Asymptotic behavior of solutions for damped wave equations with non-convex convection term on the half line

ASYMPTOTIC BEHAVIOR OF SOLUTIONS FOR DAMPED WAVE EQUATIONS WITH NON-CONVEX CONVECTION TERM ON THE HALF LINE

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Abstract. We study the asymptotic stability of nonlinear waves for damped wave equations with a convection term on the half line. In the case where the convection term satisfies the convex and sub-characteristic conditions, it is known by the work of Ueda [7] and Ueda-Nakamura-Kawashima [10] that the solution tends toward a stationary solution. In this paper, we prove that even for a quite wide class of the convection term, such a linear superposition of the stationary solution and the rarefaction wave is asymptotically stable. Moreover, in the case where the solution tends to the non-degenerate stationary wave, we derive that the time convergence rate is polynomially (resp. exponentially) fast if the initial perturbation decays polynomially (resp. exponentially) as $x \to \infty$. Our proofs are based on a technical L^2 weighted energy method.

1. Introduction

We consider the initial-boundary value problem on the half line for a damped wave equation with a nonlinear convection term:

$$\begin{cases} u_{tt} - u_{xx} + u_t + f(u)_x = 0, & x > 0,\ t > 0, \\ u(0,t) = u_-, & t > 0, \\ \lim_{x\to\infty} u(x,t) = u_+, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x > 0, \end{cases} \tag{1.1}$$

where the function $f = f(u)$ is a given C^2 function satisfying $f(0) = 0$ and $u_\pm$ are given constants with $u_- < u_+$. In this problem, we assume that the initial data $u_0(x)$ satisfies $u_0(0) = u_-$ and $\lim_{x\to\infty} u_0(x) = u_+$ as the compatibility conditions. Throughout this paper, we impose the convex and sub-characteristic conditions at the origin:

$$f''(0) > 0, \quad |f'(0)| < 1, \quad f(u) > f(0) = 0 \quad \text{for} \quad u \in [u_-, 0). \tag{1.2}$$

For the viscous conservation laws on the half line, Liu-Matsumura-Nishihara [3] investigated the case where the flux is convex and the corresponding Riemann problem for the hyperbolic part admits the transonic rarefaction wave. More precisely, it was shown in [3] that depending on the signs of the characteristic speeds,

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the large-time behavior of the solutions is classified into three cases. On the other hand, Ueda-Kawashima [9] and Ueda [7, 8] suggested that the dissipative structure of (1.1) is similar to one of viscous conservation laws. Indeed, Ueda [7] considered the problem (1.1) with $u_+ = 0$ and showed that if the flux $f(u)$ of (1.1) satisfies

$$f''(u) > 0, \quad |f'(u)| < 1 \quad \text{for} \quad u \in [u_-, 0], \tag{1.3}$$

then the solution of (1.1) tends toward the stationary solution ϕ, provided that the initial perturbation is suitably small. Here, the stationary solution $\phi = \phi(x)$ is defined by the solution of the stationary problem corresponding to (1.1):

$$\begin{cases} f(\phi) = \phi_x, & x > 0, \\ \phi(0) = u_-, \quad \lim\limits_{x\to\infty} \phi(x) = 0. \end{cases} \tag{1.4}$$

In the case where the flux is not necessarily convex, Liu-Nishihara [4] and Hashimoto-Matsumura [1] studied respectively the asymptotic stability of a viscous shock wave and superpositions of stationary solution and rarefaction wave. Especially, in order to obtain the stability result, Hashimoto-Matsumura [1] introduced a useful weight function and handled the weighted L^2 energy method.

Under the above consideration, we can expect that the asymptotic stability of the nonlinear waves holds true for the problem (1.1) under the non-convex condition (1.2). Therefore, we first treat the case $u_- < 0 < u_+$ and the condition (1.2) with $f'(0) = 0$, and derive that the solution of (1.1) tends to the superposition of the stationary solution ϕ connecting u_- and 0 and the rarefaction wave ψ^R connecting 0 and u_+. Here, the rarefaction wave $\psi^R = \psi^R(x/t)$ is concretely given by

$$\psi^R(x/t) = \begin{cases} 0, & x \le 0, \\ (f')^{-1}(x/t), & 0 \le x \le f'(u_+)t, \\ u_+, & x \ge f'(u_+)t. \end{cases} \tag{1.5}$$

We emphasize that the sub-characteristic condition is enough to be imposed only on $u = 0$.

Additionally, Ueda [7] and Ueda-Nakamura-Kawashima [11] considered the convergence rate to the stationary solution for the problem (1.1) with $u_+ = 0$. Ueda [7] derived the polynomially and exponentially convergence rate to the non-degenerate stationary solution, and Ueda-Nakamura-Kawashima [11] obtained the polynomially convergence rate to the degenerate stationary solution under the condition (1.3). At the second and third results of the present paper, we focus on the stationary solution and show the convergence rate under the non-convex condition (1.2).

This paper is organized as follows. The main theorems are given in Section 2. In Section 3, we reformulate our initial-boundary value problem (1.1) and state some preliminaries. In Section 4, we prove the asymptotic stability result under the non-convex condition (1.2) by using the weighted energy method. Finally, we focus on the stationary solution and obtain the polynomially and exponentially convergence rate of the solutions by using the space-time weighted energy method in Section 5.

Notations. We denote by $L^2 = L^2(\mathbb{R}_+)$ the usual Lebesgue space over $\mathbb{R}_+$ with the norm $\|\cdot\|_{L^2}$, and $H^1 = H^1(\mathbb{R}_+)$ the corresponding first order Sobolev space with the norm $\|\cdot\|_{H^1}$. Moreover, $H^1_0 = H^1_0(\mathbb{R}_+)$ denotes the space of functoins $f \in H^1$ with $f(0) = 0$, as a subspace of H^1.

For $\alpha > 0$, $L^2_\alpha = L^2_\alpha(\mathbb{R}_+)$ denotes the polynomially weighted L^2 space with the norm

$$\|f\|_{L^2_\alpha} := \Big(\int_0^\infty (1+x)^\alpha |f(x)|^2 dx\Big)^{1/2},$$

while $L^2_{\alpha,exp} = L^2_{\alpha,exp}(\mathbb{R}_+)$ denotes the exponentially weighted L^2 space with the norm

$$\|f\|_{L^2_{\alpha,exp}} := \Big(\int_0^\infty e^{\alpha x} |f(x)|^2 dx\Big)^{1/2}.$$

Similarly, we define the corresponding weighted Sobolev spaces $H^1_\alpha = H^1_\alpha(\mathbb{R}_+)$ and $H^1_{\alpha,exp} = H^1_{\alpha,exp}(\mathbb{R}_+)$.

For an interval I and a Banach space X, $C^k(I;X)$ denotes the space of k-times continuously differentiable functions on the interval I with values in X. Finally, letters C and c in this paper are defined as positive generic constants unless they need to be distinguished.

2. Main Theorems

In this section, we state our main results. The first theorem is the asymptotic stability of the superposition of the stationary solution and the rarefaction wave under the condition (1.2).

Theorem 2.1. *Suppose that $u_- < 0 < u_+$, $f'(0) = 0$ and (1.2) hold. Assume that $u_0 - u_+ \in H^1$ and $u_1 \in L^2$. Let $\phi(x)$ be the stationary solution satisfying the problem (1.4) and $\psi^R(x/t)$ be the rarefaction wave given by (1.5). Then there exists a positive constant ε_0 such that, if $u_+ \le \varepsilon_0$ and $\|u_0 - \phi - \psi^R(\cdot)\|_{H^1} + \|u_1\|_{L^2} \le \varepsilon_0$, then the initial-boundary value problem (1.1) has a unique global solution in time u satisfying*

$$u - u_+ \in C^0([0,\infty); H^1), \qquad u_x, u_t \in L^2(0,\infty; L^2),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} \big|u(x,t) - \phi(x) - \psi^R(x/t)\big| = 0. \tag{2.1}$$

When we consider the case $u_+ = 0$, we obtain the following corollary.

Corollary 2.2. *Suppose that $u_+ = 0$ and (1.2) hold true. Assume that $u_0 - \phi \in H^1$ and $u_1 \in L^2$. Let $\phi(x)$ be the stationary solution satisfying the problem (1.4). Then there exists a positive constant ε_1 such that, if $\|u_0 - \phi\|_{H^1} + \|u_1\|_{L^2} \le \varepsilon_1$, then the initial-boundary value problem (1.1) has a unique global solution in time u satisfying*

$$u - \phi \in C^0([0,\infty); H^1_0), \qquad (u-\phi)_x, u_t \in L^2(0,\infty; L^2),$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x>0} |u(x,t) - \phi(x)| = 0. \tag{2.2}$$

The proof of Corollary 2.2 is completely same as in Theorem 2.1 and omitted here. The second purpose of this paper is to get the convergence rates of the solution u toward the stationary wave ϕ. Both theorems are concerned with the non-degenerate case $f'(0) < 0$. Theorem 2.3 and 2.4 give the polynomial and the exponential stability result, respectively.

Theorem 2.3. *Suppose that $u_+ = 0$, $f'(0) < 0$ and (1.2) hold true. Let $\phi(x)$ be the stationary wave of the problem (1.4), and $u(x,t)$ be the global solution to the problem (1.1) which is constructed in Corollary 2.2. If $u_0 - \phi \in H^1_\alpha$ and $u_1 \in L^2_\alpha$ for $\alpha \geq 0$, then we have*

$$\|u(t) - \phi\|_{H^1} \leq CE_\alpha(1+t)^{-\alpha/2} \tag{2.3}$$

for $t \geq 0$, where C is a positive constant and $E_\alpha := \|u_0 - \phi\|_{H^1_\alpha} + \|u_1\|_{L^2_\alpha}$.

Theorem 2.4. *Suppose that the same conditions as in Theorem 2.3 hold true. Then, if $u_0 - \phi \in H^1_{\alpha,exp}$ and $u_1 \in L^2_{\alpha,exp}$ for $\alpha > 0$, then we obtain*

$$\|u(t) - \phi\|_{H^1} \leq CE_{\alpha,exp}e^{-\beta t}$$

for $t \geq 0$, where β is a positive constant depending on α, C is a positive constant and $E_{\alpha,exp} := \|u_0 - \phi\|_{H^1_{\alpha,exp}} + \|u_1\|_{L^2_{\alpha,exp}}$.

Remark.

Corollary 2.2 and Theorems 2.3, 2.4 become the extensions of the asymptotic stability result in [7].

3. Reformulation of the problem

In this section, we make preparations for the proofs of Theorem 2.1, 2.3 and 2.4. Let $\phi(x)$ be the stationary solution satisfying (1.4) and let $\psi^R(x/t)$ be the rarefaction wave given by (1.5). As in the previous works, we introduce a smooth approximation $\psi(x,t)$ of $\psi^R(x/t)$ and define

$$\Phi(x,t) = \phi(x) + \psi(x,t) \tag{3.1}$$

as an approximation of our asymptotic solution $\phi(x)+\psi^R(x/t)$. Then we reformulate our problem (1.1) by introducing the perturbation $v(x,t)$ by

$$u(x,t) = \Phi(x,t) + v(x,t). \tag{3.2}$$

This is the standard strategy for solving our stability problem.

To complete this procedure, we first review the fundamental properties of the stationary solution $\phi(x)$ which satisfies (1.4). For its proof, we refer the reader to [3, 4, 7].

Lemma 3.1. *Suppose that (1.2). Then the stationary problem (1.4) has a unique smooth solution $\phi(x)$ satisfying $u_- < \phi(x) < 0$ and $\phi_x(x) > 0$ for $x > 0$. Moreover, for the non-degenerate case $f'(0) < 0$, we have*

$$|\partial_x^k \phi(x)| \leq Ce^{-cx}, \quad x \geq 0$$

for each nonnegative integer k. On the other hand, for the degenerate case $f'(0) = 0$, we obtain

$$|\partial_x^k \phi(x)| \le C(1+x)^{-k-1}, \quad x \ge 0$$

for each nonnegative integer k.

Next we introduce a smooth approximation of our rarefaction wave $\psi^R(x/t)$. We use the approximation due to Matsumura and Nishihara [5], which is defined by

$$\psi(x,t) = (f')^{-1}(\omega(x,t))|_{x\ge 0}, \tag{3.3}$$

where $\omega(x,t)$ is the smooth solution of the following Cauchy problem for the Burgers equation:

$$\begin{cases} w_t + ww_x = 0, & x \in \mathbb{R}, \quad t > 0, \\ w(x,0) = f'(u_+)\tanh x, & x \in \mathbb{R}. \end{cases}$$

We note that our approximation $\psi(x,t)$ in (3.3) is well-defined if $f(u)$ is strictly convex on $[0,u_+]$; this is true even in the case (1.2) if u_+ is suitably small. Then, by a simple calculation, we see that $\psi(x,t)$ satisfies

$$\begin{cases} \psi_t + f(\psi)_x = 0, & x > 0, \quad t > 0, \\ \psi(0,t) = 0, & t \ge 0. \end{cases} \tag{3.4}$$

Let $\psi_0(x) := \psi(x,0) = (f')^{-1}(\omega(x,0))|_{x\ge 0}$. Furthermore, the approximation $\psi(x,t)$ satisfies the following properties which are proved in [5].

Lemma 3.2. *Suppose that* (1.2) *with $f'(0) = 0$ and $f(u)$ is strictly convex on $[0,u_+]$. Then we have:*

1) *$0 < \psi(x,t) < u_+$ and $\psi_x(x,t) > 0$ for $x > 0$ and $t > 0$.*

2) *For $1 \le p \le \infty$, there exists a positive constant C such that*

$$\begin{aligned} &\|\psi_x(t)\|_{L^p} \le C\min\{u_+,\ u_+^{1/p}(1+t)^{-1+1/p}\}, \\ &\|\psi_{xx}(t)\|_{L^p} \le C\min\{u_+,\ (1+t)^{-1}\}, \\ &\|\psi_{xxx}(t)\|_{L^p} \le C\min\{u_+,\ (1+t)^{-1}\}. \end{aligned}$$

3) *$\psi(x,t)$ is an approximation of $\psi^R(x/t)$ in the sense that*

$$\lim_{t\to\infty}\sup_{x>0}|\psi(x,t) - \psi^R(x/t)| = 0.$$

We consider our approximation $\Phi(x,t)$ defined by (3.1). By using (1.4) and (3.4), we find that $\Phi(x,t)$ satisfies

$$\begin{cases} \Phi_{tt} - \Phi_{xx} + \Phi_t + f(\Phi)_x = h & x > 0, \quad t > 0, \\ \Phi(0,t) = u_-, & t \ge 0. \end{cases} \tag{3.5}$$

where the error term h is

$$\begin{aligned} h :=& (f(\Phi) - f(\phi) - f(\psi))_x + \psi_{tt} - \psi_{xx} \\ =& (f'(\phi+\psi) - f'(\phi))\phi_x + (f'(\phi+\psi) - f'(\psi))\psi_x + \psi_{tt} - \psi_{xx} \\ =& O(|\psi||\phi_x| + |\phi||\psi_x| + |\psi_{tt}| + |\psi_{xx}|). \end{aligned} \tag{3.6}$$

Also, we note that $u_- < \Phi(x,t) < u_+$ and $\Phi_x(x,t) > 0$ for $x > 0$ and $t \geq 0$. Moreover, using the estimates in Lemmas 3.1 and 3.2, we can estimate the error term h in (3.5) as follows.

Lemma 3.3. *For the error term h defined by* (3.6)*, we estimate*

$$\|h(t)\|_{L^p} \leq C \min\{u_+,\ \sigma(t)(1+t)^{-1}\}$$

for $1 \leq p \leq \infty$, where $\sigma(t) = \log(2+t)$ for $p = 1$ and $\sigma(t) = 1$ for $1 < p \leq \infty$, and C is a positive constant independent of u_+.

We omit the proof and refer the readers to [3, 1].

Finally we introduce the perturbation $v(x,t)$ by (3.2) and rewrite our original problem (1.1) as

$$\begin{cases} v_{tt} - v_{xx} + v_t + \{f(\Phi + v) - f(\Phi)\}_x + h = 0, & x > 0, \quad t > 0, \\ v(0,t) = 0, & t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x > 0. \end{cases} \tag{3.7}$$

where we put $v_0(x) := u_0(x) - \Phi_0(x)$ with $\Phi_0(x) := \phi(x) + \psi_0(x)$ and $v_1(x) := u_1(x)$. We will discuss this reformulated problem in Sections 4 and 5 to prove our main theorems.

In order to derive the existence of the global solution in time described in Theorem 2.1, we need the local existence theorem. For this purpose, we define the solution space for any interval $I \subseteq \mathbb{R}_+$ and $M > 0$ by

$$\begin{aligned} X_M(I) := \{v \in C^0(I; H^1_0(\mathbb{R}_+))\ ;\ v_t \in C^0(I; L^2(\mathbb{R}_+)), \\ \sup_{t\in I}(\|v(t)\|_{H^1} + \|v_t(t)\|_{L^2}) \leq M\}. \end{aligned}$$

For the solution space $X_M(I)$, the local existence theorem of the solution v for (3.7) is stated as follows.

Proposition 3.4 (local existence)**.** *For any positive constant M, there exists a positive constant $t_0 = t_0(M)$ such that if $\|v_0\|_{H^1} + \|v_1\|_{L^2} \leq M$, then the initial boundary value problem* (3.7) *has a unique solution $v \in X_{2M}([0,t_0])$.*

We prove Proposition 3.4 by using a standard iterative method and omit the proof.

4. Asymptotic stability of nonlinear waves

The aim of this section is to prove Theorem 2.1. For this purpose, it is important to derive the following *a priori* estimate of solutions v for (3.7) in the Sobolev space H^1.

Proposition 4.1 (*a priori* estimate)**.** *Suppose that the same assumptions as in Theorem 2.1 hold true. Then, there exists a positive constant ε_2 such that if $v \in X_{\varepsilon_2}([0,T])$ is the solution of the problem* (3.7) *for some $T > 0$, then it holds*

$$\begin{aligned} \|v(t)\|^2_{H^1} + \|v_t(t)\|^2_{L^2} + \int_0^t (\|v_t(\tau)\|^2_{L^2} + \|v_x(\tau)\|^2_{L^2} + \|\sqrt{\Phi_x} v(\tau)\|^2_{L^2})\, d\tau \\ \leq C(\|v_0\|^2_{H^1} + \|v_1\|^2_{L^2} + |u_+|^{1/6}) \end{aligned} \tag{4.1}$$

for $t \in [0, T]$, where C is a positive constant independent of T.

Before proceeding to the proof of Proposition 4.1, we give some preparations for a weight function. Since $f''(0) > 0$ and $|f'(0)| < 1$ by (1.2), there exist positive constants r and ν such that

$$f''(u) \geq \nu \quad \text{and} \quad |f'(u)| < 1 \qquad \text{for} \qquad |u| \leq r.$$

We also assume that $u_- < 0 < u_+ < r$ throughout this section. In this situation, we choose the weight function as

$$w(u) = f(u) + \delta g(u) \quad \text{for} \quad u \in [u_-, r], \tag{4.2}$$

where $g(u)$ is defined by $g(u) = -u^{2m} + r^{2m}$, and δ and m are positive constants determined later. For the weight function (4.2), we obtain the following lemma.

Lemma 4.2 (Hashimoto-Matsumura [1])**.** *Suppose that $f(u)$ satisfies* (1.2)*. Let $w(u)$ be the weight function defined in* (4.2)*. Then, for suitably small $\delta > 0$ and suitably large integer m, there exist positive constants c and C such that*

$$c \leq w(u) \leq C, \qquad (f''w - fw'')(u) \geq c$$

for $u \in [u_-, r]$.

For the proof, readers are referred to [1]. Furthermore, we prepare the key lemma for the weight function (4.2) as follows.

Lemma 4.3. *Suppose that the same conditions as in Lemma 4.2 hold true. Then, for suitably small $\delta > 0$, we obtain the inequality*

$$(f'w - fw')(u)^2 < w(u)^2 \tag{4.3}$$

for $u \in [u_-, r]$.

Proof. By the definition of w, we rewrite (4.3) as

$$\delta^2 \big\{(f'g - fg')(u)\big\}^2 < \big\{(f + \delta g)(u)\big\}^2. \tag{4.4}$$

Thus, the inequality (4.4) is enough to derive the inequality (4.3). In order to get the inequality (4.4), we divide the interval $[u_-, r]$ into $[u_-, -r]$ and $[-r, r]$. We first consider the interval $[-r, r]$. By the condition $|f'(u)| < 1$ and $(fg)(u) \geq 0$ for $u \in [-r, r]$, we choose δ suitably small, obtaining

$$\begin{aligned}
&\big\{(f + \delta g)(u)\big\}^2 - \delta^2 \big\{(f'g - fg')(u)\big\}^2 \\
&\quad = \delta^2 g(u)^2 \big(1 - f'(u)^2\big) + f(u)^2 \big(1 - \delta^2 g'(u)^2\big) + 2\delta (fg)(u) \big(1 + \delta (f'g')(u)\big) \\
&\quad \geq \delta^2 g(u)^2 \big(1 - f'(u)^2\big) + f(u)^2 \big\{1 - \delta^2 \max_{u \in [-r,r]} |g'(u)|^2\big\} \\
&\qquad\qquad + 2\delta (fg)(u) \big\{1 - \delta \max_{u \in [-r,r]} |f'g'(u)|\big\} \\
&\quad > 0 \quad \text{for} \quad u \in [-r, r].
\end{aligned}$$

Next, we consider the interval $[u_-, -r]$. Taking δ sufficiently small, we have

$$(f+\delta g)(u) \geq \min_{u\in[u_-,-r]} f(u) - \delta \max_{u\in[u_-,-r]} |g(u)| \geq \frac{1}{2}\min_{u\in[u_-,-r]} f(u)$$

for $u \in [u_-, -r]$. Therefore, using the inequality

$$\delta^2\big\{(f'g - fg')(u)\big\}^2 \leq \delta^2 \max_{u\in[u_-,-r]} |(f'g - fg')(u)|^2$$

and choosing δ suitably small such that

$$\delta \max_{u\in[u_-,-r]} |(f'g - fg')(u)| \leq \frac{1}{2}\min_{u\in[u_-,-r]} f(u),$$

we obtain the desired inequality (4.4) for $u \in [u_-, -r]$ and complete the proof. □

Using Lemmas 4.2 and 4.3, and the technical weighted energy method given by [1], we prove Proposition 4.1.

Proof of Proposition 4.1. We introduce a new unknown function $\tilde{v}$ as

$$v(x,t) = w\big(\Phi(x,t)\big)\tilde{v}(x,t), \tag{4.5}$$

where w is the weight function defined by (4.2). Substituting (4.5) into the equation of (3.7), we obtain

$$\big(w(\Phi)\tilde{v}\big)_{tt} - \big(w(\Phi)\tilde{v}\big)_{xx} + \big(w(\Phi)\tilde{v}\big)_t + \big\{f(\Phi + w(\Phi)\tilde{v}) - f(\Phi)\big\}_x + h = 0. \tag{4.6}$$

Multiplying (4.6) by $\tilde{v}$, we get

$$\begin{aligned}&\Big\{\frac{1}{2}(w+w_t)(\Phi)\tilde{v}^2 + w(\Phi)\tilde{v}_t\tilde{v}\Big\}_t - w(\Phi)\tilde{v}_t^2 + w(\Phi)\tilde{v}_x^2 + \frac{1}{2}(w_{tt} - w_{xx} + w_t)(\Phi)\tilde{v}^2\\ &+ \Phi_x\int_0^{\tilde{v}} f'(\Phi + w(\Phi)\eta) - f'(\Phi)d\eta + \Phi_x\int_0^{\tilde{v}} f'(\Phi + w(\Phi)\eta)w'(\Phi)\eta d\eta + \mathcal{F}_x = -\tilde{v}h,\end{aligned} \tag{4.7}$$

where we define $\mathcal{F}$ as

$$\mathcal{F} = -\frac{1}{2}w(\Phi)_x\tilde{v}^2 - w(\Phi)\tilde{v}\tilde{v}_x + \big(f(\Phi + w(\Phi)\tilde{v}) - f(\Phi)\big)\tilde{v} - \int_0^{\tilde{v}} f(\Phi + w(\Phi)\eta) - f(\Phi)d\eta.$$

By using the equation (3.5) and the condition $\Phi_x = f(\Phi) + O(|\psi| + |\psi_x|)$, we find that

$$\begin{aligned}(w_{tt} - w_{xx} + w_t)(\Phi) &= w'(\Phi)(\Phi_{tt} - \Phi_{xx} + \Phi_t) + w''(\Phi)(\Phi_t^2 - \Phi_x^2)\\ &= w'(\Phi)(h - f(\Phi)_x) + w''(\Phi)(\Phi_t^2 - \Phi_x^2)\\ &= -\{w''(\Phi)\Phi_x + (f'w')(\Phi)\}\Phi_x + w''(\Phi)\Phi_t^2 + w'(\Phi)h\\ &= -(fw'' + f'w')(\Phi)\Phi_x + O(|\psi| + |\psi_x|)\Phi_x + O(|h| + |\psi_x|^2).\end{aligned} \tag{4.8}$$

Moreover, by the straightforward calculation, we have

$$\begin{aligned}\Phi_x\int_0^{\tilde v} f'(\Phi+w(\Phi)\eta)-f'(\Phi)d\eta+\Phi_x\int_0^{\tilde v} f'(\Phi+w(\Phi)\eta)w'(\Phi)\eta d\eta\\ =\frac{1}{2}(f''w+f'w')(\Phi)\,\Phi_x\tilde v^2+O(|\tilde v|)\Phi_x\tilde v^2.\end{aligned}\tag{4.9}$$

Therefore substituting (4.8) and (4.9) into the equality (4.7), we obtain

$$\begin{aligned}&\Big\{\frac{1}{2}w(\Phi)\tilde v^2+w(\Phi)\tilde v_t\tilde v+O(|\psi_x|)\tilde v^2\Big\}_t+w(\Phi)\tilde v_x^2+\frac{1}{2}(f''w-fw'')(\Phi)\Phi_x\tilde v^2\\ &\quad-w(\Phi)\tilde v_t^2+\mathcal{F}_x=-\tilde v h+O(|\tilde v|+|\psi|+|\psi_x|)\Phi_x\tilde v^2+O(|h|+|\psi_x|^2)\tilde v^2.\end{aligned}\tag{4.10}$$

Next, we multiply (4.6) by $2\tilde v_t$, obtaining

$$\mathcal{G}_t+2w(\Phi)\tilde v_t^2+\mathcal{H}-\big(2w(\Phi)\tilde v_t\tilde v_x\big)_x=-2\tilde v_t h,\tag{4.11}$$

where $\mathcal{G}$ and $\mathcal{H}$ are defined by

$$\begin{aligned}\mathcal{G}&=w(\Phi)\tilde v_t^2+w(\Phi)\tilde v_x^2+(w_x(\Phi)\tilde v^2)_x+(w_{tt}-w_{xx}+w_t)(\Phi)\tilde v^2\\ &\quad+2\Phi_x\int_0^{\tilde v} f'(\Phi+w(\Phi)\eta)-f'(\Phi)d\eta+2\Phi_x\int_0^{\tilde v} f'(\Phi+w(\Phi)\eta)w'(\Phi)\eta d\eta,\\ \mathcal{H}&=3w_t(\Phi)\tilde v_t^2-w_t(\Phi)\tilde v_x^2-(w_{tt}-w_{xx}+w_t)_t(\Phi)\tilde v^2\\ &\quad+2\big\{f'(\Phi+w(\Phi)\tilde v)w(\Phi)-w_x(\Phi)\big\}\tilde v_t\tilde v_x\\ &\quad-2\Phi_x\int_0^{\tilde v}\big\{f'(\Phi+w(\Phi)\eta)-f'(\Phi)\big\}_t d\eta-2\Phi_x\int_0^{\tilde v}\big\{f'(\Phi+w(\Phi)\eta)w'(\Phi)\eta\big\}_t d\eta.\end{aligned}$$

Applying the relations (4.8) and (4.9), we rewrite $\mathcal{G}$ as

$$\begin{aligned}\mathcal{G}=w(\Phi)\tilde v_t^2+w(\Phi)\tilde v_x^2+(w_x(\Phi)\tilde v^2)_x+(f''w-fw'')(\Phi)\Phi_x\tilde v^2\\ +O(|\tilde v|+|\psi|+|\psi_x|)\Phi_x\tilde v^2+O(|h|+|\psi_x|^2)\tilde v^2.\end{aligned}\tag{4.12}$$

On the other hand, making use of the equality

$$f'(\Phi+w(\Phi)\tilde v)w(\Phi)-w_x(\Phi)=(f'w-fw')(\Phi)+O(|\tilde v|+|\psi|+|\psi_x|),$$

we have

$$\begin{aligned}\mathcal{H}&=2(f'w-fw')(\Phi)\tilde v_t\tilde v_x+O(|\tilde v|+|\psi|+|\psi_x|)\tilde v_t\tilde v_x+O(|\tilde v|^2)\Phi_t\Phi_x\\ &\quad+O(|\psi_x|)(\tilde v_t^2+\tilde v_x^2)+O(|\psi_x|^2+|\psi_{xx}|+|\psi_x|^3+|\psi_x\psi_{xx}|+|\psi_{xxx}|)\tilde v^2.\end{aligned}\tag{4.13}$$

Summing up (4.10) and (4.11), and substituting (4.12) and (4.13) into the resultant equation, we obtain

$$\big(\tilde E+R_1\big)_t+\tilde D+\tilde F_x=R_1+R_2-(\tilde v+2\tilde v_t)h,\tag{4.14}$$

where $\tilde{E}$, $\tilde{D}$, $\tilde{F}$, $\tilde{R}_1$ and $\tilde{R}_2$ are defined by

$$\begin{aligned}
\tilde{E} &= w(\Phi)\Big(\frac{1}{2}\tilde{v}^2 + \tilde{v}_t^2 + \tilde{v}_x^2 + \tilde{v}\tilde{v}_t\Big) + (f''w - fw'')(\Phi)\Phi_x\tilde{v}^2,\\
\tilde{D} &= w(\Phi)(\tilde{v}_x^2 + \tilde{v}_t^2) + 2(f'w - fw')(\Phi)\tilde{v}_t\tilde{v}_x + \frac{1}{2}(f''w - fw'')(\Phi)\Phi_x\tilde{v}^2,\\
\tilde{F} &= -\frac{1}{2}w(\Phi)_x\tilde{v}^2 - w(\Phi)(\tilde{v}\tilde{v}_x + 2\tilde{v}_t\tilde{v}_x)\\
&\qquad + \big(f(\Phi + w(\Phi)\tilde{v}) - f(\Phi)\big)\tilde{v} - \int_0^{\tilde{v}} f(\Phi + w(\Phi)\eta) - f(\Phi)d\eta,\\
R_1 &= O(|\tilde{v}| + |\psi| + |\psi_x|)\Phi_x\tilde{v}^2 + O(|h| + |\psi_x|^2)\tilde{v}^2,\\
R_2 &= O(|\tilde{v}| + |\psi| + |\psi_x|)\tilde{v}_t\tilde{v}_x + O(|\psi_x|)(\tilde{v}_t^2 + \tilde{v}_x^2)\\
&\qquad + O(|\psi_{xx}| + |\psi_x|^3 + |\psi_x\psi_{xx}| + |\psi_{xxx}|)\tilde{v}^2.
\end{aligned}$$

Therefore, integrating the equation (4.14) over $\mathbb{R}_+$, we get the energy equality

$$\frac{d}{dt}\int_0^\infty \tilde{E} + R_1 dx + \int_0^\infty \tilde{D}\,dx = \int_0^\infty R_1 + R_2 - (\tilde{v} + 2\tilde{v}_t)h\,dx. \tag{4.15}$$

Here, calculating the discriminants and using Lemmas 4.2 and 4.3, we have the condition

$$\int_0^\infty \tilde{E}\,dx \sim \|(\tilde{v}, \tilde{v}_x, \tilde{v}_t, \sqrt{\Phi_x}\tilde{v})\|_{L^2}^2, \quad \int_0^\infty \tilde{D}\,dx \sim \|(\tilde{v}_x, \tilde{v}_t, \sqrt{\Phi_x}\tilde{v})\|_{L^2}^2. \tag{4.16}$$

We next consider the remainder terms. We first estimate the third term on the right hand side of (4.15). Using Lemma 3.3 and the Sobolev and Young inequalities, we obtain

$$\begin{aligned}
\int_0^\infty |(\tilde{v} + 2\tilde{v}_t)h|dx &\le \int_0^\infty |\tilde{v}h|dx + 2\int_0^\infty |\tilde{v}_t h|dx\\
&\le C\|\tilde{v}\|_{L^2}^{1/2}\|\tilde{v}_x\|_{L^2}^{1/2}\|h\|_{L^1} + \varepsilon\|\tilde{v}_t\|_{L^2}^2 + C_\varepsilon\|h\|_{L^2}^2\\
&\le \varepsilon(\|\tilde{v}_x\|_{L^2}^2 + \|\tilde{v}_t\|_{L^2}^2) + C_\varepsilon(\|\tilde{v}\|_{L^2}^{2/3}\|h\|_{L^1}^{4/3} + \|h\|_{L^2}^2)\\
&\le \varepsilon(\|\tilde{v}_x\|_{L^2}^2 + \|\tilde{v}_t\|_{L^2}^2)\\
&\qquad + C_\varepsilon\big\{\|\tilde{v}\|_{L^2}^{2/3}|u_+|^{1/6}(1+t)^{-7/6}\log^{6/7}(2+t) + |u_+|^{1/2}(1+t)^{-3/2}\big\}
\end{aligned} \tag{4.17}$$

for any $\varepsilon > 0$, where C_ε is a positive constant depending on ε. By Lemmas 3.1, 3.2 and 3.3, and the same computation as in (4.17), we estimate R_1 and R_2 as

$$\begin{aligned}
\int_0^\infty |R_1|\,dx &\le C\int_0^\infty (|\tilde{v}| + |\psi| + |\psi_x|)\Phi_x\tilde{v}^2\,dx + C\int_0^\infty (|h| + |\psi_x|^2)\tilde{v}^2\,dx\\
&\le C\big(\|\tilde{v}\|_{L^\infty} + |u_+|\big)\|\sqrt{\Phi_x}\tilde{v}\|_{L^2}^2 + C\|\tilde{v}\|_{L^2}\|\tilde{v}_x\|_{L^2}\big(\|h\|_{L^1} + \|\psi_x\|_{L^2}^2\big)\\
&\le C\big(\|\tilde{v}\|_{L^\infty} + |u_+|\big)\|\sqrt{\Phi_x}\tilde{v}\|_{L^2}^2 + \varepsilon\|\tilde{v}_x\|_{L^2}^2\\
&\qquad + C_\varepsilon\|\tilde{v}\|_{L^2}^2\big\{|u_+|^{1/6}(1+t)^{-11/6}\log^{11/6}(2+t) + |u_+|^2(1+t)^{-2}\big\}
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
\int_0^\infty |R_2|\ dx &\le C\int_0^\infty (|\tilde v| + |\psi| + |\psi_x|)\tilde v_t\tilde v_x\ dx + C\int_0^\infty |\psi_x|(\tilde v_t^2 + \tilde v_x^2)\ dx \\
&\qquad + C\int_0^\infty (|\psi_{xx}| + |\psi_x|^3 + |\psi_x\psi_{xx}| + |\psi_{xxx}|)\tilde v^2\ dx \\
&\le C\big(\|\tilde v\|_{L^\infty} + |u_+|\big)\big(\|\tilde v_t\|_{L^2}^2 + \|\tilde v_x\|_{L^2}^2\big) + C\|\tilde v\|_{L^2}^{3/2}\|\tilde v_x\|_{L^2}^{1/2}\big(\|\psi_{xx}\|_{L^2} + \|\psi_{xxx}\|_{L^2}\big) \\
&\qquad + C\|\tilde v\|_{L^2}\|\tilde v_x\|_{L^2}\big(\|\psi_x\|_{L^3}^3 + \|\psi_x\|_{L^\infty}\|\psi_{xx}\|_{L^1}\big) \\
&\le C\big(\|\tilde v\|_{L^\infty} + |u_+|\big)\big(\|\tilde v_t\|_{L^2}^2 + \|\tilde v_x\|_{L^2}^2\big) + \varepsilon\|\tilde v_x\|_{L^2}^2 \\
&\qquad + C_\varepsilon\|\tilde v\|_{L^2}^2\big(\|\psi_{xx}\|_{L^2}^{4/3} + \|\psi_{xxx}\|_{L^2}^{4/3} + \|\psi_x\|_{L^3}^6 + \|\psi_x\|_{L^\infty}^2\|\psi_{xx}\|_{L^1}^2\big) \\
&\le C\big(\|\tilde v\|_{L^\infty} + |u_+|\big)\big(\|\tilde v_t\|_{L^2} + \|\tilde v_x\|_{L^2}\big) + \varepsilon\|\tilde v_x\|_{L^2}^2 \\
&\qquad + C_\varepsilon\|\tilde v\|_{L^2}^2\big\{|u_+|^{1/6}(1+t)^{-7/6} + |u_+|^3(1+t)^{-3} + |u_+|^2(1+t)^{-2}\big\}
\end{aligned} \tag{4.19}$$

for any $\varepsilon > 0$, where C_ε is a positive constant depending on ε. Therefore, integrating (4.15) over $(0,t)$, substituting (4.17), (4.18) and (4.19) into the resultant equality and taking ε and $\sup_{0\le t\le T}\|v(t)\|_{H^1} + |u_+|$ sufficiently small, we obtain

$$\|\tilde v\|_{H^1}^2 + \|\tilde v_t\|_{L^2}^2 + \int_0^t \|\tilde v_x\|_{L^2}^2 + \|\tilde v_t\|_{L^2}^2 + \|\sqrt{\Phi_x}\tilde v\|_{L^2}^2 d\tau \le C\big(\|\tilde v_0\|_{H^1}^2 + \|\tilde v_1\|_{L^2}^2 + |u_+|^{1/6}\big).$$

Finally, by the positivity of w in Lemma 4.8 and the simple relations $v_x = w_x\tilde v + w\tilde v_x$ and $v_t = w_t\tilde v + w\tilde v_t$, we find that $\|v\|_{L^2} \sim \|\tilde v\|_{L^2}$ and

$$\begin{aligned}
&\|v_x\|_{L^2} \le C(\|\sqrt{\Phi_x}\tilde v\|_{L^2} + \|\tilde v_x\|_{L^2}), \quad \|\tilde v_x\|_{L^2} \le C(\|\sqrt{\Phi_x}v\|_{L^2} + \|v_x\|_{L^2}), \\
&\|v_t\|_{L^2} \le C(\|\sqrt{\Phi_x}\tilde v\|_{L^2} + \|\tilde v_t\|_{L^2}), \quad \|\tilde v_t\|_{L^2} \le C(\|\sqrt{\Phi_x}v\|_{L^2} + \|v_t\|_{L^2}).
\end{aligned}$$

Thus, by using the above inequalities, we have the desired estimate (4.1) and complete the proof of Proposition 4.1. □

Proof of Theorem 2.1. The global existence of solutions to the initial-boundary value problem (1.1) can be proved by the continuation argument based on a local existence result in Proposition 3.4 combined with the corresponding *a priori* estimate in Proposition 4.1. We omit the details and refer the readers to [1, 7]. □

5. Convergence rates of stationary solutions

In this section, we prove Theorems 2.3 and 2.4. The main idea of the proofs are due to Ueda [7]. We use the space-time weighted energy method introduced in Kawashima-Matsumura [2]. Before stating the proofs, we give a preparation. The following lemma is concerning the inequality of the nonlinear term f and the weight function w.

Lemma 5.1. *Suppose that $f(u)$ satisfies (1.2) and $f'(0) < 0$. Let $w(u)$ be the weight function defined by (4.2). Then, for suitably large integer m, there exists a positive*

constant c such that

$$(fw' - f'w)(u) \geq c \tag{5.1}$$

for $u \in [u_-, 0]$.

Proof. By the definition of weight function w, we have

$$(fw' - f'w)(u) = \delta(fg' - f'g)(u).$$

In order to derive the desired inequality, we decompose the interval $[u_-, 0]$ into $[u_-, -r]$, $[-r, -r/2]$ and $[-r/2, 0]$. We first consider the case $[u_-, -r]$. For $u \in [u_-, -r]$, we have

$$\begin{aligned}(fg' - f'g)(u) &= -\{2mu^{2m-1}f(u) + f'(u)(-u^{2m} + r^{2m})\} \\ &= -2mu^{2m-1}\Big\{f'(u)\Big(-1 + \Big|\frac{r}{u}\Big|^{2m}\Big)\frac{u}{2m} + f(u)\Big\}.\end{aligned} \tag{5.2}$$

Here, we note that $|r/u| \leq 1$ and $f(u) \geq c_0$, $|f'(u)| < C$ for $u \in [u_-, -r]$, where c_0 and C are positive constants. Thus we can choose m sufficiently large such that

$$f'(u)\Big(-1 + \Big|\frac{r}{u}\Big|^{2m}\Big)\frac{u}{2m} + f(u) \geq \frac{c_0}{2}. \tag{5.3}$$

Therefore, (5.2) and (5.3) imply the following inequality

$$(fg' - f'g)(u) \geq c_0 m r^{2m-1} > 0. \tag{5.4}$$

For the case $u \in [-r, -r/2]$, since $f > 0, g' > 0$ and $f' < 0 \leq g$, it immediately holds

$$(fg' - f'g)(u) \geq (fg')(u) \geq (fg')(-r/2) > 0. \tag{5.5}$$

Finally, for the case $u \in [-r/2, 0]$, since $f > 0, g' \geq 0$ and $f' < 0 < g$, we get

$$(fg' - f'g)(u) \geq (f'g)(u) \geq \min_{u \in [-r/2, 0]} |(f'g)(u)| > 0. \tag{5.6}$$

Thus combining (5.4), (5.5) and (5.6), we obtain the desired estimate (5.1). □

Proof of Theorem 2.3. When we consider the case $u_+ = 0$, the solution of (1.1) converges to the stationary solution ϕ. In this case, applying the weighted energy method, we obtain the equation (4.14) with $\psi = 0$. More precisely, we get the following differential equality.

$$\big(\bar{E} + \bar{R}_1\big)_t + \bar{D} + \bar{F}_x = \bar{R}_1 + \bar{R}_2, \tag{5.7}$$

where $\bar{E}$, $\bar{D}$, $\bar{F}$, $\bar{R}_1$ and $\bar{R}_2$ are defined by

$$\begin{aligned}
\bar{E} &= w(\phi)\Big(\frac{1}{2}\tilde{v}^2 + \tilde{v}_t^2 + \tilde{v}_x^2 + \tilde{v}\tilde{v}_t\Big) + (f''w - fw'')(\phi)\phi_x\tilde{v}^2,\\
\bar{D} &= w(\phi)(\tilde{v}_x^2 + \tilde{v}_t^2) + 2(f'w - fw')(\phi)\tilde{v}_t\tilde{v}_x + \frac{1}{2}(f''w - fw'')(\phi)\phi_x\tilde{v}^2,\\
\bar{F} &= -\frac{1}{2}w(\phi)_x\tilde{v}^2 - w(\phi)(\tilde{v}\tilde{v}_x + 2\tilde{v}_t\tilde{v}_x)\\
&\qquad + \big(f(\phi + w(\phi)\tilde{v}) - f(\phi)\big)\tilde{v} - \int_0^{\tilde{v}} f(\phi + w(\phi)\eta) - f(\phi)d\eta,\\
\bar{R}_1 &= O(|\tilde{v}|)\phi_x\tilde{v}^2, \qquad \bar{R}_2 = O(|\tilde{v}|)\tilde{v}_t\tilde{v}_x.
\end{aligned}$$

Here, we note that the perturbation v is defined by $v = u - \phi$ and $\tilde{v}$ is defined by $v = w(\phi)\tilde{v}$. Applying Lemma 5.1 to $\bar{F}$, we calculate $\bar{F}$ as

$$\begin{aligned}
-\bar{F} &= \frac{1}{2}(fw' - f'w)(\phi)\tilde{v}^2 + w(\phi)(\tilde{v}\tilde{v}_x + 2\tilde{v}_t\tilde{v}_x) + O(|\tilde{v}|^3)\\
&\geq c\tilde{v}^2 - C(\tilde{v}_x^2 + \tilde{v}_t^2) + O(|\tilde{v}|^3),
\end{aligned} \tag{5.8}$$

where c and C are positive constants.

Let γ and β be any positive constants satisfying $0 \leq \gamma, \beta \leq \alpha$. We multiply the equality (5.7) by $(1+t)^\gamma(1+x)^\beta$, obtaining

$$\begin{aligned}
&\{(1+t)^\gamma(1+x)^\beta(\bar{E} + \bar{R}_1)\}_t - \gamma(1+t)^{\gamma-1}(1+x)^\beta(\bar{E} + \bar{R}_1) + (1+t)^\gamma(1+x)^\beta\bar{D}\\
&+ \{(1+t)^\gamma(1+x)^\beta\bar{F}\}_x - \beta(1+t)^\gamma(1+x)^{\beta-1}\bar{F} = (1+t)^\gamma(1+x)^\beta(\bar{R}_1 + \bar{R}_2).
\end{aligned} \tag{5.9}$$

Substituting (5.8) into (5.9), integrating the resultant inequality over $\mathbb{R}_+ \times (0,t)$ and taking $\sup_{0\leq t\leq T}\|v(t)\|_{L^\infty}$ sufficiently small, we have

$$\begin{aligned}
&(1+t)^\gamma\|(\tilde{v},\tilde{v}_t,\tilde{v}_x)(t)\|_{L^2_\beta}^2 + \int_0^t (1+\tau)^\gamma\big(\|(\tilde{v}_t,\tilde{v}_x,\sqrt{\phi_x}\tilde{v})(\tau)\|_{L^2_\beta}^2 + \beta\|\tilde{v}(\tau)\|_{L^2_{\beta-1}}^2\big)d\tau\\
&\leq CE_\beta^2 + \gamma C\int_0^t (1+\tau)^{\gamma-1}\|(\tilde{v},\tilde{v}_t,\tilde{v}_x)(\tau)\|_{L^2_\beta}^2 d\tau + \beta C\int_0^t (1+\tau)^\gamma\|(\tilde{v}_t,\tilde{v}_x)(\tau)\|_{L^2_{\beta-1}}^2 d\tau
\end{aligned}$$

for an arbitrary γ and β with $0 \leq \gamma,\ \beta \leq \alpha$, where C is a constant independent of γ and β. For the above estimate, applying the induction argument, we can obtain the desired estimate (2.3) in Theorem 2.3. For the details, we refer the readers to [6, 7]. $\square$

Finally, we prove Theorem 2.4 by using the space-time weighted energy method.

Proof of Theorem 2.4. Let $\alpha,\ \beta > 0$. Multiplying (5.7) by $e^{\beta t}e^{\alpha x}$, we obtain

$$\begin{aligned}
&\{e^{\beta t}e^{\alpha x}(\bar{E} + \bar{R}_1)\}_t - \beta e^{\beta t}e^{\alpha x}(\bar{E} + \bar{R}_1) + e^{\beta t}e^{\alpha x}\bar{D}\\
&+ \{e^{\beta t}e^{\alpha x}\bar{F}\}_x - \alpha e^{\beta t}e^{\alpha x}\bar{F} = e^{\beta t}e^{\alpha x}(\bar{R}_1 + \bar{R}_2).
\end{aligned} \tag{5.10}$$

Substituting (5.8) into (5.10), integrating the resultant inequality over $\mathbb{R}_+ \times (0,t)$ and taking $\sup_{0\le t\le T}\|v(t)\|_{L^\infty}$ sufficiently small, we get

$$e^{\beta t}\|(\tilde{v}_t,\tilde{v}_x,\tilde{v})(t)\|^2_{L^2_{\alpha,exp}} + \int_0^t e^{\beta\tau}\|(\tilde{v}_t,\tilde{v}_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau + \alpha\int_0^t e^{\beta\tau}\|\tilde{v}(\tau)\|^2_{L^2_{\alpha,exp}}d\tau$$
$$\le CE^2_{\alpha,exp} + \beta C_0\int_0^t e^{\beta\tau}\|\tilde{v}(\tau)\|^2_{L^2_{\alpha,exp}}d\tau + (\alpha+\beta)C_1\int_0^t e^{\beta\tau}\|(\tilde{v}_t,\tilde{v}_x)(\tau)\|^2_{L^2_{\alpha,exp}}d\tau,$$

where C_0, C_1 and C are positive constants independent of α and β. Taking $\alpha > 0$ and $\beta > 0$ suitably small such that $\beta C_0 \le \alpha$ and $(\alpha+\beta)C_1 \le 1$, we obtain the desired estimate in Theorem 2.4 and complete the proof. □

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