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A group-theoretic characterization of the space obtained by omitting the coordinate hyperplanes from the complex Euclidean space, II

Dedicated to the memory of Professor Nobuyuki Suita.

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Abstract. In this paper, we prove that the holomorphic automorphism groups of the spaces $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$ and $(\boldsymbol{C}^k - \{0\}) \times (\boldsymbol{C}^*)^{n-k}$ are not isomorphic as topological groups. By making use of this fact, we establish the following characterization of the space $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$: Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that the holomorphic automorphism group of M is isomorphic to the holomorphic automorphism group of $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$ as topological groups. Then M itself is biholomorphically equivalent to $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$. This was first proved by us in [**5**] under the stronger assumption that M is a Stein manifold.

Introduction.

This is a continuation of our previous paper [**5**], and discusses a characterization of $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell$ by its holomorphic automorphism group. The problems related to the structure of the holomorphic automorphism group $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell)$ of $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell$ are in general very difficult to study. One reason is that $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell)$ is terribly big when $k + \ell \geq 2$, and can not have the structure of a Lie group with respect to the compact-open topology. But, by looking at topological subgroups with Lie group structures of the topological group $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell)$ equipped with the compact-open topology, we can find a lead to apply the Lie group theory to the investigation of such problems. Besides $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell$ admits a natural $(k + \ell)$-dimensional compact torus action given as the rotations along each coordinate axis. This fact enables us to use the machinery associated with torus actions, for example, the theory of Reinhardt domains [**7**], [**8**], as was shown in [**5**]. Under these points of view, we first prove the following theorem on the topological group structure of $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k})$ as our main result:

THEOREM 1. *Let k be an arbitrary integer with $1 \leq k \leq n$. Then $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k})$ is not isomorphic to $\mathrm{Aut}((\boldsymbol{C}^k - \{0\}) \times (\boldsymbol{C}^*)^{n-k})$ as topological groups.*

In the case of $k = 1$, this was already verified in the proof of our Main Theorem in [**5**]. On the other hand, when $k = n$, this fact is a special case of a result of Isaev and

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Kruzhilin (Theorem 5.1 in [**4**]). Therefore, taking these into account, we will carry out the proof of this theorem in the case where $1 < k < n$.

Our Theorem 1 has an interesting application. In fact, making use of the fact obtained in Theorem 1 and employing the same technique as in our previous paper [**5**], we can establish the following characterization of the space $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$:

THEOREM 2. *Let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* $\mathrm{Aut}(M)$ *is isomorphic to* $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k})$ *as topological groups, where the groups* $\mathrm{Aut}(M)$ *and* $\mathrm{Aut}(\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k})$ *are equipped with the compact-open topology. Then M itself is biholomorphically equivalent to $\boldsymbol{C}^k \times (\boldsymbol{C}^*)^{n-k}$.*

This is a natural generalization of our previous result [**5**, Main Theorem]. Indeed, exactly the same conclusion in Theorem 2 was shown there under the stronger assumption that M is a Stein manifold. When $k = n$, that is, for the case of a characterization of $\boldsymbol{C}^n$, our theorem is an immediate consequence of Isaev and Kruzhilin [**4**, Theorem 5.1]. Moreover, it should be remarked that, if M is a domain in $\boldsymbol{C}^n$, then it admits a smooth envelope of holomorphy (cf. [**6**, Chapter 6]). Hence, our Theorem 2 can always be applied when M is a domain in $\boldsymbol{C}^n$.

This paper is organized as follows. In Section 1, we collect some preliminary facts. In particular, two main tools for our study are given. One is a tool to obtain the normal form of some compact group action on a Reinhardt domain studied in Shimizu [**8**], and the other is a tool for the standardization of torus actions on complex manifolds due to Barrett, Bedford and Dadok [**2**]. Also, a group-theoretic characterization of some subgroups of the holomorphic automorphism group of a Reinhardt domain is given, which plays a key role in our study. Sections 2 and 3 are devoted to the proofs of our Thorems 1 and 2, respectively.

1. Reinhardt domains and torus actions.

We begin by recalling a basic fact on Lie group actions on complex manifolds (cf. [**1**]). Let M be a complex manifold. An *automorphism* of M means a biholomorphic mapping of M onto itself. We denote by $\mathrm{Aut}(M)$ the topological group of all automorphisms of M equipped with the compact-open topology. Let G be a Lie group. When a continuous group homomorphism $\rho : G \to \mathrm{Aut}(M)$ of G into $\mathrm{Aut}(M)$ is given, the mapping

$$G \times M \ni (g, p) \longmapsto (\rho(g))(p) \in M$$

is of class C^ω, and we say that *G acts on M as a Lie transformation group through ρ*. Also, the action of G on M is called *effective* if ρ is injective.

We now recall basic concepts and results on Reinhardt domains (cf. [**7**], [**8**]). For an element $\alpha = (\alpha_1, \cdots, \alpha_n)$ of $(\boldsymbol{C}^*)^n$, we denote by π_α an element of $\mathrm{Aut}(\boldsymbol{C}^n)$ given by

$$\boldsymbol{C}^n \ni (z_1, \cdots, z_n) \longmapsto (\alpha_1 z_1, \cdots, \alpha_n z_n) \in \boldsymbol{C}^n.$$

Let D be a Reinhardt domain in $\boldsymbol{C}^n$. Then, by just the definition of a Reinhardt domain,

π_α maps D onto itself and induces a holomorphic automorphism of D for every element α of the n-dimensional compact torus $T^n = (U(1))^n$ given as the direct product of n copies of $U(1)$, where $U(1)$ denotes the multiplicative group of complex numbers with absolute value 1. The mapping ρ_D sending α to π_α is an injective continuous group homomorphism of the torus T^n into the topological group $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^n)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$. Furthermore, we denote by $\Pi(D)$ the topological subgroup of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that φ has the form $\varphi = \pi_\gamma$, where γ is an element of $(\boldsymbol{C}^*)^n$. And we denote by $\mathrm{Aut}_{\mathrm{alg}}(D)$ the topological subgroup of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that each component of φ is given by a Laurent monomial, that is, φ has the form

$$\begin{aligned} &\varphi : (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n), \\ &w_i = \alpha_i z_1^{a_{i1}} \cdots z_n^{a_{in}}, \quad i = 1, \cdots, n, \end{aligned}$$

where $(a_{ij}) \in GL(n, \boldsymbol{Z})$ and $(\alpha_i) \in (\boldsymbol{C}^*)^n$.

The groups $\Pi(D)$ and $\mathrm{Aut}_{\mathrm{alg}}(D)$ have an important meaning, because they are characterized group-theoretically as follows:

Lemma 1.1. *The centralizer of the torus $T(D)$ in the group $\mathrm{Aut}(D)$ is given by $\Pi(D)$, while the normalizer of $T(D)$ in $\mathrm{Aut}(D)$ is given by $\mathrm{Aut}_{\mathrm{alg}}(D)$.*

Proof. The first statement is [**5**, Lemma 1.2]. The second statement is a consequence of [**8**, Section 2, Proposition 1]. □

Note that $T(D)$ is contained in $\Pi(D)$, and $\Pi(D)$ is contained in $\mathrm{Aut}_{\mathrm{alg}}(D)$. Moreover, $\mathrm{Aut}_{\mathrm{alg}}(D)$ is a Lie group whose identity component is $\Pi(D)$.

The following proposition and corollary play a crucial role in our study.

Proposition 1.1 (see [**5**, Section 1]). *Let D be a bounded Reinhardt domain in $\boldsymbol{C}^n$ and suppose that*

$$\begin{aligned} &D \cap \{z_i = 0\} \neq \varnothing, \quad 1 \leq i \leq m, \\ &D \cap \{z_i = 0\} = \varnothing, \quad m+1 \leq i \leq n. \end{aligned}$$

If G is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$, then there exists a transformation

$$\varphi : \boldsymbol{C}^m \times (\boldsymbol{C}^*)^{n-m} \ni (z_1, \cdots, z_n) \longmapsto (w_1, \cdots, w_n) \in \boldsymbol{C}^m \times (\boldsymbol{C}^*)^{n-m},$$
$$\begin{cases} w_i = r_i z_{\sigma'(i)} (z'')^{\nu_i''}, & 1 \leq i \leq m, \\ w_i = r_i z_{\sigma''(i)}, & m+1 \leq i \leq n, \end{cases}$$

such that, for $\tilde{D} := \varphi(D)$ and $\tilde{G} := \varphi G \varphi^{-1} \subset \mathrm{Aut}(\tilde{D})$, one has

$$\begin{aligned} &\tilde{G} = U(k_1) \times \cdots \times U(k_s) \times U(k_{s+1}) \times \cdots \times U(k_t), \\ &k_1 + \cdots + k_s + k_{s+1} + \cdots + k_t = n, \end{aligned}$$

$$k_1 + \cdots + k_s = m,$$
$$k_{s+1} = \cdots = k_t = 1,$$

where $r_1, \cdots, r_n$ *are positive constants,* σ' *and* σ'' *are permutations of* $\{1, \cdots, m\}$ *and* $\{m+1, \cdots, n\}$*, respectively,* z'' *denotes the coordinates* $(z_{m+1}, \cdots, z_n)$*, and* $\nu_1'', \cdots, \nu_m''$ *are elements of* $\boldsymbol{Z}^{n-m}$*.*

Corollary (see [**5**, Section 1]). *In the above proposition, if* G *is isomorphic to* $U(k) \times (U(1))^{n-k}$ *as topological groups and if* $k \geq 2$*, then* $m \geq k$*.*

Roughly speaking, the corollary above implies that if a bounded Reinhardt domain D admits an effective $U(k)$-action, then it has the non-empty intersection with at least k coordinate hyperplanes.

Finally, we recall the fundamental result on torus actions on complex manifolds.

Standardization Theorem (see [**2**, Theorem 1]). *Let* M *be a connected complex manifold of dimension* n *that is holomorphically separable and admits a smooth envelope of holomorphy. Assume that* T^n *acts effectively on* M *as a Lie transformation group through* ρ*. Then there exist a biholomorphic mapping* F *of* M *into* $\boldsymbol{C}^n$ *and a continuous group automorphism* θ *of the torus* T^n *such that*

$$F((\rho(\alpha))(p)) = \theta(\alpha) \cdot F(p) \quad \text{for all } \alpha \in T^n \text{ and all } p \in M.$$

Consequently, $D := F(M)$ *is a Reinhardt domain in* $\boldsymbol{C}^n$*, and one has* $F\rho(T^n)F^{-1} = T(D)$*.*

2. Proof of Theorem 1.

Throughout this section, we use the following notation: For the given integer k with $1 \leq k \leq n$ and a point $(z_1, \cdots, z_n) \in \boldsymbol{C}^n$, we set

$$\ell = n - k, \quad \Omega_k = \boldsymbol{C}^k \times (\boldsymbol{C}^*)^\ell, \quad \Omega_k^0 = (\boldsymbol{C}^k - \{0\}) \times (\boldsymbol{C}^*)^\ell,$$
$$z = (z_1, z_2, \cdots, z_k), \quad z' = (z_2, \cdots, z_k) \text{ and } (w_1, \cdots, w_\ell) = (z_{k+1}, \cdots, z_n).$$

We shall often use the natural identifications given by

$$SU(k) = SU(k) \times \{(1, \cdots, 1)\} \subset U(k) \times (U(1))^\ell,$$
$$SU(k-1) = \left\{ \begin{pmatrix} 1 & O \\ O & A \end{pmatrix} \middle| A \in SU(k-1) \right\} \subset U(k),$$

where $SU(m)$ stands for the special unitary group of degree m. Note that Ω_k^0 is a Reinhardt domain in $\boldsymbol{C}^n$ invariant under the standard action of $U(k) \times (U(1))^\ell$ on $\boldsymbol{C}^n$; and accordingly, one may regard the group $U(k) \times (U(1))^\ell$ as a subgroup of $\mathrm{Aut}(\Omega_k^0)$. And, the group $T(\Omega_k^0)$ can be identified with the n-dimensional torus $T^n = (U(1))^n$.

Let us start the proof. As mentioned in the introduction, we have only to prove the theorem in the case where $k \geq 2$ and $\ell \geq 1$. Since $k \geq 2$, we may assume that $\mathrm{Aut}(\Omega_k^0)$ is the group of all elements $f \in \mathrm{Aut}(\Omega_k)$ satisfying $f(\{0\} \times (\boldsymbol{C}^*)^\ell) = \{0\} \times (\boldsymbol{C}^*)^\ell$.

Now, assume contrarily that there exists a topological group isomorphism $\Psi : \mathrm{Aut}(\Omega_k^0) \to \mathrm{Aut}(\Omega_k)$. First of all, without loss of generality we may assume here that Ψ satisfies the following two conditions:

$$\Psi(T^n) = T^n \quad \text{and} \quad \Psi(U(k) \times (U(1))^\ell) = U(k) \times (U(1))^\ell; \tag{2.1}$$

consequently, it follows that $\Psi(SU(k)) = SU(k)$. Indeed, these can be seen as follows. Since Ω_k^0 is a Reinhardt domain in $\boldsymbol{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k^0} : T^n \to \mathrm{Aut}(\Omega_k^0)$. Considering the composition of $\rho_{\Omega_k^0}$ and Ψ, we now obtain an injective continuous group homomorphism $\Psi \circ \rho_{\Omega_k^0} : T^n \to \mathrm{Aut}(\Omega_k)$. Thus, by the Standardization Theorem, there exists a biholomorphic mapping F of Ω_k into $\boldsymbol{C}^n$ such that $D := F(\Omega_k)$ is a Reinhardt domain in $\boldsymbol{C}^n$ and $F(\Psi \circ \rho_{\Omega_k^0})(T^n)F^{-1} = T(D)$. Moreover, we know that the image domain $D = F(\Omega_k)$ is literally equal to Ω_k by [**5**, Lemma 1.3]. Therefore, considering $F \circ \Psi \circ F^{-1}$ instead of Ψ if necessary, we may assume that $\Psi(T(\Omega_k^0)) = T(\Omega_k)$, which is the first one of (2.1). Next, recalling that $\mathrm{Aut}(\Omega_k^0)$ contains $U(k) \times (U(1))^\ell$ as its subgroup, we set $G = \Psi(U(k) \times (U(1))^\ell)$. Then G is a connected compact subgroup of $\mathrm{Aut}(\Omega_k)$ containing $T(\Omega_k)$, because $U(k) \times (U(1))^\ell \supset T(\Omega_k^0)$ and $\Psi(T(\Omega_k^0)) = T(\Omega_k)$. Take a bounded domain U in $\boldsymbol{C}^n$ whose closure $\overline{U}$ is contained in Ω_k and put

$$D_0 = \{g(z) \in \Omega_k \mid g \in G,\ z \in U\} = \bigcup_{g \in G} g(U) = \bigcup_{z \in U} G \cdot z.$$

Then it is easily seen that D_0 is a bounded Reinhardt domain in $\boldsymbol{C}^n$ contained in Ω_k and G can be regarded as a connected compact subgroup of the Lie group $\mathrm{Aut}(D_0)$ containing $T(D_0)$. Recalling that G is isomorphic to $U(k) \times (U(1))^\ell$ and $k \geq 2$, we can apply Proposition 1.1 and its corollary to D_0 and G. As a consequence, the intersection of D_0 and the coordinate hyperplane given by $\{z_i = 0\}$ is not empty for all i with $1 \leq i \leq k$. Indeed, otherwise, such an intersection is empty for some i with $1 \leq i \leq k$. This implies that the number s of coordinate hyperplanes having the nonempty intersection with D_0 is less than k, because $D_0 \subset \Omega_k$ and hence $D_0 \cap \{z_j = 0\} = \varnothing$ for all j with $k+1 \leq j \leq n$. But, since D_0 admits an effective $U(k)$-action, by the corollary to Proposition 1.1, D_0 must have the non-empty intersection with at least k coordinate hyperplanes, which implies that $s \geq k$. This is a contradiction, because we have seen that $s < k$. We see that the integer m in Proposition 1.1 is given by k in the case now as well. By applying Proposition 1.1 as $m = k$, after a change of coordinates by some element of $\mathrm{Aut}_{\mathrm{alg}}(\Omega_k)$, we have that $G = U(k) \times (U(1))^\ell$ as sets in $\mathrm{Aut}(D_0)$, and hence in $\mathrm{Aut}(\Omega_k)$ by analytic continuation. This is nothing but the second equality of (2.1), as desired.

From now on, let us denote by $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(\Omega_k^0)$ the inverse isomorphism of $\Psi : \mathrm{Aut}(\Omega_k^0) \to \mathrm{Aut}(\Omega_k)$. It follows then from (2.1) that Φ gives rise to a continuous group isomorphism of $U(k) \times (U(1))^\ell$ onto itself with $\Phi(T^n) = T^n$. Hence there exists an element (a_{ij}) of $GL(n, \boldsymbol{Z})$ such that

$$\begin{aligned}&\Phi\big(\big(\exp 2\pi\sqrt{-1}\theta_1,\cdots,\exp 2\pi\sqrt{-1}\theta_n\big)\big)\\&\quad=\left(\exp 2\pi\sqrt{-1}\left(\sum_{j=1}^{n}a_{1j}\theta_j\right),\cdots,\exp 2\pi\sqrt{-1}\left(\sum_{j=1}^{n}a_{nj}\theta_j\right)\right)\end{aligned}$$

for all $\theta_1,\cdots,\theta_n\in\boldsymbol{R}$. Here consider an arbitrary element $\rho\in(U(1))^n$ having the form

$$\rho=\big(\exp 2\pi\sqrt{-1}\theta,\cdots,\exp 2\pi\sqrt{-1}\theta,\exp 2\pi\sqrt{-1}\theta_1,\cdots,\exp 2\pi\sqrt{-1}\theta_\ell\big).$$

Then, for any $A\in SU(k)$, we have $\rho\circ A=A\circ\rho$ as automorphisms of Ω_k; so that $\Phi(\rho)\circ A=A\circ\Phi(\rho)$ for all $A\in SU(k)$. This gives us that

$$\begin{aligned}&\sum_{j=1}^{k}a_{1j}=\cdots=\sum_{j=1}^{k}a_{kj},\\&a_{1(k+i)}=\cdots=a_{k(k+i)},\quad 1\le i\le\ell.\end{aligned}$$

Therefore, putting

$$\begin{aligned}&a=\sum_{j=1}^{k}a_{1j},\quad b_i=a_{1(k+i)},\quad c_i=\sum_{j=1}^{k}a_{(k+i)j},\quad 1\le i\le\ell,\\&(d_{ij})_{1\le i,j\le\ell}=\big(a_{(k+i)(k+j)}\big)_{1\le i,j\le\ell}\end{aligned}$$

and writing $\Phi(\rho)=(\tilde\rho_1,\cdots,\tilde\rho_n)$, we obtain that

$$\begin{aligned}\tilde\rho_i&=\exp 2\pi\sqrt{-1}\left(a\theta+\sum_{j=1}^{\ell}b_j\theta_j\right),\quad 1\le i\le k,\\\tilde\rho_{k+i}&=\exp 2\pi\sqrt{-1}\left(c_i\theta+\sum_{j=1}^{\ell}d_{ij}\theta_j\right),\quad 1\le i\le\ell.\end{aligned}$$

Accordingly, for an arbitrary element

$$g=\big(\big(\exp 2\pi\sqrt{-1}\theta\big)A,\exp 2\pi\sqrt{-1}\theta_1,\cdots,\exp 2\pi\sqrt{-1}\theta_\ell\big)\in U(k)\times(U(1))^\ell \tag{2.2}$$

with $A\in SU(k)$ and θ, $\theta_i\in\boldsymbol{R}$, $1\le i\le\ell$, our isomorphism Φ of $U(k)\times(U(1))^\ell$ onto itself is given by

$$\begin{aligned}\Phi(g)=\Bigg(&\left(\exp 2\pi\sqrt{-1}\left(a\theta+\sum_{j=1}^{\ell}b_j\theta_j\right)\right)\Phi_s(A),\\&\exp 2\pi\sqrt{-1}\left(c_1\theta+\sum_{j=1}^{\ell}d_{1j}\theta_j\right),\cdots,\exp 2\pi\sqrt{-1}\left(c_\ell\theta+\sum_{j=1}^{\ell}d_{\ell j}\theta_j\right)\Bigg),\end{aligned} \tag{2.3}$$

where Φ_s is the automorphism of $SU(k)$ given by the restriction of Φ to $SU(k)$.

It should be made a few remarks here. Firstly, the existence of the inverse isomorphism Ψ of Φ guarantees that

$$\begin{pmatrix} a & b_1 & \cdots & b_\ell \\ c_1 & d_{11} & \cdots & d_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ c_\ell & d_{\ell 1} & \cdots & d_{\ell\ell} \end{pmatrix} \quad \in \quad GL(\ell+1, \boldsymbol{Z}). \tag{2.4}$$

Secondly, we may assume that

$$\Phi_s(A) = A \quad \text{or} \quad \Phi_s(A) = \bar{A} \quad \text{for all } A \in SU(k), \tag{2.5}$$

where $\bar{A}$ denotes the complex conjugate of A. Indeed, it is known that there exists an element $\sigma \in SU(k)$ such that

$$\Phi_s(A) = \sigma^{-1} A \sigma \quad \text{or} \quad \Phi_s(A) = \sigma^{-1} \bar{A} \sigma \quad \text{for every } A \in SU(k).$$

By using this σ, we define a biholomorphic mapping $\varphi : \Omega_k^0 \to \Omega_k^0$ by

$$(\tilde{z}, \tilde{w}) := \varphi(z, w) = (\sigma z, w), \quad (z, w) \in \Omega_k^0$$

(think of z as column vectors). Then, considering the new coordinates $(\tilde{z}, \tilde{w})$ instead of (z, w) if necessary, we may assume that Φ satisfies the condition (2.5).

Here we claim that

$$\det(d_{ij}) \neq 0. \tag{2.6}$$

Indeed, taking a special element $g_o := \left(\exp \frac{2\pi\sqrt{-1}}{k}\right) E_k$, which is clearly contained in the centers of both $U(k)$ and $SU(k)$, we have

$$\left(\exp \frac{2\pi\sqrt{-1}a}{k}\right) E_k = \Phi_s(g_o) \quad \text{and} \quad \exp \frac{2\pi\sqrt{-1}c_i}{k} = 1, \quad 1 \le i \le \ell.$$

So the integers a and c_i may be written in the form

$$\begin{aligned} &c_i = \tilde{c}_i k, \quad 1 \le i \le \ell, \text{ and} \\ &a = 1 + k_o k \quad \text{or} \quad a = -1 + k_o k \end{aligned} \tag{2.7}$$

according as $\Phi_s(A) = A$ or $\Phi_s(A) = \bar{A}$ in (2.5), where $\tilde{c}_i$ and k_o are integers. Therefore, expanding the determinant of the matrix in (2.4) according to the first column, we obtain

$$\pm 1 = a \det(d_{ij}) + k\tilde{k} \quad \text{for some } \tilde{k} \in \boldsymbol{Z}.$$

Since $k \geq 2$, this means (2.6).

Finally, note that the expression in the form (2.2) of a given element $g \in U(k) \times (U(1))^\ell$ is not unique. Indeed, the same element g can be also represented in the form

$$g = \Bigg(\Bigg(\exp 2\pi\sqrt{-1}\Big(\theta + p - \frac{q}{k}\Big)\Bigg) \Bigg(\Big(\exp \frac{2\pi\sqrt{-1}q}{k}\Big) A \Bigg), \\ \exp 2\pi\sqrt{-1}(\theta_1 + p_1), \cdots, \exp 2\pi\sqrt{-1}(\theta_\ell + p_\ell) \Bigg) \tag{2.8}$$

with $p, p_i, q \in \boldsymbol{Z}$ and $0 \leq q \leq k-1$. However, by using the fact (2.7), one can check easily that the right hand side of (2.3) does not depend on the choice of representation of g as in (2.8). Thus it should be emphasized that, in computing the image $\Phi(G)$ of a given subgroup G of $U(k) \times (U(1))^\ell$, we are not worried about the representation of an element $g \in G$ as in (2.2).

We now have two cases to consider.

Case 1. $(b_1, \cdots, b_\ell) = (0, \cdots, 0)$.

In this case, $a = \pm 1$ and $(d_{ij}) \in GL(\ell, \boldsymbol{Z})$ in (2.4). Hence, after a change of coordinates by some element of $\mathrm{Aut}_{\mathrm{alg}}(\Omega_k^0)$ if necessary, we may assume that $(d_{ij}) = E_\ell$. The proof will be divided into two cases as follows:

Case (1-1). $k \geq 3$: In this case, by (2.3) and (2.5) we obtain

$$\Phi(SU(k-1) \times (U(1))^\ell) = SU(k-1) \times (U(1))^\ell.$$

Therefore, since $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(\Omega_k^0)$ is a group isomorphism, we see that Φ maps the centralizer C of $SU(k-1) \times (U(1))^\ell$ in $\mathrm{Aut}(\Omega_k)$ onto the centralizer C_0 of $SU(k-1) \times (U(1))^\ell$ in $\mathrm{Aut}(\Omega_k^0)$; and consequently, their commutator groups $Z = [C, C]$ and $Z_0 = [C_0, C_0]$ have to be isomorphic.

In order to derive a contradiction, we claim that Z_0 is abelian, while Z is not. To prove our claim, take an arbitrary element f belonging to C or to C_0, and represent $f = (f_1, \cdots, f_n)$ by coordinates. Then one has

$$f_1\big(z_1, Az', \big(\exp 2\pi\sqrt{-1}\theta_1\big)w_1, \cdots, \big(\exp 2\pi\sqrt{-1}\theta_\ell\big)w_\ell\big) = f_1(z, w) \quad \text{on} \quad \Omega_k$$

for all $A \in SU(k-1)$ and all $\theta_i \in \boldsymbol{R}$ (think of z' as column vectors). Since $k - 1 \geq 2$, this implies that f_1 does not depend on the variables z' and w (cf. [**3**, p. 142]); so f_1 has the form $f_1(z, w) = f_1(z_1)$ and it induces a holomorphic automorphism of $\boldsymbol{C}$. Note that $f_1(0) = 0$ in the case when $f \in C_0$.

Next, by using the Laurent series of f_{k+j}, one can find that it has the form

$$f_{k+j}(z, w) = \gamma_j(z_1)w_j, \quad 1 \leq j \leq \ell,$$

where $\gamma_j(z_1)$ is a nowhere vanishing holomorphic function on $\boldsymbol{C}$.

Finally we assert that $(f_2, \cdots, f_k)$ can be written in the form

$$(f_2(z,w),\cdots,f_k(z,w)) \;=\; b(z_1)z', \tag{2.9}$$

where $b(z_1)$ is a nowhere vanishing holomorphic function on $\boldsymbol{C}$. Indeed, it is easily checked that each function f_j, $2 \le j \le k$, is independent on w; and hence, it has the form $f_j(z,w) = f_j(z_1,z')$. Take now a point $z_1^o \in \boldsymbol{C}$ arbitrarily and define a mapping $L : \boldsymbol{C}^{k-1} \to \boldsymbol{C}^{k-1}$ by setting

$$L(z') = {}^t\big(f_2(z_1^o,z'),\cdots,f_k(z_1^o,z')\big), \quad z' \in \boldsymbol{C}^{k-1},$$

where ${}^t\mathfrak{a}$ denotes the transpose of a given element $\mathfrak{a} \in M(1,k-1,\boldsymbol{C})$. Then L is a holomorphic automorphism of $\boldsymbol{C}^{k-1}$ satisfying the condition

$$L(Az') = A\cdot L(z') \quad \text{for every } A \in SU(k-1). \tag{2.10}$$

Thus $L(z') = 0$ if and only if $z' = 0$. For an arbitrary given point $z'_o \neq 0$ of $\boldsymbol{C}^{k-1}$, let us put $r_o = \|z'_o\|$ and $R_o = \|L(z'_o)\|$, and consider a biholomorphic mapping $\tilde{L} : \boldsymbol{C}^{k-1} \to \boldsymbol{C}^{k-1}$ defined by

$$\tilde{L}(z') = (r_o/R_o)L(z'), \quad z' \in \boldsymbol{C}^{k-1}.$$

It then follows from (2.10) that $\tilde{L}$ gives rise to a holomorphic automorphism of the open ball $B(0,r_o)$ in $\boldsymbol{C}^{k-1}$ of radius r_o with center 0, and $\tilde{L}(0) = 0$. Hence $\tilde{L}$ has to be a unitary transformation of $\boldsymbol{C}^{k-1}$; and so there exists an element $U \in U(k-1)$ such that $L(z') = (R_o/r_o)Uz'$ on $\boldsymbol{C}^{k-1}$. Moreover, the relation (2.10) tells us that U is a scalar matrix (depending only on z_1^o), from which we obtain the assertion (2.9).

As a result, we have seen that C, C_0 are the groups of all elements $f \in \mathrm{Aut}(\Omega_k)$, $f_0 \in \mathrm{Aut}(\Omega_k^0)$ having the forms

$$\begin{aligned} f(z_1,z',w_1,\cdots,w_\ell) &= (\alpha z_1+\beta, b(z_1)z', \gamma_1(z_1)w_1,\cdots,\gamma_\ell(z_1)w_\ell),\\ f_0(z_1,z',w_1,\cdots,w_\ell) &= (\alpha z_1, b(z_1)z', \gamma_1(z_1)w_1,\cdots,\gamma_\ell(z_1)w_\ell) \end{aligned} \tag{2.11}$$

respectively, where $\alpha \in \boldsymbol{C}^*$, $\beta \in \boldsymbol{C}$ and $b(z_1)$, $\gamma_j(z_1)$ are all nowhere vanishing holomorphic functions on $\boldsymbol{C}$. Therefore, in exactly the same way as in the proof of [**5**, Main Theorem], it can be shown that Z_0 is an abelian group, while Z is not; completing the proof in the case of $k \ge 3$.

Case (1-2). $k=2$: Since every automorphism of $SU(2)$ is inner, one has $\Phi_s(A) = A$ for all $A \in SU(2)$ in (2.5). Now let us consider a subgroup G of $U(2)\times(U(1))^\ell$ defined by

$$G = \left\{ \left(\begin{pmatrix} 1 & 0 \\ 0 & \exp 4\pi\sqrt{-1}\theta \end{pmatrix}, \exp 2\pi\sqrt{-1}\theta_1, \cdots, \exp 2\pi\sqrt{-1}\theta_\ell \right) \,\middle|\, \theta,\theta_1,\cdots,\theta_\ell \in \boldsymbol{R} \right\}. \tag{2.12}$$

Note that

$$g(\theta) := \begin{pmatrix} \exp(-2\pi\sqrt{-1}\theta) & 0 \\ 0 & \exp 2\pi\sqrt{-1}\theta \end{pmatrix} \in SU(2) \tag{2.13}$$

and

$$\begin{pmatrix} 1 & 0 \\ 0 & \exp 4\pi\sqrt{-1}\theta \end{pmatrix} = \bigl(\exp 2\pi\sqrt{-1}\theta\bigr) g(\theta).$$

Then, considering the new coordinates $(\tilde{z}_1, \tilde{z}_2) := (z_2, z_1)$ and replacing $(c_1, \cdots, c_\ell)$ by $(-c_1, \cdots, -c_\ell)$ if necessary, we may assume by (2.3) that

$$\begin{aligned} \Phi(G) = \Biggl\{ \Biggl(\begin{pmatrix} 1 & 0 \\ 0 & \exp 4\pi\sqrt{-1}\theta \end{pmatrix}, \exp 2\pi\sqrt{-1}(c_1\theta + \theta_1), \cdots \\ \cdots, \exp 2\pi\sqrt{-1}(c_\ell\theta + \theta_\ell) \Biggr) \Biggm| \theta, \theta_1, \cdots, \theta_\ell \in \boldsymbol{R} \Biggr\}. \end{aligned}$$

Let C and C_0 be the centralizers of G and $\Phi(G)$ in $\mathrm{Aut}(\Omega_k)$ and in $\mathrm{Aut}(\Omega_k^0)$ respectively, and let $Z = [C, C]$ and $Z_0 = [C_0, C_0]$ be their commutator groups. Then, the proof is now reduced to showing that Z_0 is abelian and Z is not abelian. But this can be shown with exactly the same argument as in Case (1-1). Indeed, straightforward computations give us that arbitrary elements $f \in C$ and $f_0 \in C_0$ have the same descriptions as in (2.11). Therefore, we conclude that Z_0 is an abelian group, while Z is not an abelian group, as desired.

Case 2. $(b_1, \cdots, b_\ell) \neq (0, \cdots, 0)$.

First of all, by (2.4) there exist integers $\lambda, \lambda_1, \cdots, \lambda_\ell$ such that

$$(1, 0, \cdots, 0) = \lambda(a, b_1, \cdots, b_\ell) - \sum_{i=1}^{\ell} \lambda_i (c_i, d_{i1}, \cdots, d_{i\ell})$$

or equivalently

$$\lambda a - \sum_{i=1}^{\ell} \lambda_i c_i = 1 \quad \text{and} \quad \sum_{i=1}^{\ell} \lambda_i (d_{i1}, \cdots, d_{i\ell}) = \lambda(b_1, \cdots, b_\ell). \tag{2.14}$$

Since $\det(d_{ij}) \neq 0$ by (2.6), we see that $\lambda \neq 0$ and $\lambda_i \neq 0$ for some i. Now, each integer λ_i can be written uniquely in the form

$$\lambda_i = q_i\lambda + r_i \quad \text{with} \quad q_i,\ r_i \in \boldsymbol{Z}, \quad 0 \leq |r_i| < |\lambda| \quad \text{and} \quad \lambda r_i \leq 0.$$

It then follows from (2.14) that

$$\lambda\Big(a-\sum_{i=1}^{\ell} q_i c_i\Big) = 1+\sum_{i=1}^{\ell} r_i c_i. \tag{2.15}$$

Using these integers q_i, we now define a transformation φ of Ω_k^0 by

$$\varphi : (z_1,\cdots,z_k,w_1,\cdots,w_\ell) \longmapsto (\tilde{z}_1,\cdots,\tilde{z}_k,\tilde{w}_1,\cdots,\tilde{w}_\ell)$$

$$\begin{cases} \tilde{z}_i = z_i w_1^{-q_1}\cdots w_\ell^{-q_\ell}, & 1\le i\le k, \\ \tilde{w}_j = w_j, & 1\le j\le \ell, \end{cases}$$

and take an arbitrary element g of $U(k)\times(U(1))^\ell$ written in the form (2.2). Then, replacing (z,w) by the new coordinates $(\tilde{z},\tilde{w})$ if necessary, one can see by (2.14) and (2.15) that the transformation $\varPhi(g)$ of Ω_k^0 in (2.3) is given by the following correspondence:

$$\varPhi(g) : \begin{cases} z \longmapsto \Big(\exp\dfrac{2\pi\sqrt{-1}}{\lambda}\Big\{\Big(1+\displaystyle\sum_{i=1}^{\ell} r_i c_i\Big)\theta + \sum_{i,j=1}^{\ell} d_{ij} r_i \theta_j\Big\}\Big)\varPhi_s(A)z, \\ w_i \longmapsto \Big(\exp 2\pi\sqrt{-1}\Big(c_i\theta+\displaystyle\sum_{j=1}^{\ell} d_{ij}\theta_j\Big)\Big)w_i, \quad 1\le i\le \ell \end{cases} \tag{2.16}$$

(think of z as column vectors).

Since $\gcd(\lambda,\lambda_1,\cdots,\lambda_\ell)=1$ by (2.14), we see that $|\lambda|=1$ if and only if $r_i=0$ for all i; and, in such a case, the proof can be reduced to Case 1. Therefore, in the following, we assume that

$$|\lambda|\ge 2 \quad \text{or equivalently} \quad (r_1,\cdots,r_\ell)\ne(0,\cdots,0).$$

Thus, renaming the indices if necessary, we may further assume that

$$r_1\cdots r_s \ne 0, \text{ while } r_{s+1}=\cdots=r_\ell=0$$

for some integer s with $1\le s\le \ell$ (where it is understood that every $r_i\ne 0$ if $s=\ell$). Again we divide the proof into two cases.

Case (2-1). $k\ge 3$: In this case, it follows from (2.5), (2.6) and (2.16) that

$$\varPhi\big(SU(k-1)\times(U(1))^\ell\big) = \Big\{\Big(\Big(\exp 2\pi\sqrt{-1}\Big(\sum_{i=1}^{s}\frac{r_i\theta_i}{\lambda}\Big)\Big)\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}, \exp 2\pi\sqrt{-1}\theta_1,\cdots$$
$$\cdots,\exp 2\pi\sqrt{-1}\theta_\ell\Big)\Big| A\in SU(k-1),\ \theta_1,\cdots,\theta_\ell\in \boldsymbol{R}\Big\}.$$

Let C_0 be the centralizer of $\varPhi(SU(k-1)\times(U(1))^\ell)$ in $\mathrm{Aut}(\Omega_k^0)$ and let Z_0 be the commutator group of C_0. Once it is shown that Z_0 is an abelian group, we arrive at the same contradiction as in Case (1-1). Therefore we have only to show that Z_0 is abelian. To this end, take an arbitrary element $f = (f_1, \cdots, f_n) \in C_0$. To simplify the notation, we put $\Theta = \sum_{i=1}^{s} \frac{r_i \theta_i}{\lambda}$ and write

$$X_{z,w,A,\theta_1,\cdots,\theta_\ell} = \Big(\big(\exp 2\pi\sqrt{-1}\Theta\big)z_1, \big(\exp 2\pi\sqrt{-1}\Theta\big)Az', \\ \big(\exp 2\pi\sqrt{-1}\theta_1\big)w_1, \cdots, \big(\exp 2\pi\sqrt{-1}\theta_\ell\big)w_\ell\Big).$$

Then we obtain

$$f_1\big(X_{z,w,A,\theta_1,\cdots,\theta_\ell}\big) = \big(\exp 2\pi\sqrt{-1}\Theta\big)f_1(z,w) \quad \text{on} \quad \Omega_k$$

for all $A \in SU(k-1)$ and all $\theta_1, \cdots, \theta_\ell \in \boldsymbol{R}$. Thus it follows that f_1 is independent on z'; and hence it has the form $f_1(z,w) = f_1(z_1,w)$. Let $f_1(z_1,w) = \sum A_{\mu\nu} z_1^{\mu} w^{\nu}$ be the Laurent expansion of f_1 on $\boldsymbol{C} \times (\boldsymbol{C}^*)^{\ell}$, where $0 \le \mu \in \boldsymbol{Z}$ and $w^{\nu} = w_1^{\nu_1} \cdots w_\ell^{\nu_\ell}$ for the coordinate $w = (w_1, \cdots, w_\ell)$ of $\boldsymbol{C}^{\ell}$ and $\nu = (\nu_1, \cdots, \nu_\ell) \in \boldsymbol{Z}^{\ell}$. Substituting this into the preceding identity and comparing the coefficient of $z_1^{\mu} w^{\nu}$, we have the following by the uniqueness of expansion:

$$\left(\sum_{i=1}^{s} \frac{r_i \theta_i}{\lambda}\right)(\mu - 1) + \sum_{i=1}^{\ell} \theta_i \nu_i = 0$$

for all $\theta_1, \cdots, \theta_\ell \in \boldsymbol{R}$, provided that $A_{\mu\nu} \neq 0$. This is equivalent to

$$(\nu_1, \cdots, \nu_s, \nu_{s+1}, \cdots, \nu_\ell) = (\mu - 1)\left(-\frac{r_1}{\lambda}, \cdots, -\frac{r_s}{\lambda}, 0, \cdots, 0\right).$$

Thus, putting

$$\begin{aligned} n_1 &= \min\left\{ n \in \boldsymbol{N} \,\middle|\, -\frac{r_i}{\lambda} n \in \boldsymbol{Z}, \quad 1 \le i \le s \right\}, \\ m_i &= -\frac{r_i}{\lambda} n_1, \quad 1 \le i \le s, \end{aligned} \tag{2.17}$$

we obtain that $n_1 \ge 2$, $m_i \ge 1$ and

$$(\mu - 1, \nu_1, \cdots, \nu_s, \nu_{s+1}, \cdots, \nu_\ell) = m(n_1, m_1, \cdots, m_s, 0, \cdots, 0)$$

for $m = 0, 1, 2, \cdots$. This assures us that f_1 has the form

$$f_1(z,w) = \alpha\big(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}\big) z_1,$$

where $\alpha(u)$ is a holomorphic function on $\boldsymbol{C}$.

Next consider the functions f_{k+j}, $1 \le j \le \ell$. We have now the identity

$$f_{k+j}\big(X_{z,w,A,\theta_1,\cdots,\theta_\ell}\big) = \big(\exp 2\pi\sqrt{-1}\theta_j\big) f_{k+j}(z,w) \quad \text{on} \quad \Omega_k$$

for all $A \in SU(k-1)$ and all $\theta_1, \cdots, \theta_\ell \in \boldsymbol{R}$. Therefore, by the same reasoning as above, f_{k+j} has to be of the form $f_{k+j}(z,w) = f_{k+j}(z_1,w)$. So, denoting by $f_{k+j}(z_1,w) = \sum C_{\mu\nu} z_1^{\mu} w^{\nu}$ the Laurent series of f_{k+j}, we obtain this time

$$\sum_{i=1}^{s} \left(\frac{r_i}{\lambda}\mu + \nu_i\right)\theta_i + \sum_{i=s+1}^{\ell} \theta_i \nu_i = \theta_j$$

for all $\theta_1, \cdots, \theta_\ell \in \boldsymbol{R}$ whenever $C_{\mu\nu} \neq 0$. Thus

$$\begin{cases} \nu_j - 1 = -\dfrac{r_j}{\lambda}\mu, \\ \nu_i = -\dfrac{r_i}{\lambda}\mu, \quad 1 \le i \le s,\ i \neq j, \\ \nu_i = 0, \quad s+1 \le i \le \ell \end{cases} \quad \text{or} \quad \begin{cases} \nu_j = 1, \\ \nu_i = -\dfrac{r_i}{\lambda}\mu, \quad 1 \le i \le s, \\ \nu_i = 0, \quad s+1 \le i \le \ell,\ i \neq j \end{cases}$$

according as $1 \le j \le s$ or $s+1 \le j \le \ell$. Hence, these relations together yield that each function f_{k+j} has the form

$$f_{k+j}(z,w) = \gamma_j\big(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}\big) w_j, \quad 1 \le j \le \ell,$$

where $\gamma_j(u)$ is a holomorphic function on $\boldsymbol{C}$ and n_1, m_i are the same positive integers appearing in (2.17). At this point, notice that all the functions $\alpha(u)$ and $\gamma_j(u)$ are nowhere vanishing on $\boldsymbol{C}$, because f has the inverse mapping contained also in C_0. Moreover, it should be noted that f_1 and f_{k+j} extend to holomorphic functions on the whole space $\boldsymbol{C}^n$, since n_1 and m_i are all positive integers.

Our next task is to determine the form of $(f_2, \cdots, f_k)$. For this purpose, choose a point $(z_1^o, w_o) \in \boldsymbol{C} \times (\boldsymbol{C}^*)^\ell$ arbitrarily and define a mapping $L : \boldsymbol{C}^{k-1} \to \boldsymbol{C}^{k-1}$ by

$$L(z') = {}^t\big(f_2(z_1^o, z', w_o), \cdots, f_k(z_1^o, z', w_o)\big), \quad z' \in \boldsymbol{C}^{k-1}.$$

Then L is a holomorphic automorphism of $\boldsymbol{C}^{k-1}$ such that $L(Az') = A \cdot L(z')$ for all $A \in SU(k-1)$; accordingly, we may conclude by the same reasoning as in Case (1-1) that $(f_2, \cdots, f_k)$ can be written in the form

$$(f_2(z,w), \cdots, f_k(z,w)) = b(z_1,w) z'.$$

Accordingly $b(z_1,w)$ has to satisfy the condition:

$$b\big(\big(\exp 2\pi\sqrt{-1}\Theta\big) z_1, \big(\exp 2\pi\sqrt{-1}\theta_1\big) w_1, \cdots, \big(\exp 2\pi\sqrt{-1}\theta_\ell\big) w_\ell\big) = b(z_1,w)$$

for all $\theta_1, \cdots, \theta_\ell \in \boldsymbol{R}$. Now, by the Laurent series argument as above, one can see that $b(z_1,w)$ is expressed as

$$b(z_1,w) = \beta\big(z_1^{n_1} w_1^{m_1} \cdots w_s^{m_s}\big)$$

and hence

$$(f_2(z,w),\cdots,f_k(z,w)) = \beta\big(z_1^{n_1}w_1^{m_1}\cdots w_s^{m_s}\big)z',$$

where $\beta(u)$ is a holomorphic function on $\boldsymbol{C}$ and n_1, m_i are the same positive integers as in (2.17).

Summarizing our result obtained so far, we have shown that every element f of C_0 has the form

$$f(z,w) = \big(\alpha(u)z_1, \beta(u)z', \gamma_1(u)w_1, \cdots, \gamma_\ell(u)w_\ell\big), \quad u = z_1^{n_1}w_1^{m_1}\cdots w_s^{m_s}, \tag{2.18}$$

where $\alpha(u), \beta(u)$ and $\gamma_j(u)$ are nowhere vanishing holomorphic functions on $\boldsymbol{C}$, and n_1 and m_i are positive integers depending only on C_0. In particular, every element $f \in C_0$ can be regarded as a holomorphic automorphism of $\boldsymbol{C}^n$.

Finally we would like to show that the commutator group Z_0 of C_0 is abelian. We verify this only in the case when $\ell = 1$, since the verification in the general case is almost identical. To simplify discussion, we change the notations and write $(n_1, m_1) = (s,t)$ and $w_1 = w$. Now, take an arbitrary element $F \in C_0$. Then, by (2.18) F can be expressed in the form

$$F(z,w) = \big(\alpha(z_1^s w^t)z_1,\ \beta(z_1^s w^t)z',\ \gamma(z_1^s w^t)w\big),$$

where $\alpha(u), \beta(u)$ and $\gamma(u)$ are nowhere vanishing holomorphic functions on $\boldsymbol{C}$; and moreover, we know that F can be regarded as a holomorphic automorphism of $\boldsymbol{C}\times\boldsymbol{C}^{k-1}\times\boldsymbol{C} = \boldsymbol{C}^{k+1}$. Here we first assert that

$$\alpha(u)^s\gamma(u)^t = \alpha(0)^s\gamma(0)^t \quad \text{for every } u \in \boldsymbol{C}. \tag{2.19}$$

Indeed, define a holomorphic function f on $\boldsymbol{C}$ by

$$f(u) = \alpha(u)^s\gamma(u)^t u, \quad u \in \boldsymbol{C}$$

and consider the set

$$M_c = \big\{(z_1, 0, w) \in \boldsymbol{C}\times\boldsymbol{C}^{k-1}\times\boldsymbol{C} \big| z_1^s w^t = c\big\}, \quad c \in \boldsymbol{C}.$$

Then it is easy to see that $F(M_c) \subset M_{f(c)}$ for all $c \in \boldsymbol{C}$. This, combined with the fact that the inverse F^{-1} of F is also contained in C_0, yields at once that f is an automorphism of $\boldsymbol{C}$ with $f(0) = 0$. Consequently, there exists a non-zero constant A such that $f(u) = Au$ on $\boldsymbol{C}$. Clearly this implies our assertion (2.19), as desired. Thanks to (2.19), if we set $A = \alpha(0)^s\gamma(0)^t$, then F^{-1} may be given by

$$F^{-1}(z,w) = \big(\alpha(A^{-1}z_1^s w^t)^{-1}z_1,\ \beta(A^{-1}z_1^s w^t)^{-1}z',\ \gamma(A^{-1}z_1^s w^t)^{-1}w\big).$$

Let us take another element $G \in C_0$ having the form

$$G(z,w) = \big(\lambda(z_1^s w^t)z_1,\ \mu(z_1^s w^t)z',\ \nu(z_1^s w^t)w\big)$$

and set $B = \lambda(0)^s \nu(0)^t$. Then we can compute explicitly the commutator $[G,F] := G^{-1}\circ F^{-1}\circ G\circ F$ of G and F as follows:

$$[G,F](z,w) = \left(\frac{\lambda(Au)\alpha(u)}{\lambda(u)\alpha(Bu)}z_1,\ \frac{\mu(Au)\beta(u)}{\mu(u)\beta(Bu)}z',\ \frac{\nu(Au)\gamma(u)}{\nu(u)\gamma(Bu)}w\right), \quad u = z_1^s w^t.$$

Let us choose another pair $\tilde{F},\tilde{G}$ of elements belonging to C_0, where we denote the objects relative to $\tilde{F},\tilde{G}$ by the corresponding symbols of F,G with tilde. Then, by routine computations it follows that

$$\begin{aligned}[\tilde{G},\tilde{F}]\circ[G,F](z,w) = \Bigg(&\frac{\tilde{\lambda}(\tilde{A}u)\tilde{\alpha}(u)\lambda(Au)\alpha(u)}{\tilde{\lambda}(u)\tilde{\alpha}(\tilde{B}u)\lambda(u)\alpha(Bu)}z_1,\ \frac{\tilde{\mu}(\tilde{A}u)\tilde{\beta}(u)\mu(Au)\beta(u)}{\tilde{\mu}(u)\tilde{\beta}(\tilde{B}u)\mu(u)\beta(Bu)}z',\\ &\frac{\tilde{\nu}(\tilde{A}u)\tilde{\gamma}(u)\nu(Au)\gamma(u)}{\tilde{\nu}(u)\tilde{\gamma}(\tilde{B}u)\nu(u)\gamma(Bu)}w\Bigg), \quad u = z_1^s w^t,\end{aligned}$$

from which we may conclude that Z_0 is, in fact, abelian. Therefore the proof is completed in the case when $k \geq 3$.

Case (2-2). $k = 2$: In order to derive a contradiction, we again consider the subgroup G of $U(2)\times(U(1))^\ell$ defined in (2.12). Since $\Phi_s(A) = A$ for all $A \in SU(2)$ in (2.5), we obtain by (2.6) and (2.16) that

$$\begin{aligned}\Phi(G) = \Bigg\{\Bigg(&\Big(\exp\frac{2\pi\sqrt{-1}}{\lambda}\Big\{\Big(1+\sum_{i=1}^{s} r_i c_i\Big)\theta + \sum_{i=1}^{s} r_i\theta_i\Big\}\Big)g(\theta),\\ &\exp 2\pi\sqrt{-1}(c_1\theta+\theta_1),\cdots,\exp 2\pi\sqrt{-1}(c_\ell\theta+\theta_\ell)\Bigg)\,\Bigg|\,\theta,\theta_1,\cdots,\theta_\ell \in \boldsymbol{R}\Bigg\},\end{aligned}$$

where $g(\theta)$ is the element of $SU(2)$ appearing in (2.13). Let C, C_0 and Z, Z_0 be the same groups introduced in Case (1-2). Then we know already that Z and Z_0 are isomorphic, and Z is a non-abelian group. Therefore, the only thing which has to be proved now is that Z_0 is abelian. But this can be achieved by the same technique used in Case (2-1). Indeed, define the positive integers n_1, n_2 and m_i by

$$n_1 = \min\left\{n \in \boldsymbol{N}\,\middle|\,\frac{\lambda-1}{\lambda+1}n \in \boldsymbol{Z},\ \ -\frac{2r_i}{\lambda+1}n \in \boldsymbol{Z},\ \ 1 \leq i \leq s\right\},$$

$$n_2 = \frac{\lambda-1}{\lambda+1}n_1 \quad \text{and} \quad m_i = -\frac{2r_i}{\lambda+1}n_1, \quad 1 \leq i \leq s.$$

Then, in exactly the same way as in Case (2-1), one can show that every element $f \in C_0$ has the form

$$f(z,w) = (\alpha(u)z_1, \beta(u)z_2, \gamma_1(u)w_1, \cdots, \gamma_\ell(u)w_\ell), \quad u = z_1^{n_1}z_2^{n_2}w_1^{m_1}\cdots w_s^{m_s},$$

where all the functions $\alpha(u), \beta(u)$ and $\gamma_j(u)$ are nowhere vanishing holomorphic functions on $\boldsymbol{C}$. In particular, every $f \in C_0$ can be regarded as a holomorphic automorphism of $\boldsymbol{C}^n$. Thus, repeating the same computations as in Case (2-1), we conclude eventually that Z_0 is an abelian group, as desired. This completes the proof of Theorem 1.

3. Proof of Theorem 2.

We retain the notation in the previous section.

Now, as in Theorem 2 stated in the introduction, let M be a connected complex manifold of dimension n that is holomorphically separable and admits a smooth envelope of holomorphy, and assume that there exists an isomorphism $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(M)$ between the topological groups $\mathrm{Aut}(\Omega_k)$ and $\mathrm{Aut}(M)$. Since Ω_k is a Reinhardt domain in $\boldsymbol{C}^n$, we have the injective continuous group homomorphism $\rho_{\Omega_k} : T^n \to \mathrm{Aut}(\Omega_k)$. Considering the composition of ρ_{Ω_k} and Φ, we obtain an injective continuous group homomorphism $\Phi \circ \rho_{\Omega_k} : T^n \to \mathrm{Aut}(M)$. Thus, by the same reasoning as in the proof of the first equality of (2.1), we may assume that M is a Reinhardt domain D in $\boldsymbol{C}^n$ and we have a topological group isomorphism $\Phi : \mathrm{Aut}(\Omega_k) \to \mathrm{Aut}(D)$ such that $\Phi(T(\Omega_k)) = T(D)$.

In the following, we will show our Theorem 2 in three steps. Firstly, we will show that D has the form $D = \Omega_{k'} - \bigcup_\lambda V_\lambda$ for some k' with $0 \leq k' \leq n$ by looking at the centralizers of the standard tori $T(\Omega_k)$ and $T(D)$, where $\bigcup_\lambda V_\lambda$ is the union of some of coordinate vector subspaces V_λ of codimension greater than or equal to two. Precisely speaking, V_λ is a vector subspace of $\boldsymbol{C}^n$ of codimension greater than or equal to two defined by the condition that some of the coordinate components of $\boldsymbol{C}^n$ are equal to 0. Secondly, we will show $k' = k$ by looking at the normalizers of the standard tori $T(\Omega_k)$ and $T(D)$. Finally, we will show that $\bigcup_\lambda V_\lambda$ must be empty, and hence D coincides with Ω_k by our Theorem 1.

3.1. First step. Since the group isomorphism Φ maps $T(\Omega_k)$ onto $T(D)$, it maps the centralizer of $T(\Omega_k)$ in $\mathrm{Aut}(\Omega_k)$ onto the centralizer of $T(D)$ in $\mathrm{Aut}(D)$. By Lemma 1.1, the centralizers of $T(\Omega_k)$ and $T(D)$ are given by $\Pi(\Omega_k)$ and $\Pi(D)$, respectively. Therefore $\Pi(\Omega_k)$ and $\Pi(D)$ are isomorphic as Lie groups. Since $\Pi(\Omega_k)$ is isomorphic to $(\boldsymbol{C}^*)^n$, and hence has real dimension $2n$, $\Pi(D)$ also has real dimension $2n$. This implies that π_α maps D onto D for all elements α of $(\boldsymbol{C}^*)^n$, and consequently, π_α maps the complement D^c of D in $\boldsymbol{C}^n$ onto itself for all α in $(\boldsymbol{C}^*)^n$. In particular, for a point z_0 of $\boldsymbol{C}^n$, the set $\Pi \cdot z_0$ consisting of all elements $\pi_\alpha(z_0)$, α running through all of $(\boldsymbol{C}^*)^n$, is contained in either D or D^c. We can take as z_0 a point of D belonging to $(\boldsymbol{C}^*)^n$. Then D contains $\Pi \cdot z_0$, which is $(\boldsymbol{C}^*)^n$. Therefore D^c is a closed subset contained in the union of all coordinate hyperplanes of $\boldsymbol{C}^n$. Take any point z_0 of D^c. Then some of the coordinate components of z_0 are 0, and the closure of $\Pi \cdot z_0$ is a coordinate vector subspace of $\boldsymbol{C}^n$, which is contained in D^c. This shows that D^c is the union of some of coordinate vector subspaces. By a permutation of coordinates, we may assume that D^c is the union of the $n - k'$ coordinate hyperplanes $\{z_i = 0\}$, $i = k'+1, \cdots, n$, and some of coordinate vector subspaces of codimension greater than or equal to two. Then D has the form $D = \Omega_{k'} - \bigcup_\lambda V_\lambda$, as required.

3.2. Second step. We need the following lemma.

LEMMA 3.1. *Let $\Omega_p^\sharp$ and $\Omega_{p'}^\sharp$ be Reinhardt domains in $\boldsymbol{C}^n$ of the form*

$$\Omega_p^\sharp = \Omega_p - \bigcup_\lambda V_\lambda \quad \text{and} \quad \Omega_{p'}^\sharp = \Omega_{p'} - \bigcup_{\lambda'} V'_{\lambda'},$$

where $\bigcup_\lambda V_\lambda$ and $\bigcup_{\lambda'} V'_{\lambda'}$ are the unions of some of coordinate vector subspaces of codimension greater than or equal to two, respectively. If the groups $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)$ and $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)$ are isomorphic as topological groups, then $p = p'$.

This lemma is applied as follows. Since the group isomorphism Φ maps $T(\Omega_k)$ onto $T(D)$, it maps the normalizer of $T(\Omega_k)$ in $\mathrm{Aut}(\Omega_k)$ onto the normalizer of $T(D)$ in $\mathrm{Aut}(D)$. By Lemma 1.1, the normalizers of $T(\Omega_k)$ and $T(D)$ are given by $\mathrm{Aut}_{\mathrm{alg}}(\Omega_k)$ and $\mathrm{Aut}_{\mathrm{alg}}(D)$, respectively. Therefore $\mathrm{Aut}_{\mathrm{alg}}(\Omega_k)$ and $\mathrm{Aut}_{\mathrm{alg}}(D)$ are isomorphic as topological groups. Since $D = \Omega_{k'}^\sharp$ by the first step, it follows from Lemma 3.1 that $k = k'$, and the second step is shown.

We now prove Lemma 3.1. We begin with some preliminary observations. For an element $z = (z_i)$ of $\boldsymbol{C}^n$ and an element $\mathrm{N} = (\nu_{ij})$ of $GL(n, \boldsymbol{Z})$, we denote by z^{N} the element of $\boldsymbol{C}^n$ whose i-th component is given by $z_1^{\nu_{i1}} \cdots z_n^{\nu_{in}}$ (when it is well-defined). For elements $z = (z_i)$ and $w = (w_i)$ of $\boldsymbol{C}^n$, we denote by $z \cdot w$ the element of $\boldsymbol{C}^n$ whose i-th component is given by $z_i w_i$. When $z, w \in \boldsymbol{C}^n$ and $\mathrm{M}, \mathrm{N} \in GL(n, \boldsymbol{Z})$, it holds that

$$(z^{\mathrm{N}})^{\mathrm{M}} = z^{\mathrm{MN}}, \quad (z \cdot w)^{\mathrm{N}} = z^{\mathrm{N}} \cdot w^{\mathrm{N}}. \tag{3.1}$$

Let D be a Reinhardt domain in $\boldsymbol{C}^n$ and let φ be an element of $\mathrm{Aut}(D)$. Then it follows from the proof of [**8**, Section 2, Proposition 1] that φ has the form $\varphi(z) = \gamma \cdot z^{\mathrm{N}}$ if and only if $\varphi T(D) \varphi^{-1} = T(D)$ and we have

$$\varphi \circ \pi_\alpha \circ \varphi^{-1} = \pi_{\alpha^{\mathrm{N}}} \quad \text{for } \alpha \in T^n = (U(1))^n,$$

where $\gamma \in (\boldsymbol{C}^*)^n$, $\mathrm{N} \in GL(n, \boldsymbol{Z})$.

Under the assumption of Lemma 3.1, suppose that $p \neq p'$, and we will derive a contradiction. We may assume without loss of generality that $p < p'$. By assumption, there exists an isomorphism $\Phi : \mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp) \to \mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)$ between the topological groups $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)$ and $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)$. Since $\Pi(\Omega_p^\sharp)$ and $\Pi(\Omega_{p'}^\sharp)$ are the identity components of $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)$ and $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)$, respectively, and since $T(\Omega_p^\sharp)$ and $T(\Omega_{p'}^\sharp)$ are the unique maximal compact subgroups of $\Pi(\Omega_p^\sharp)$ and $\Pi(\Omega_{p'}^\sharp)$, respectively, we have

$$\Phi\big(\Pi\big(\Omega_p^\sharp\big)\big) = \Pi\big(\Omega_{p'}^\sharp\big), \tag{3.2}$$

$$\Phi\big(T\big(\Omega_p^\sharp\big)\big) = T\big(\Omega_{p'}^\sharp\big). \tag{3.3}$$

By (3.2), $\Phi : \mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp) \to \mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)$ induces a group isomorphism between the quotient groups $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)/\Pi(\Omega_p^\sharp)$ and $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)/\Pi(\Omega_{p'}^\sharp)$, which we denote also by Φ. Here consider the group homomorphism $\varpi : \mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp) \to GL(n, \boldsymbol{Z})$ that sends an element φ of $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)$ written in the form $\varphi(z) = \gamma \cdot z^{\mathrm{N}}$ to the element N of $GL(n, \boldsymbol{Z})$, where $\gamma \in (\boldsymbol{C}^*)^n$, $\mathrm{N} \in GL(n, \boldsymbol{Z})$. Then we have $\ker \varpi = \Pi(\Omega_p^\sharp)$ and $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)/\Pi(\Omega_p^\sharp)$ is isomorphic to the subgroup $\mathscr{G} := \varpi(\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp))$ of $GL(n, \boldsymbol{Z})$. It is readily verified that $\mathscr{G}$ consists of all matrices N in $GL(n, \boldsymbol{Z})$ of the form

$$\mathrm{N} = \begin{pmatrix} A & B \\ O & C \end{pmatrix}, \quad A \in G, \quad B \in M(p, n-p, \boldsymbol{Z}), \quad C \in GL(n-p, \boldsymbol{Z}), \tag{3.4}$$

where G is a subgroup of the linear symmetric group S_p of degree p and $M(p, n-p, \boldsymbol{Z})$ is the set of all p by $(n-p)$ integral matrices. Similarly, $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)/\Pi(\Omega_{p'}^\sharp)$ is isomorphic to the subgroup $\mathscr{G}'$ of $GL(n, \boldsymbol{Z})$ consisting of all matrices N' of the form

$$\mathrm{N}' = \begin{pmatrix} A' & B' \\ O & C' \end{pmatrix}, \quad A' \in G', \quad B' \in M(p', n-p', \boldsymbol{Z}), \quad C' \in GL(n-p', \boldsymbol{Z}), \tag{3.5}$$

where G' is a subgroup of the linear symmetric group $S_{p'}$ of degree p' and $M(p', n-p', \boldsymbol{Z})$ is the set of all p' by $(n-p')$ integral matrices.

Sublemma. *Under the isomorphisms above of* $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)/\Pi(\Omega_p^\sharp)$ *onto* $\mathscr{G}$ *and of* $\mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)/\Pi(\Omega_{p'}^\sharp)$ *onto* $\mathscr{G}'$*, the group isomorphism* $\tilde{\Phi} : \mathscr{G} \to \mathscr{G}'$ *induced from* $\Phi :$ $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)/\Pi(\Omega_p^\sharp) \to \mathrm{Aut}_{\mathrm{alg}}(\Omega_{p'}^\sharp)/\Pi(\Omega_{p'}^\sharp)$ *has the form* $\tilde{\Phi}(\mathrm{N}) = \mathrm{LNL}^{-1}$ *for* $\mathrm{N} \in \mathscr{G}$, *where* L *is some element of* $GL(n, \boldsymbol{Z})$.

Proof. By (3.3), Φ gives a continuous group isomorphism between the tori $T(\Omega_p^\sharp)$ and $T(\Omega_{p'}^\sharp)$. Therefore we have

$$\Phi(\pi_\alpha) = \pi_{\alpha^{\mathrm{L}}} \quad \text{for } \alpha \in T^n = (U(1))^n, \tag{3.6}$$

where L is some element of $GL(n, \boldsymbol{Z})$. Let N be any element of $\mathscr{G}$ and let φ be an element of $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)$ such that the coset belonging to φ corresponds to N under the isomorphism between $\mathrm{Aut}_{\mathrm{alg}}(\Omega_p^\sharp)/\Pi(\Omega_p^\sharp)$ and $\mathscr{G}$. Then φ satisfies the condition that

$$\varphi \circ \pi_\alpha \circ \varphi^{-1} = \pi_{\alpha^{\mathrm{N}}} \quad \text{for } \alpha \in T^n = (U(1))^n. \tag{3.7}$$

Applying Φ to the both sides of (3.7) and using (3.6) and (3.1), we obtain

$$\begin{aligned} \Phi(\varphi) \circ \pi_{\alpha^{\mathrm{L}}} \circ \Phi(\varphi)^{-1} &= \Phi(\varphi) \circ \Phi(\pi_\alpha) \circ \Phi(\varphi)^{-1} \\ &= \Phi(\pi_{\alpha^{\mathrm{N}}}) = \pi_{(\alpha^{\mathrm{N}})^{\mathrm{L}}} = \pi_{\alpha^{\mathrm{LN}}} \end{aligned}$$

for $\alpha \in T^n = (U(1))^n$. By putting $\beta = \alpha^{\mathrm{L}}$, this implies that

$$\Phi(\varphi) \circ \pi_\beta \circ \Phi(\varphi)^{-1} = \pi_{\left(\beta^{\mathrm{L}^{-1}}\right)^{\mathrm{LN}}} = \pi_{\beta^{\mathrm{LNL}^{-1}}}$$

for $\beta \in T^n = (U(1))^n$, and the coset belonging to $\Phi(\varphi)$ corresponds to LNL^{-1} under the isomorphism between $\mathrm{Aut}_{\mathrm{alg}}(\Omega^{\sharp}_{p'})/\Pi(\Omega^{\sharp}_{p'})$ and $\mathscr{G}'$. We thus conclude that $\tilde{\Phi}(\mathrm{N}) = \mathrm{LNL}^{-1}$. □

When $p = 0$, we have $\mathscr{G} = GL(n, \boldsymbol{Z})$ by (3.4). It follows from Sublemma that

$$\mathscr{G}' = \tilde{\Phi}(\mathscr{G}) = \mathrm{L}GL(n, \boldsymbol{Z})\mathrm{L}^{-1} = GL(n, \boldsymbol{Z}).$$

But, since $p' > p = 0$, this contradicts (3.5). So let us consider the case of $p > 0$. For $i = 1, \cdots, p$, $j = 1, \cdots, n-p$, and $\ell \in \boldsymbol{Z}$, we denote by $\mathrm{N}_{ij}(\ell)$ the matrix in $\mathscr{G}$ of the form

$$\begin{pmatrix} E_p & \ell B_{ij} \\ O & E_{n-p} \end{pmatrix}$$

where B_{ij} is the p by $(n-p)$ integral matrix whose (i, j) entry is equal to 1 and whose entries except for (i, j) are all equal to 0, and the notation E_k denotes the identity matrix of degree k for an integer $k \geq 1$. Also, we denote by τ the group homomorphism of $\mathscr{G}'$ into $GL(p', \boldsymbol{C})$ that sends N' written in the form (3.5) to A'. Note that $\mathrm{N}_{ij}(\ell) = \mathrm{N}_{ij}(1)^{\ell}$. Write

$$A'_0 = \tau(\tilde{\Phi}(\mathrm{N}_{ij}(1))). \tag{3.8}$$

Since A'_0 is an element of the linear symmetric group $S_{p'}$, there exists an element P' of $GL(p', \boldsymbol{C})$ such that

$$P' {A'_0}^{\ell} {P'}^{-1} = \begin{pmatrix} \lambda_1^{\ell} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \lambda_{p'}^{\ell} \end{pmatrix} \quad \text{for every } \ell \in \boldsymbol{Z}, \tag{3.9}$$

where λ_i, $i = 1, \cdots, p'$, are complex constants of absolute value 1. On the other hand, by (3.8) and Sublemma, we have

$${A'_0}^{\ell} = \tau\big(\mathrm{LN}_{ij}(\ell)\mathrm{L}^{-1}\big). \tag{3.10}$$

From the equation

$$\begin{pmatrix} \lambda_1^{\ell} & & \mathrm{O} \\ & \ddots & \\ \mathrm{O} & & \lambda_{p'}^{\ell} \end{pmatrix} = {P'}^{-1} \tau\big(\mathrm{LN}_{ij}(\ell)\mathrm{L}^{-1}\big) P'$$

obtained by combining (3.9) with (3.10), it follows that

$$\lambda_i^{\ell} = a_i \ell + b_i, \quad i = 1, \cdots, p', \tag{3.11}$$

where a_i, b_i, $i = 1, \cdots, p'$, are complex constants independent on ℓ. If $a_i \neq 0$ in (3.11), then, letting $\ell \to +\infty$, we see that the left hand side of (3.11) is bounded, while the right hand side of (3.11) is unbounded, which is a contradiction. Hence we have $a_i = 0$, or λ_i^ℓ is equal to the constant b_i for every $\ell \in \boldsymbol{Z}$. This imples that λ_i must be 1, so that A'_0 is the identity matrix. We have thus shown that

$$\tau\bigl(\mathrm{LN}_{ij}(\ell)\mathrm{L}^{-1}\bigr) = E_{p'} \quad \text{for every } \ell \in \boldsymbol{Z}. \tag{3.12}$$

Write

$$\mathrm{L} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad \text{and} \quad \mathrm{L}^{-1} = \begin{pmatrix} P^* & Q^* \\ R^* & S^* \end{pmatrix}$$

where P, P^* are p by p integral matrices, Q, Q^* are p by $(n-p)$ integral matrices, R, R^* are $(n-p)$ by p integral matrices, and S, S^* are $(n-p)$ by $(n-p)$ integral matrices. Note that we have

$$RP^* + SR^* = O. \tag{3.13}$$

The equation (3.12) above yields that, for every $B \in M(p, n-p, \boldsymbol{Z})$, the matrix

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} E_p & B \\ O & E_{n-p} \end{pmatrix} \begin{pmatrix} P^* & Q^* \\ R^* & S^* \end{pmatrix}$$

has the form

$$\begin{pmatrix} E_{p'} & B' \\ O & C' \end{pmatrix},$$

where $B' \in M(p', n-p', \boldsymbol{Z})$ and $C' \in GL(n-p', \boldsymbol{Z})$. Since $p' > p$, it follows from this fact and (3.13) that

$$RBR^* = O \quad \text{for every } B \in M(p, n-p, \boldsymbol{Z}),$$

which implies that $R = R^* = O$. Therefore, by noting (3.4), we see that $\tilde{\varPhi}(\mathscr{G}) = \mathrm{L}\mathscr{G}\mathrm{L}^{-1}$ consists of all matrices M in $GL(n, \boldsymbol{Z})$ of the form

$$\mathrm{M} = \begin{pmatrix} A & B \\ O & C \end{pmatrix}, \quad A \in PGP^{-1}, \quad B \in M(p, n-p, \boldsymbol{Z}), \quad C \in GL(n-p, \boldsymbol{Z}).$$

On the other hand, $\tilde{\varPhi}(\mathscr{G})$ must coincide with $\mathscr{G}'$ as sets. It is easily seen that, when $p' > p$, this is impossible, and the proof of Lemma 3.1 is completed.

3.3. Third and final step. To begin with, when k is 0 or 1, the set $\bigcup_\lambda V_\lambda$ is clearly empty and hence $D = \varOmega_k$, because $(\boldsymbol{C}^*)^n$ or $\boldsymbol{C} \times (\boldsymbol{C}^*)^{n-1}$ can not contain a coordinate vector subspace of codimension greater than or equal to two. So, let us consider the case of $k \geq 2$. Noting that $\mathrm{Aut}(\Omega_k)$ contains the subgroup $U(k) \times (U(1))^{\ell}$, we set $G = \Phi(U(k) \times (U(1))^{\ell})$. Then G is a connected compact subgroup of $\mathrm{Aut}(D)$ containing $T(D)$. Thus, repeating exactly the same argument as in the proof of the second equality of (2.1), we can assume that G coincides with just $U(k) \times (U(1))^{\ell}$ as sets; consequently, $\mathrm{Aut}(D)$ contains $U(k) \times (U(1))^{\ell}$. By considering the $U(k) \times (U(1))^{\ell}$-orbits on D, we now obtain that $D \supset \Omega_k^0$. Since $\{0\} \times (\boldsymbol{C}^*)^{\ell} \subset D$ if $(\{0\} \times (\boldsymbol{C}^*)^{\ell}) \cap D \neq \varnothing$, we have the following two cases: $D = \Omega_k$ and $D = \Omega_k^0$. If $D = \Omega_k^0$, then our Theorem 1 tells us that $\mathrm{Aut}(D)$ is not isomorphic to $\mathrm{Aut}(\Omega_k)$ as topological groups. Therefore we conclude that $D = \Omega_k$; completing the proof of Theorem 2.

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