(\lambda)$ has at most one zero.

(iv) $p(\Gamma(\lambda))$ *is monotonously decreasing.*

(v) $p(\Gamma(\lambda)) = m_+$ *for* $|\lambda|$ *large enough.*

(vi) The number of the zeros of $\det \Gamma(\lambda)$ *is equal to the total number of the zeros of all* $\mu_i(\lambda)$ *taking into account the multiplicity.*

Proof. (i) Theorem 6.8 in [8] or Proposition 1 in [9] implies this assertion. We remark that $G_0(x, y; \lambda)$ converges to $G_0(x, y; -1)$ as $\lambda \to -1-0$ if $d \geq 3$. (ii) Proposition 2.6 implies that $\Gamma(\lambda_1) > \Gamma(\lambda_2)$ if $\lambda_1 > \lambda_2$. Therefore, the minimax principle [8, Theorem 6.44] implies $\mu_i(\lambda_1) > \mu_i(\lambda_2)$ for all i. Parts (iii) and (iv) follow from (ii). Part (v) follows from the fact $\lim_{\lambda \to -\infty} \Gamma(\lambda) = A^{-1}$. Part (vi) follows from the fact $\det \Gamma(\lambda) = \mu_1(\lambda)\mu_2(\lambda) \cdots \mu_n(\lambda)$. $\square$

3 The number of the discrete eigenvalues

We will prove our main theorems in Section 1 by combining the known methods with the aid of the positive definiteness of the derivative of the auxiliary matrix $d\Gamma(\lambda)/d\lambda$, which is established in the previous section. Note that all of zeros of $\det \Gamma(\lambda)$ lie in $\mathbb{R} \cap \rho(-P) = \mathbb{R} \setminus [-1, 1]$.

We first discuss the simplest cases, $\mathcal{G} = \mathbb{Z}^d$ for $d \geq 3$ and $\mathcal{G} = T^r$ for $r \geq 3$.

Proof of Theorem 1.2. Since $g_{i,j} = G_0(x_i, x_j; -1)$, we have that $M = \Gamma(-1)$ and $p(M)$ is the number of positive $\mu_i(-1)$. Since m_+ is the number of positive $\mu_i(\lambda)$ for $|\lambda|$ large enough and all of $\mu_i(\lambda)$ are monotonously increasing functions, the total number of the zeros of all $\mu_i(\lambda)$ on $(-\infty, -1)$ is equal to $p(M) - m_+$. Therefore, the number of the zeros of $\det \Gamma(\lambda)$ is equal to $p(M) - m_+$. Thus, Theorem 2.5 implies that $N_-(L) = p(M) - m_+$. The second statement is trivial from the first. $\square$

Proof of Theorem 1.3. Since $d\Gamma(\lambda)/d\lambda$ is positive definite, $\lim_{\lambda \to 0-} \Gamma(\lambda) = M$ and $\lim_{\lambda \to -\infty} \Gamma(\lambda) = A^{-1}$, we can obtain the theorem in a similar way in the proof of Theorem 1.2. $\square$

Remark 3.1. In [3], K. Ando and one of the authors have already studied L of T^r with a uniform potential V on $\{x \in T^r; |x - y| = c\}$ with a root vertex y and a constant c. The spectrum of L is completely identified.

We now turn to the case where $\mathbb{Z}^1$ and $\mathbb{Z}^2$. Then, $\lim_{\lambda \to -1-0} G(x, y; \lambda)$ diverges, but $\lim_{\lambda \to -1-0}[G_0(x, x; \lambda) - G_0(0, 0; \lambda)]$ converges for any $x, y \in \mathbb{Z}^d$. Therefore, we can put

$$g_{i,j} = \lim_{\lambda \to -1-0} [G_0(x_i, x_j; \lambda) - G_0(0, 0; \lambda)],$$

where these constants are the same as in Theorem 1.1.

PROPOSITION 3.2. *Let $d = 1$ or $d = 2$. Then, there exists $\lambda_0 < -1$ such that $p(\Gamma(\lambda)) = p(QMQ) + 1$ for all $\lambda \geq \lambda_0$.*

Proof. Put $p = p(QMQ)$ for brevity.

We first prove that $p(\Gamma(\lambda)) \leq p + 1$. Since $d\Gamma(\lambda)/d\lambda$ is positive definite on $(-\infty, -1)$ by Proposition 2.6, $dQ\Gamma(\lambda)Q/d\lambda = Q(d\Gamma(\lambda)/d\lambda)Q$ is non-negative definite. This implies that $p(Q\Gamma(\lambda)Q)$ is monotonously non-decreasing on $(-\infty, -1)$. Since Q is the orthogonal projection onto the orthogonal complement space of the all-one vector $\mathbf{1}$, we have that $JQ = 0$ and

$$\Gamma(\lambda)Q = [\Gamma(\lambda) - G_0(0, 0; \lambda)J]Q \to MQ \quad (\lambda \to -1 - 0).$$

Therefore, the eigenvalues, $\tilde{\mu}_i(\lambda)$, of

$$\tilde{\Gamma}(\lambda) = \begin{cases} Q\Gamma(\lambda)Q & (\lambda < -1), \\ QMQ & (\lambda = -1), \end{cases}$$

are continuous and monotonously non-decreasing on $(-\infty, -1]$. Thus, we obtain that $p(\tilde{\Gamma}(\lambda_1)) \leq p(\tilde{\Gamma}(\lambda_2))$ for $\lambda_1 \leq \lambda_2 \leq -1$. In particular, it holds that $p(Q\Gamma(\lambda)Q) \leq p(QMQ) = p$. Since $Q\Gamma(\lambda)Q$ is a compression of $\Gamma(\lambda)$ to an $(n-1)$-dimensional subspace, we obtain that $p(\Gamma(\lambda)) \leq p + 1$ by Cauchy's interlacing theorem [4, Corollary III.1.5].

Let us prove the existence of λ_0 such that $p(\Gamma(\lambda)) \geq p + 1$ for all $\lambda \geq \lambda_0$. Let $I(\lambda; \psi) = \langle \Gamma(\lambda)\psi, \psi \rangle$ and $\xi = \mathbf{1}/\sqrt{n}$, and let $\varphi_1, \varphi_2, \ldots, \varphi_p$ be linearly independent normalized eigenvectors of QMQ belonging to its positive eigenvalues. It is sufficient to prove the existence of λ_0 such that $I(\lambda; \psi) > 0$ for any $\lambda \geq \lambda_0$ and any linear combination ψ of ξ and φ_i.

Assuming that $p = 0$, we have that

$$\begin{aligned} I(\lambda; \xi) &= \langle A^{-1}\xi, \xi \rangle + \langle (G_0(x_i, x_j; \lambda))\xi, \xi \rangle \\ &= \sum_{k=1}^{n} \frac{1}{\alpha_k} + \frac{-1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\langle \xi, K(\theta)\xi \rangle}{\frac{1}{d}\sum_{k=1}^{d} \cos\theta_k + \lambda} \, d\theta \end{aligned}$$

with the same matrix $K(\theta)$ in Proposition 2.6. Since $\langle \xi, K(\theta)\xi \rangle > 0$ a.e. θ, this quantity $I(\lambda; \xi) \to \infty$ as $\lambda \to -1 - 0$. Thus, we can obtain λ_0 such that $I(\lambda; \xi) > 0$ for any $\lambda \geq \lambda_0$.

Assume that $p \geq 1$. We denote by ν the smallest positive eigenvalue of QMQ. Let φ be a normalized linear combination of φ_i. Note that $Q\varphi = \varphi$.

Since there exists λ_1 such that $\|QMQ - Q\Gamma(\lambda)Q\| < \nu/2$ for any $\lambda > \lambda_1$, we have

$$\begin{aligned} I(\lambda;\varphi) &= \langle QMQ\varphi, \varphi\rangle + \langle (Q\Gamma(\lambda)Q - QMQ)\varphi, \varphi\rangle \\ &\geq \nu - \|Q\Gamma(\lambda)Q - QMQ\| \\ &\geq \nu/2 \end{aligned}$$

for any $\lambda > \lambda_1$.

Noting that $Q\varphi = \varphi$, we have that

$$|\langle \Gamma(\lambda)\varphi, \xi\rangle| = |\langle \Gamma(\lambda)Q\varphi, \xi\rangle| \leq \|\Gamma(\lambda)Q\| \to \|MQ\|$$

as $\lambda \to -1-0$. Since Q is a projection, we have $\|MQ\| \leq \|M\|$, and thus, there exists λ_2 such that $|\langle \Gamma(\lambda)\varphi, \xi\rangle| \leq 2\|M\|$ for any $\lambda > \lambda_2$.

Let $(a,b) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $\lambda > \max\{\lambda_1, \lambda_2\}$. We have

$$\begin{aligned} I(\lambda; a\varphi + b\xi) &= |a|^2 I(\lambda;\varphi) + \langle \Gamma(\lambda)a\varphi, b\xi\rangle + \langle \Gamma(\lambda)b\xi, a\varphi\rangle + |b|^2 I(\lambda;\xi) \\ &\geq (\nu/2)|a|^2 - 4|ab|\|M\| + |b|^2 I(\lambda;\xi) \\ &= (\nu/2)(|a| - 4|b|\|M\|/\nu)^2 + |b|^2(I(\lambda;\xi) - 8\|M\|^2/\nu). \end{aligned}$$

The first term is non-negative. Since $I(\lambda;\xi) \to \infty$ as $\lambda \to -1-0$, there exists λ_0 such that the second term is positive for all $\lambda \geq \lambda_0$. Since both $I(\lambda;\xi)$ and $8\|M\|^2/\nu$ are independent of a, b, and φ, so is λ_0. Therefore, $I(\lambda;\psi) > 0$ for all $\lambda \geq \lambda_0$ and any nonzero linear combination ψ of ξ and φ_i. Thus, we obtain that $p(\Gamma(\lambda)) \geq p+1$ for all $\lambda \geq \lambda_0$. This completes the proof. □

Proposition 3.3. *For $d=1$ or $d=2$, there exists λ_0 such that $N_-(L) = p(\Gamma(\lambda)) - m_+$ for all $\lambda \geq \lambda_0$. In particular, $N_-(L) \leq m_-$, and $N_-(L) = m_-$ if and only if $\Gamma(\lambda_0)$ is positive definite for some λ_0.*

Proof. We can prove this proposition in a similar way to that in the proof of Theorem 1.2; take λ_0 greater than any zeros of $\mu_i(\lambda)$ if they exist, or $\lambda_0 = -2$ otherwise. Then, $p(\Gamma(\lambda_0))$ and m_+ are the numbers of positive $\mu_i(\lambda_0)$ and positive $\mu_i(\lambda)$ for $|\lambda|$ large enough, respectively. Since all $\mu_i(\lambda)$ are monotonously increasing functions, the total number of the zeros of all $\mu_i(\lambda)$ on $(-\infty, \lambda_0)$ is equal to $p(M) - m_+$. Since there is no zero on $[\lambda_0, -1)$, the number of the zeros of $\det\Gamma(\lambda)$ is equal to $p(M) - m_+$ taking into account the multiplicity. Thus, Theorem 2.5 implies that $N_-(L) = p(\Gamma(\lambda_0)) - m_+$. The second statement is trivial from the first. □

Proof of Theorem 1.1. Propositions 3.2 and 3.3 imply $N_-(L) = p(QMQ) + 1 - m_+$. Since $p(QMQ) \leq n-1$, we have that $N_-(L) \leq n - m_+ = m_-$. In addition, since $\dim\ker Q^\perp = n-1$, we have that $QMQ\lfloor_{\ker Q^\perp}$ is positive definite if and only if $p(QMQ) = n-1$. This implies the last assertion. □

4 Application: Discrete analogue of Albeverio and Nizhnik's result

In this section, we give another method to obtain $N_-(L)$ when $d = 1$ as an application of Theorem 1.1. This method is inspired by Albeverio and Nizhnik's algorithm stated below.

Albeverio and Nizhnik have investigated the number of the negative discrete eigenvalues of a one-dimensional *continuous* Schrödinger operator with finite point-interactions,

$$H\psi(x) = -\frac{d^2}{dx^2}\psi(x) + \sum_{j=1}^{n} \tilde{\alpha}_j \delta(x - \tilde{x}_j)\psi(x)$$

Here, $\delta(x)$ is the Dirac's delta function, $\tilde{\alpha}_j \in \mathbb{R} \setminus \{0\}$ and $\tilde{x}_1 < \tilde{x}_2 < \cdots < \tilde{x}_n$. The domain $D(H)$ of H is as follows:

$$D(H) = \{\psi \in W^{2,1}(\mathbb{R}); \psi'(\tilde{x}_j + 0) - \psi'(\tilde{x}_j - 0) = \tilde{\alpha}_j \psi(\tilde{x}_j)\}.$$

They established an excellent algorithm to obtain the number $N(H)$ of the negative eigenvalues of H.

Theorem 4.1 (Lemma 4 in [2]). *Let $\tilde{\phi}$ be the solution of the differential equation, $H\tilde{\phi}(x) = 0$ and $\tilde{\phi}(x) = 1$ $(x < \tilde{x}_1)$. Put*

$$\tilde{w} = (\tilde{\phi}(\tilde{x}_1), \tilde{\phi}(\tilde{x}_2), \ldots, \tilde{\phi}(\tilde{x}_n), (1 + \tilde{\alpha}_n(\tilde{x}_n - \tilde{x}_{n-1}))\tilde{\phi}(\tilde{x}_n) - \tilde{\phi}(\tilde{x}_{n-1})). \quad (1)$$

Then, we have that $N(H) = \operatorname{sig} \tilde{w}$. Here, $\operatorname{sig}(a_1, a_2, \ldots, a_n)$ stands for the number of sign changes.

Note that $\tilde{\phi}(x)$ is a piecewise linear continuous function. Consequently, $\tilde{\phi}(x_k)$ satisfies the following recurrence formula (cf. Eqs. (8) and (9) in [2]);

$$\frac{\tilde{\phi}(\tilde{x}_{k+1}) - \tilde{\phi}(\tilde{x}_k)}{\tilde{x}_{k+1} - \tilde{x}_k} - \frac{\tilde{\phi}(\tilde{x}_k) - \tilde{\phi}(\tilde{x}_{k-1})}{\tilde{x}_k - \tilde{x}_{k-1}} = \tilde{\phi}'(\tilde{x}_k + 0) - \tilde{\phi}'(\tilde{x}_k - 0) = \tilde{\alpha}_k \tilde{\phi}(\tilde{x}_k) \quad (2)$$

and the following initial value: $\tilde{\phi}(\tilde{x}_0) = 0$, and $\tilde{\phi}(\tilde{x}_1) = 1$, where $\tilde{x}_0 = \tilde{x}_1 - 1$.

Remark 4.2. We have that

$$\begin{aligned} \tilde{\phi}'(\tilde{x}_n + 0) &= \tilde{\phi}'(\tilde{x}_n - 0) + \tilde{\alpha}_n \tilde{\phi}(\tilde{x}_n) \\ &= \frac{\tilde{\phi}(\tilde{x}_n) - \tilde{\phi}(\tilde{x}_{n-1})}{\tilde{x}_n - \tilde{x}_{n-1}} + \tilde{\alpha}_n \tilde{\phi}(\tilde{x}_n) \\ &= \frac{(1 + \tilde{\alpha}_n(\tilde{x}_n - \tilde{x}_{n-1}))\tilde{\phi}(\tilde{x}_n) - \tilde{\phi}(\tilde{x}_{n-1})}{\tilde{x}_n - \tilde{x}_{n-1}}. \end{aligned}$$

Thus, $\operatorname{sig} \tilde{w} = \operatorname{sig}(\tilde{\phi}(\tilde{x}_1), \tilde{\phi}(\tilde{x}_2), \ldots, \tilde{\phi}(\tilde{x}_n), \tilde{\phi}'(\tilde{x}_n + 0))$.

A finitely supported potential on $\mathbb{Z}^1$ in the above is naturally considered as a discrete analogue of finite point-interactions on $\mathbb{R}$. Thus our interest is whether a similar result to Theorem 4.1 holds; the answer is positive.

THEOREM 4.3. *Let $d = 1$ and ϕ be the solution of the difference equation, $(L+1)\phi(x) = 0$ and $\phi(x) = 1$ $(x < x_1)$. Put*

$$w = (\phi(x_1), \phi(x_2), \dots, \phi(x_n), (1 + 2\alpha_n(x_n - x_{n-1}))\phi(x_n) - \phi(x_{n-1})). \quad (3)$$

Then, we have that $N_-(L) = \operatorname{sig} w$.

Note that $\phi(x)$ satisfies that $\phi(x+1) - \phi(x) = \phi(x) - \phi(x-1)$ for $x \neq x_k$ and $\phi(x_k+1) + \phi(x_k-1) - 2\phi(x_k) = 2\alpha_k\phi(x_k)$. These imply the following recurrence formula;

$$\frac{\phi(x_{k+1}) - \phi(x_k)}{x_{k+1} - x_k} - \frac{\phi(x_k) - \phi(x_{k-1})}{x_k - x_{k-1}} = 2\alpha_k\phi(x_k) \quad (4)$$

and the following initial value: $\phi(x_0) = 0$, and $\phi(x_1) = 1$, where $x_0 = x_1 - 1$.

REMARK 4.4. We have that

$$\begin{aligned}
\phi(x_n+1) - \phi(x_n) &= 2(1+\alpha_n)\phi(x_n) - \phi(x_n) - \phi(x_n - 1) \\
&= 2\alpha_n\phi(x_n) + (\phi(x_n) - \phi(x_n - 1)) \\
&= 2\alpha_n\phi(x_n) + \frac{\phi(x_n) - \phi(x_{n-1})}{x_n - x_{n-1}} \\
&= \frac{(1 + 2\alpha_n(x_n - x_{n-1}))\phi(x_n) - \phi(x_{n-1})}{x_n - x_{n-1}}.
\end{aligned}$$

Thus, $\operatorname{sig} w = \operatorname{sig}(\phi(x_1), \phi(x_2), \dots, \phi(x_n), \phi(x_n+1) - \phi(x_n))$. We see that $\phi(x_n+1) - \phi(x_n)$ is corresponding to $\tilde{\phi}'(\tilde{x}_n + 0)$.

We can prove Theorem 4.3 by virtue of Theorem 4.1, Theorem 1.1 and the following theorem about H;

THEOREM 4.5 (Theorem 1 in [10]). *We have that*

$$N(H) = n(Q\tilde{M}Q) + 1 - \tilde{m}_+ \quad \textit{with} \quad \tilde{M} = \left(-\frac{\delta_{i,j}}{\tilde{\alpha}_j} + \frac{|\tilde{x}_i - \tilde{x}_j|}{2}\right)_{i,j}.$$

Here, $\tilde{m}_+ = |\{\tilde{\alpha}_j; \tilde{\alpha}_j > 0\}|$ and $n(T)$ is the number of negative eigenvalues of a real symmetric matrix T.

Proof of Theorem 4.3. Theorem 1.1 with Remark 2.3 for $d = 1$ implies that

$$N_-(L) = p(QMQ) + 1 - m_+ \quad \text{with} \quad M = \left(\frac{\delta_{i,j}}{\alpha_j} - |x_i - x_j|\right)_{i,j}.$$

Theorems 4.1 and 4.5 implies that $\operatorname{sig}\tilde{w} = n(Q\tilde{M}Q) + 1 - \tilde{m}_+$. Now, let us take $\tilde{x}_j = x_j$ and $\tilde{\alpha}_j = 2\alpha_j$. Then, we have that $\tilde{m}_+ = m_+$, $\tilde{M} = (-1/2)M$ and thus $n(Q\tilde{M}Q) = p(QMQ)$. In addition, the recurrence formulas (2) and (4) imply that $\tilde{\phi}(\tilde{x_j}) = \phi(x_j)$ for any j. Therefore, $\tilde{w} = w$. Consequently, we obtain that

$$N_-(L) = p(QMQ) + 1 - m_+ = n(Q\tilde{M}Q) + 1 - \tilde{m}_+ = \operatorname{sig}\tilde{w} = \operatorname{sig} w.$$

This is the desired result. $\square$

5 Examples and Discussions

In this section, we give some concrete examples and some discussions.

We note the r-regular tree T^r for $r \geq 3$ always belongs to the case $d \geq 3$ in examples stated below.

EXAMPLE 5.1. Let $n = 1$. We have

$$N_-(L) = \begin{cases} m_-, & \text{if } d = 1, 2, \\ 1, & \text{if } d \geq 3 \text{ and } \alpha_1 < -\frac{1}{g_{1,1}}, \\ 0, & \text{if } d \geq 3 \text{ and } \alpha_1 > -\frac{1}{g_{1,1}}. \end{cases}$$

Let us consider the two points case.

EXAMPLE 5.2. Let $n = 2$ and $\alpha_1 = \alpha_2 = \alpha$.

(1) If $d = 1$ or 2, we have that

$$N_-(L) = \begin{cases} 2, & \text{if } \alpha < -\frac{1}{g_{1,2}}, \\ 1, & \text{if } -\frac{1}{g_{1,2}} \leq \alpha < 0, \\ 0, & \text{otherwise}. \end{cases}$$

(2) If $d \geq 3$, we have that

$$N_-(L) = \begin{cases} 2, & \text{if } \alpha < -\frac{1}{g_{1,1} - g_{1,2}}, \\ 1, & \text{if } -\frac{1}{g_{1,1} - g_{1,2}} \leq \alpha < -\frac{1}{g_{1,1} + g_{1,2}}, \\ 0, & \text{otherwise}. \end{cases}$$

Example 5.3. Let $n = 2$ and $\alpha_1 = -\alpha_2 = \alpha$.

(1) If $d = 1$ or 2, it always holds that $N_\pm(L) = 1$.

(2) If $d \geq 3$, we have that

$$N_\pm(L) = \begin{cases} 1, & \text{if } |\alpha| > (g_{1,1}^2 - g_{1,2}^2)^{-1/2}, \\ 0, & \text{otherwise.} \end{cases}$$

In this paper we obtained some explicit expressions for the number of the discrete eigenvalues of $-P + V$, where V is a finitely supported potential on $\mathbb{Z}^d$ or T^r. Thus, we have completely identified the numbers of them, but we do not know the location yet. Our current knowledge is only that the discrete eigenvalues lay in

$$[\min\{-1 + \alpha_i; i = 1, 2, \dots, n\}, -1) \cup (1, \max\{1 + \alpha_i; i = 1, 2, \dots, n\}],$$

which we can obtain by min-max principle. Though we know that the discrete eigenvalues are the roots of the equation, $\det \Gamma(z) = 0$, it seems to be difficult to solve it completely. In another point of view, more detailed information of $g_{i,j}$ would help us but any closed form of the so-called Watson integral,

$$I(x; z) = \int_{\mathbb{T}^d} \frac{\cos(x \cdot \theta)}{\frac{1}{d} \sum_{i=1}^d \cos \theta_i + z} \, d\theta$$

is unknown in general.

Another interesting problem is the existence of embedded eigenvalue of L [6] for arbitrary n. We will discuss it in our next paper under preparation.

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