Computation of correlation functions by tensor renormalization group method

Sci. Rep. Kanazawa Univ.
Vol. 60, pp. 11–25, 2016

Computation of correlation functions by tensor renormalization group method

Norihiro NAKAMOTO and Shinji TAKEDA*

Institute of Theoretical Physics, Kanazawa University, Kanazawa 920-1192, Japan

(Received June 28, 2016 and accepted in revised form August 26, 2016)

Abstract We demonstrate how to compute the correlation function by using the tensor renormalization group for 2D Ising model. From the correlation function around the critical temperature, we extract the correlation length and the spontaneous magnetization and then determine some critical exponents. Attractive feature of our method to extract the critical exponents is that numerical derivative, that causes a loss of significant digits, and additional simulation parameters, that demand more computational cost and complicated analysis, are *not* required. On the other hand, our approach requires an additional treatment of impurity tensors, but the additional cost is still tolerable.

Key words. Tensor renormalization group, Correlation function, Phase transition, Critical exponents

1 Introduction

The original idea of the tensor renormalization group (TRG) method was proposed by Levin and Nave [1] for the Ising model on the triangular lattice. And then it has been applied to many other models in two dimensional square lattice: ϕ^4 theory [2], Schwinger model [3, 4] and finite fermion density system [5]. A semi-analytic study for Ising model on square lattice was done by Aoki et al., [6] who showed that the critical temperature and the critical exponent ν can be reproduced with a relatively high accuracy even though the bond dimension is restricted to only two. Although the original idea was limited to the two dimensional system, a new coarse graining method suited for any higher dimensional system was invented in Ref. [7]. The new method uses the higher order singular value decomposition, thus it is called higher order tensor renormalization group (HOTRG). The authors in [8] seriously analyzed the 3D Ising model by using the HOTRG and obtained the critical temperature and the exponents precisely. The HOTRG was also applied in two dimensional systems; XY model [9], O(3) model [10] and CP(1) model [11].

A key point of TRG method is to use the singular value decomposition (SVD) in the coarse graining procedure, and then one naturally sets a bond dimension which controls the truncation in the decomposition. Thanks to this, one can reduce the size of tensor which would grow exponentially along the renormalization steps unless a truncation is applied, and can make the computational cost manageable while maintaining the important information of the system. It is in

*Corresponding author Email: takeda@hep.kanazawa-u.ac.jp

fact known that the TRG and HOTRG show very precisely results in the off-critical region while quite large bond dimension is required near critical point to maintain the accuracy. To overcome this difficulty, the tensor network renormalization (TNR) [12] and the loop optimization for tensor network renormalization (Loop-TNR) [13] have been recently developed. These methods first recast the tensor network such that an entanglement is reduced and then a successive coarse graining can be done very efficiently, namely the hierarchy of SVD becomes sharp and the required bond dimension can be reduced.

In this way, the efficient coarse graining methods even around critical point have been developed successfully. Next important issue is how to obtain the information of dynamics, for instance spectrum of theory, matrix elements, amplitudes by the TRG-related method. For that purpose, one needs to compute the correlation function encoding the above information. The idea of how to compute the correlation function using TRG was originally introduced in Ref. [14], but an actual implementation of the algorithm has not been done yet for the classical 2D Ising model to our best knowledge[1]. Therefore, in this paper we demonstrate the computation of spin-spin correlation function by using the algorithm for the model in details. And then we will extract the critical temperature and the critical exponents as a bench mark test. Actually, the extraction of the critical exponents is sometimes a nontrivial task for TRG-related method since usually one needs to introduce numerical derivative or additional simulation parameters, say the external magnetic field, in order to compute physical quantities. Since numerical results at some finite bond dimension is already affected by systematic error, a further loss of significant digits due to the numerical derivative makes the situation much worse. An additional parameter simply requires a wider parameter region to survey and then the cost and analysis tend to be hard. On the other hand, as we will see later soon the computation of the correlation function requires only an additional treatment of an impurity tensor and the total cost is only a few times larger than that of the partition function. Actually our method to extract the critical exponents is in sharp contract to the Monte Carlo method in which the finite size scaling is often used. In fact, the TRG method, whose cost is proportional to the logarithm of a system size, can easily deal with extremely large volume. Therefore one can safely compute the correlation length even around critical point and the critical exponent ν, which dictates the scaling behavior, can be directly obtained from the correlation length. In this way, we believe that this method is useful and enhances the utility of the TRG method.

The organization of this paper is as follows. In Sec. 2, a setup of the model, notations and tensor network representation are provided. The computation of the correlation function is explained in Sec. 3. In Sec. 4, we show numerical results. Our concluding remarks are summarized in Sec. 5.

2 Tensor network representation

The Hamiltonian for 2D Ising model is given by

$$H = -\sum_{\langle a,b\rangle} s_a s_b, \tag{2.1}$$

where $s_a \in \{+1,-1\}$ is a spin variable at a site a on 2D square lattice and the summation is taken over all nearest-neighbor lattice sites. In the following, we exclusively consider the vanishing

[1]There were some calculations of the correlation function in Ref. [10, 15] with the TRG-related methods. In Ref. [10], spin correlation function is computed in HOTRG method. The authors in Ref. [15] studied the phase diagram of the quenched bond-diluted Ising model on the triangular lattice by using the TRG method. The long-distance behavior of the correlation functions plays an important role in the study, but their algorithm to compute the correlation function is not explicitly written and they did not refer to [14] in their paper.

external magnetic field ($h = 0$). The partition function of the model is given by

$$Z = \sum_{\{s\}} e^{-\beta H} = \sum_{\{s\}} e^{\beta \sum_{\langle a,b\rangle} s_a s_b}, \tag{2.2}$$

where β denotes the inverse temperature ($T = 1/\beta$). To obtain the tensor network representation of the partition function, first we use an expansion for the Boltzmann weight

$$e^{\beta s_a s_b} = \cosh\beta + s_a s_b \sinh\beta = \cosh\beta \sum_{x=0}^{1} (s_a s_b \tanh\beta)^x. \tag{2.3}$$

After substituting the above expansion to the all Boltzmann weights in the partition function, one can carry out the summation of spin variables exactly and then the partition function can be written as

$$Z = 2^V (\cosh\beta)^{2V} \sum_{\{x,y,x',y'\}} \prod_a A_{x_a y_a x'_a y'_a}, \tag{2.4}$$

where V is the lattice volume and the explicit form of the tensor is given by

$$A_{ijkl} = (\sqrt{\tanh\beta})^{i+j+k+l} \delta_{\mathrm{mod}(i+j+k+l,2),0}. \tag{2.5}$$

Next, let us consider the one-point function for a single spin variable at a site a, namely the magnetization,

$$m = \langle s_a \rangle = \sum_{\{s\}} s_a e^{-\beta H}/Z. \tag{2.6}$$

In a similar way to the partition function, one can consider a tensor network representation of the one-point function by using the "impurity" tensor [14],

$$\check{A}_{ijkl} = (\sqrt{\tanh\beta})^{i+j+k+l} \delta_{\mathrm{mod}(i+j+k+l+1,2),0}, \tag{2.7}$$

where a slight difference appears only in the index of Kronecker delta. The numerator of the one-point function is now expressed in terms of the impurity tensor,

$$\sum_{\{s\}} s_a e^{-\beta H} = 2^V (\cosh\beta)^{2V} \sum_{\{x,y,x',y'\}} \left[\check{A}_{x_a y_a x'_a y'_a} \prod_{c\neq a} A_{x_c y_c x'_c y'_c} \right]. \tag{2.8}$$

The two-point function of spin variables ($a \neq b$) is given by,

$$\langle s_a s_b \rangle = \sum_{\{s\}} s_a s_b e^{-\beta H}/Z = \langle s_0 s_r \rangle, \tag{2.9}$$

where $r = |a-b|$ is a distance between two sites a and b. Similarly, the numerator of the two-point function is also expressed in terms of the impurity tensors

$$\sum_{\{s\}} s_a s_b e^{-\beta H} = 2^V (\cosh\beta)^{2V} \sum_{\{x,y,x',y'\}} \left[\check{A}_{x_a y_a x'_a y'_a} \check{A}_{x_b y_b x'_b y'_b} \prod_{c\neq a,b} A_{x_c y_c x'_c y'_c} \right]. \tag{2.10}$$

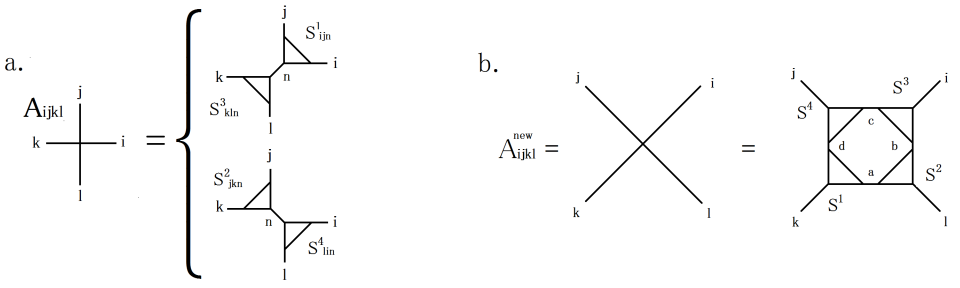

Figure 1: Basic steps of coarse graining: (a) The decomposition of the normal tensor and (b) Producing the new tensor by contraction of the original indices.

3 Coarse graining for correlation function

Before explaining a coarse graining of the tensor network for the correlation function, first of all, we summarize the coarse graining for the partition function in eq.(2.4) which contains only the normal tensor A. The normal coarse graining consists of two steps: the decomposition and the contraction [1]. Figure 1-(a) shows the decomposition step which uses singular value decomposition (SVD). There are two ways of decomposition:

$$A_{ijkl} = M_{(i\otimes j)(k\otimes l)} = \sum_n U_{(i\otimes j)n}\Sigma_n (V^\dagger)_{n(k\otimes l)} = \sum_n S^1_{ijn} S^3_{kln}, \tag{3.1}$$

$$S^1_{ijn} = U_{(i\otimes j)n}\sqrt{\Sigma_n}, \qquad S^3_{kln} = V^*_{(k\otimes l)n}\sqrt{\Sigma_n}, \tag{3.2}$$

and

$$A_{ijkl} = \tilde{M}_{(j\otimes k)(l\otimes i)} = \sum_n \tilde{U}_{(j\otimes k)n}\tilde{\Sigma}_n (\tilde{V}^\dagger)_{n(l\otimes i)} = \sum_n S^2_{jkn} S^4_{lin}, \tag{3.3}$$

$$S^2_{jkn} = \tilde{U}_{(j\otimes k)n}\sqrt{\tilde{\Sigma}_n}, \qquad S^4_{lin} = \tilde{V}^*_{(l\otimes i)n}\sqrt{\tilde{\Sigma}_n}, \tag{3.4}$$

where M and $\tilde{M}$ are matrix representation of the normal tensor, Σ and $\tilde{\Sigma}$ are diagonal matrix whose diagonal elements are singular values, $U, \tilde{U}, V$ and $\tilde{V}$ are unitary matrix and $S^{1,2,3,4}$ are considered as a rank-3 tensor. The new integer n is regarded as a new variable. In an actual implementation, the summation of n is truncated and the bond dimension $D_{\rm cut}$ is introduced. Due to the truncation, the rotational symmetry is broken thus $S^{1,2,3,4}$ are considered as independent, although they should be the same if the truncation is not applied. Figure 1-(b) shows the contraction step where the contractions of original indices are carried out and the new tensor is produced

$$A^{\rm new}_{ijkl} = \sum_{a,b,c,d} S^1_{adk} S^2_{bal} S^3_{cbi} S^4_{dcj}. \tag{3.5}$$

By repeating the decomposition and the contraction steps iteratively, the coarse graining of the tensor network can be done as shown in Fig. 2, and then finally one can obtain the partition function by contracting a small number of tensors.

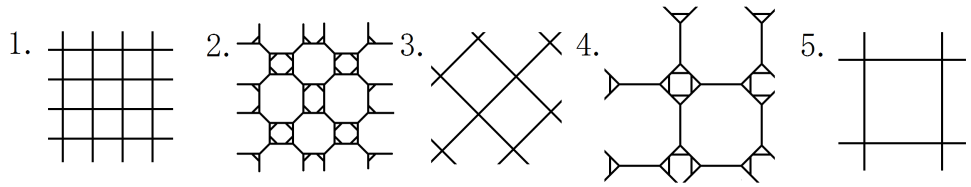

Figure 2: Coarse graining of tensor network. Step from 1 to 2 (3 to 4) is the decomposition. Step from 2 to 3 (4 to 5) is the contraction.

Let us explain the coarse graining for the correlation functions [14], namely a coarse graining of tensor network including the impurity tensor. As a concrete example, we address the one- and two-point function namely we consider two types of renormalization procedures in the following

First, we consider how to renormalize the impurity tensor for one spin variable whose procedure is shown in Fig. 3. We define "first sweep" which consists of four steps from 1-1 to 1-5 in Fig. 3 where the linear lattice extent shrinks a half size. At the end of the first sweep, one obtains three new impurity tensors. Similarly, the second sweep is composed of steps from 2-1 to 2-5 in Fig. 3 and there are four new impurity tensors at the end of this sweep. On the other hand, in the third sweep, the number of impurity tensors remains intact thus after the third sweep the impurity does not diffuse more than four sites and note that there is a cycle composed of steps from 3-1 to 3-5 in Fig. 3. This is a practically important feature to have a sustainable renormalization procedure. Finally, one can perform a contraction for a small number of tensors and obtain the numerator of the one-point function.

As the second example, Figure 4 shows the renormalization of the impurity tensor for two spin variables. As seen in the previous paragraph, the number of impurity tensor for one spin variable turns out to four after several sweeps. The initial configuration in Fig. 4-1-1 represents that the "two" impurity tensors for one spin variable after several sweeps are placed next to each other, thus in total there are eight impurity tensors in the network. At the end of the first sweep, the number of impurity tensor is reduced to six (see Fig. 4-1-5). In the second sweep, the number of impurity tensor does not change any more (see Fig. 4-2-5). In contrast to the previous case, the cycle appears in the second sweep. The sustainable renormalization is also attained for the two-point function.

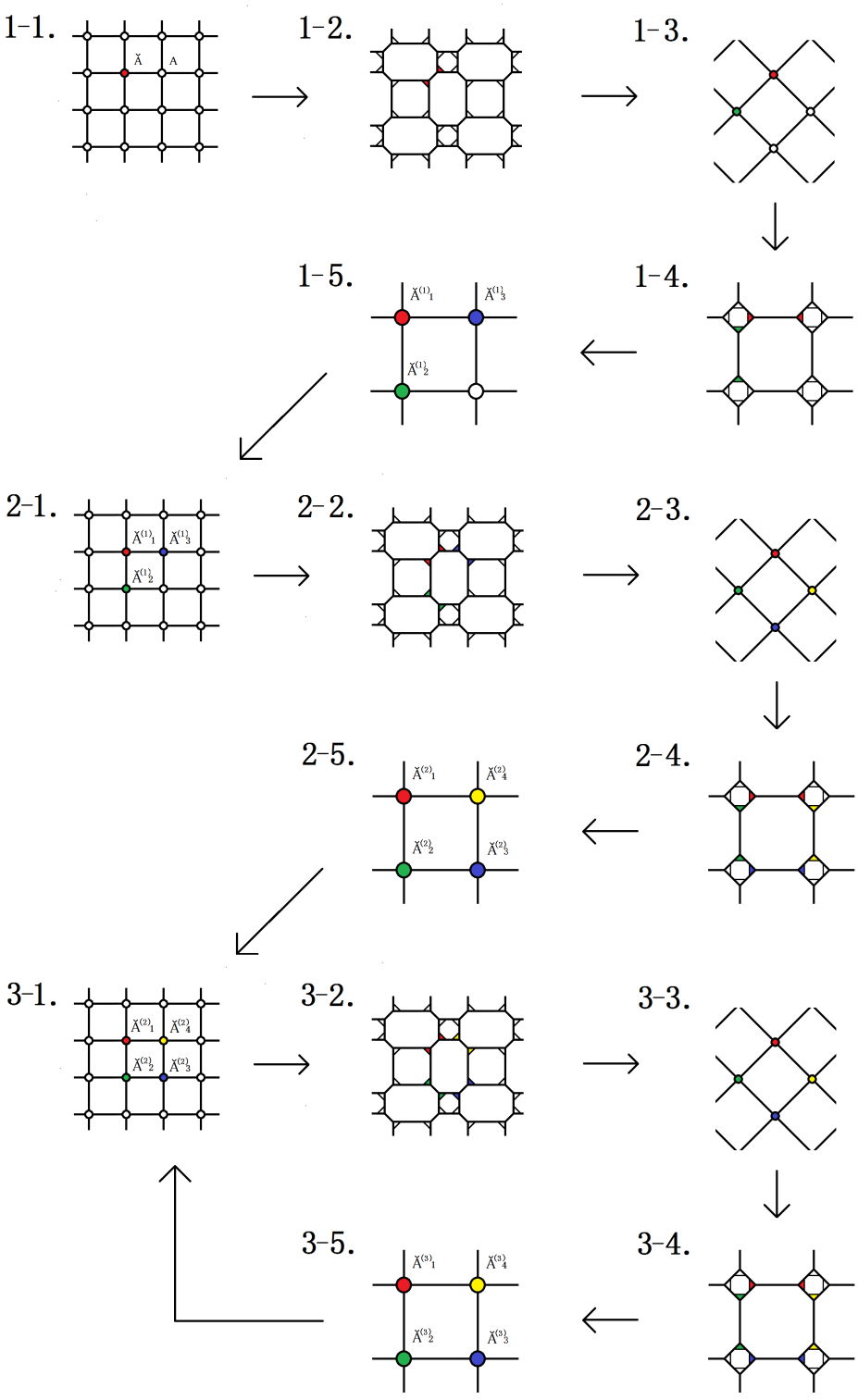

Figure 3: Renormalization of the impurity tensor for one spin variable. The open circle (white point) represents the normal tensor. The colored point denotes the impurity tensor. In the first and second sweep, the number of impurity tensor increases while it remains intact from the third sweep and a cycle (from 3-1 to 3-5) is seen.

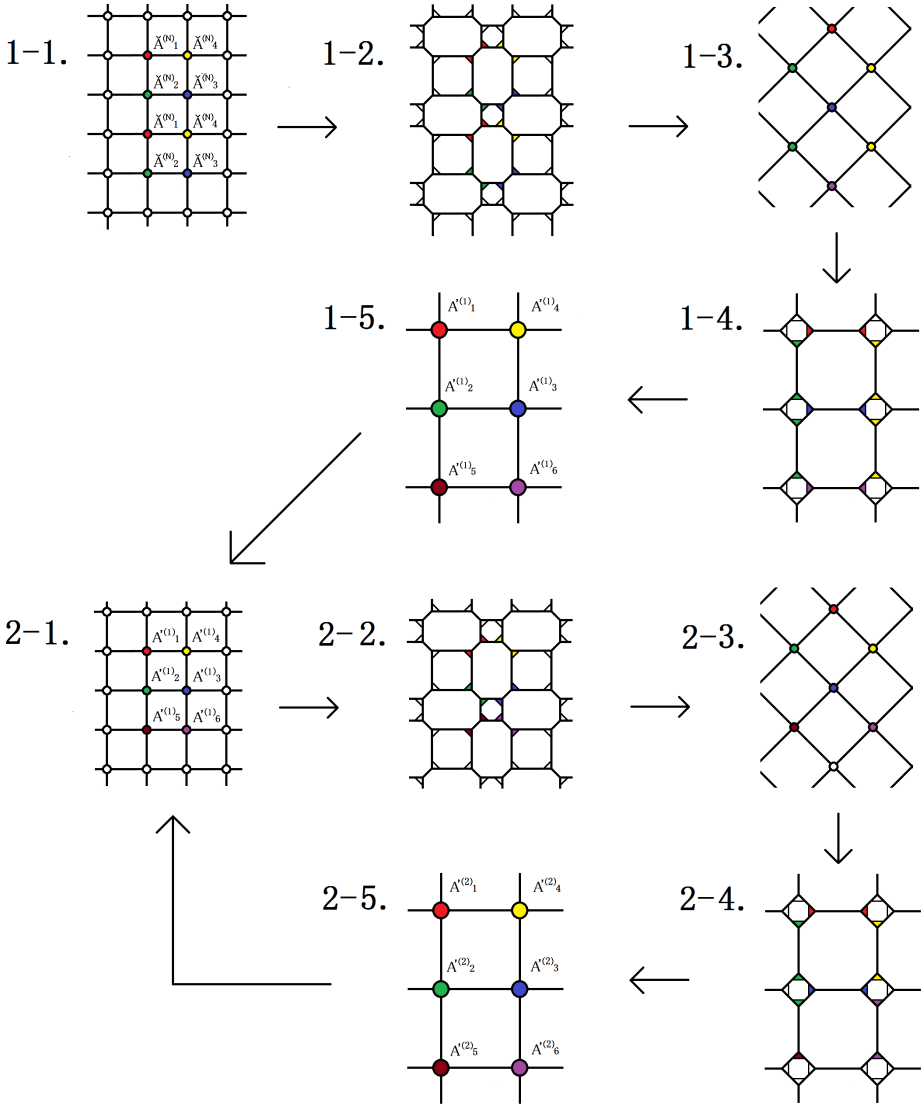

Figure 4: Renormalization of the impurity tensor for two spin variables. The symbols are the same as those of Fig. 3. The initial configuration represents the two impurity tensors for one spin variable after several sweeps. In the first sweep, the number of impurity tensor decreases from eight to six, while it remains intact from the second sweep and a cycle (from 2-1 to 2-5) is seen.

4 Result

In this section we show the numerical results of the correlation function in 2D Ising model and then present the analysis to obtain the critical exponents.

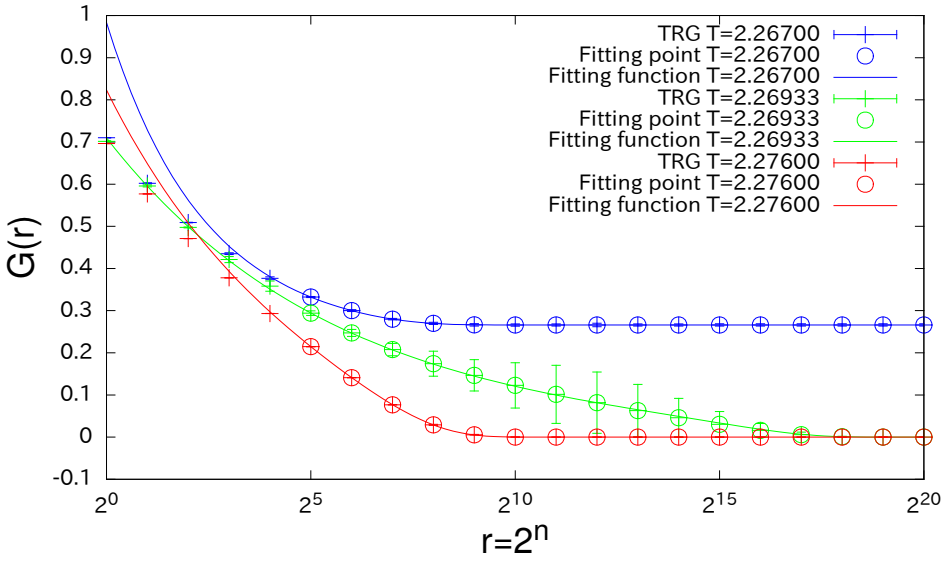

Figure 5: The correlation function as a function of distance r with fixed $D_{\text{cut}} = 32$ for 2D Ising model together with the fitting curve in eq.(4.4). Selected three temperatures are shown, $T = 2.26700$ (the low temperature phase), 2.26933 (around exact $T_{\text{c}} = 2.269185\cdots$), 2.27600 (the high temperature phase). The error bar is just the difference given in eq.(4.2).

We calculate the two-point function

$$G(r) = \langle s_0 s_r \rangle, \tag{4.1}$$

with the distance $r = 2^0, 2^1, 2^2, \cdots, 2^{20}$ on a fixed lattice volume $V = L^2 = 2^{21} \times 2^{21}$. The periodic boundary condition is imposed on the system. The numerical results of the $G(r)$ at fixed $D_{\text{cut}} = 32$ are shown in Fig. 5 for some selected temperatures. The error bar shown in the figure is just the difference of the correlation function at $D_{\text{cut}} = 24$ and 32,

$$\delta = |\langle O \rangle_{D_{\text{cut}}=32} - \langle O \rangle_{D_{\text{cut}}=24}|. \tag{4.2}$$

At higher temperature than the critical temperature T_{c}, the correlation function shows exponential decay clearly. Such an exponential decay is also observed in the low temperature side $T < T_{\text{c}}$, but it stays a nonzero constant for larger distance. This indicates a development of the nontrivial spontaneous magnetization and we will address this issue later on. The error bars at the temperature in the off-critical region are very small, this means that the D_{cut} dependence is negligible. On the other hand, around the critical point $T = 2.26933$, $G(r)$ shows a long range correlation as expected and D_{cut} dependence is clearly visible especially around $r \approx 2^{10}$ at this temperature. Therefore one should be careful that the temperature should not be too close to the critical otherwise a large contamination due to the truncation error ruins the result.

Let us see how to extract the magnetization and the correlation length from $G(r)$. Around the critical temperature $T \approx T_c$, the "connected" two-point function $G_c(r)$ should behave as

$$G_c(r) = \langle s_r s_0 \rangle_c = \langle s_r s_0 \rangle - \langle s_r \rangle \langle s_0 \rangle = \frac{A}{r^E} e^{-r/\xi}, \quad \text{for } r \to \infty, \tag{4.3}$$

where ξ is the correlation length and A is just a proportional constant. The exponent E is also a fitting parameter which is supposed to be the critical exponent $\eta = 1/4$ at the critical temperature ($\xi^{-1} = 0$) for 2D Ising model. To compute the connected correlation function, the magnetization $m = \langle s_0 \rangle$ is required. Instead of using $G_c(r)$, however, we fit $G(r)$ with the following functional form at each temperature,

$$G(r) = C^2 + \frac{A}{r^E} e^{-r/\xi}, \tag{4.4}$$

where a constant term C^2, a fitting parameter, represents the tadpole contribution in eq.(4.3) and C is regarded as the magnetization[2]. In the fitting procedure, we take eq.(4.2) as an error but note that this does not mean that the systematic error originated from the finite D_{cut} is incorporated. The fitting range of r is determined by the following way. Initially, a fit is done including all available r data points. When the resulting chi square is large, say $\chi^2/\text{d.o.f.} > 10$, we discard data point at the smallest r and then perform a next fit. This procedure is repeated until we obtain a reasonable chi square of $O(1)$ and we also check a stability of fitting parameters when changing the fit range. Furthermore, we also monitor that the chosen lattice volume can afford a long range correlation $\xi \ll L$ for all simulated temperatures to avoid the finite size effect on the correlation function.

Finally, let us show how to extract the critical exponents β, ν and the critical temperature. As mentioned, the fitting parameter C in eq.(4.4) is the spontaneous magnetization m, thus one can extract the β and T_c by fitting with the following temperature dependence[3],

$$C = m(T) = D|T - T_c|^{\beta}, \quad \text{for } T < T_c. \tag{4.5}$$

The critical exponent ν and the another estimate of T_c can be obtained from the correlation length that diverges at the critical point[4],

$$\xi(T) = B|T - T_c|^{-\nu}. \tag{4.6}$$

The fitting should be done at higher and lower temperature regions separately. Figure 6 shows the m, ξ and E as a function of the temperature together with the fitting line at $D_{\text{cut}} = 32$. The magnetization m stays zero in the symmetric phase (the high temperature phase) while it has a nonzero value in the broken phase (the low temperature phase). The correlation length ξ diverges at the critical point. We observe that E converges to $1/4$ near the critical temperature as expected.

The fitting range of temperature for m and ξ to obtain T_c and the critical exponents should be carefully chosen such that it is not too close to the critical point, around which D_{cut} truncation error is large, while the temperature should be in the scaling region. To find a proper region, we plot the T_c and ν in the high temperature phase as a function of the fitting range in Fig. 7 for $D_{\text{cut}} = 32$.

[2] In fact, we directly compute the one-point function $\langle s_0 \rangle$ by following the procedure given in the previous section and then it turns out that the value is exactly zero at a small number of the iteration but it takes significantly nonzero value after several times of the iteration. Actually, the value of $\langle s_0 \rangle$ is consistent with C in eq.(4.4) within errors for each volume.

[3] Practically, it is preferred to transfer to $m^{1/\beta_{\text{exp}}}$, where β_{exp} is a proper constant number (say, $\beta_{\text{exp}} = 1/8$ for 2D Ising model), and then fit with the form, $m^{1/\beta_{\text{exp}}} \propto |T - T_c|^{\beta/\beta_{\text{exp}}}$.

[4] In the actual fitting procedure, it is useful to fit the inverse correlation length, $\xi^{-1} \propto |T - T_c|^{\nu}$.

$D_{\rm cut}$	$T_{\rm c}$ from ξ ($T > T_{\rm c}$)	ν from ξ ($T > T_{\rm c}$)	$T_{\rm c}$ from m ($T < T_{\rm c}$)	β from m ($T < T_{\rm c}$)
26	2.26877(3)	0.904(11)	2.26882(1)	0.1176(5)
28	2.269030(7)	0.977(4)	2.26903(1)	0.1179(8)
30	2.269290(3)	0.945(6)	2.269320(6)	0.1244(2)
32	2.269330(8)	0.931(6)	2.269330(6)	0.1253(3)
34	2.26920(5)	1.035(21)	2.26938(2)	0.1286(12)
36	2.26935(1)	0.997(11)	2.269440(6)	0.1270(2)
38	2.269270(5)	0.967(7)	2.26928(2)	0.1244(11)
40	2.26917(4)	1.013(16)	2.26924(1)	0.1249(4)
exact	$2.2691853\cdots$	1	$2.2691853\cdots$	$0.125 = 1/8$

Table 1: The resulting $T_{\rm c}$ and the critical exponents determined from the magnetization m and the correlation length ξ for 2D Ising model. The exact critical temperature is $T_{\rm c} = \frac{2}{\log(1+\sqrt{2})}$.

The x-axis of the figure is the value of the lowest temperature of the fitting range in which the high temperature end is fixed. We take a representative value of ν and $T_{\rm c}$ from the saturation region. We repeat the same procedure for different $D_{\rm cut} = 26$, 28, 30,..., 40. The resulting $T_{\rm c}$ and ν for each $D_{\rm cut}$ are summarized in Table. 1 and plotted in Fig. 9. We roughly observe that they converge to the exact values for larger $D_{\rm cut}$. A similar analysis is attempted for the low temperature phase. The window analysis, however, turns out to be unstable since ξ (and also E) in the low temperature phase is flattering as seen in Fig. 6, thus we abandon to present the result.

We also carry out the similar analysis to obtain β and $T_{\rm c}$ from the magnetization m in the low temperature phase. We first check the finite size effect on the magnetization by performing additional calculations on the larger lattice $2^{31} \times 2^{31}$, and then it turns out that it is completely negligible in the temperature region we investigated. The window analysis for β and $T_{\rm c}$ is shown in Fig. 8 for $D_{\rm cut} = 32$. The resulting values of $T_{\rm c}$ and β for various $D_{\rm cut}$ are given in Table. 1. They tend to converge to the exact values for large $D_{\rm cut}$ as shown in Fig. 9.

5 Summary

In this paper, we have demonstrated how to compute the correlation function in the framework of TRG. This technique opens up a possibility to gain some dynamical information of a system from the correlation function. Furthermore we have extracted the critical temperature and the universal critical exponents from the correlation function for 2D Ising model, and have shown that they are correctly reproduced. The method demonstrated here requires no numerical derivative and no additional parameters, thus it will be a powerful tool to determine the critical exponents for more complicated systems, where using a small $D_{\rm cut}$ is unavoidable due to a computational cost. When using the method, however, one should be careful to ensure a proper window where the temperature should not be too close to the critical point, but being in the scaling region.

So far, we have restricted ourself within the framework of TRG. As future work, it would be interesting to establish a method to compute the correlation function in the framework of TNR and Loop-TNR. This will open up further possibilities to capture more insights of dynamics in more precise and less costly way.

Acknowledgments

We would like to thank H. Kawauchi and R. Sakai for helpful advice and useful discussion. N. N. is grateful to S. Kumamoto and S. Onai for useful conversation. This work was supported by Kanazawa University SAKIGAKE Project.

References

[1] M. Levin and C. P. Nave, Phys. Rev. Lett. 99, 120601 (2007)

[2] Y. Shimizu, Mod. Phys. Lett. A 27, 1250035 (2012)

[3] Y. Shimizu and Y. Kuramashi, Phys. Rev. D **90**, 014508 (2014)

[4] Y. Shimizu and Y. Kuramashi, Phys. Rev. D **90**, 074503 (2014)

[5] S. Takeda and Y. Yoshimura, PTEP **2015** (2015) no.4, 043B01

[6] K. Aoki, T. Kobayashi and H. Tomita, Int. Jour. Mod. Phys. B 23, 18 (2009).

[7] Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang and T. Xiang, Phys. Rev. B **86**, 045139 (2012)

[8] S. Wang, Z. Y. Xie, J. Chen, B. Normand and T. Xiang, Chin. Phys. Lett. **31**, 070503 (2014)

[9] J. F. Yu, Z. Y. Xie, Y. Meurice, Y. Liu, A. Denbleyker, H. Zou, M. P. Qin, J. Chen and T. Xiang, Phys. Rev. E **89**, 013308 (2014)

[10] J. Unmuth-Yockey, Y. Meurice, J. Osborn, H. Zou, arXiv:1411.4213.

[11] H. Kawauchi and S. Takeda, Phys. Rev. D **93**, no. 11, 114503 (2016)

[12] G. Evenbly and G. Vidal, Phys. Rev. Lett. 115, 180405 (2015)

[13] S. Yang, Z. Gu, and X. Wen, arXiv:1512.04938v1

[14] Z. Gu, M. Levin and X. Wen, Phys. Rev. B 78, 205116 (2008)

[15] C. Gueven, M. Hinczewski and A. N. Berker, Phys. Rev. E **82**, 051110 (2010)

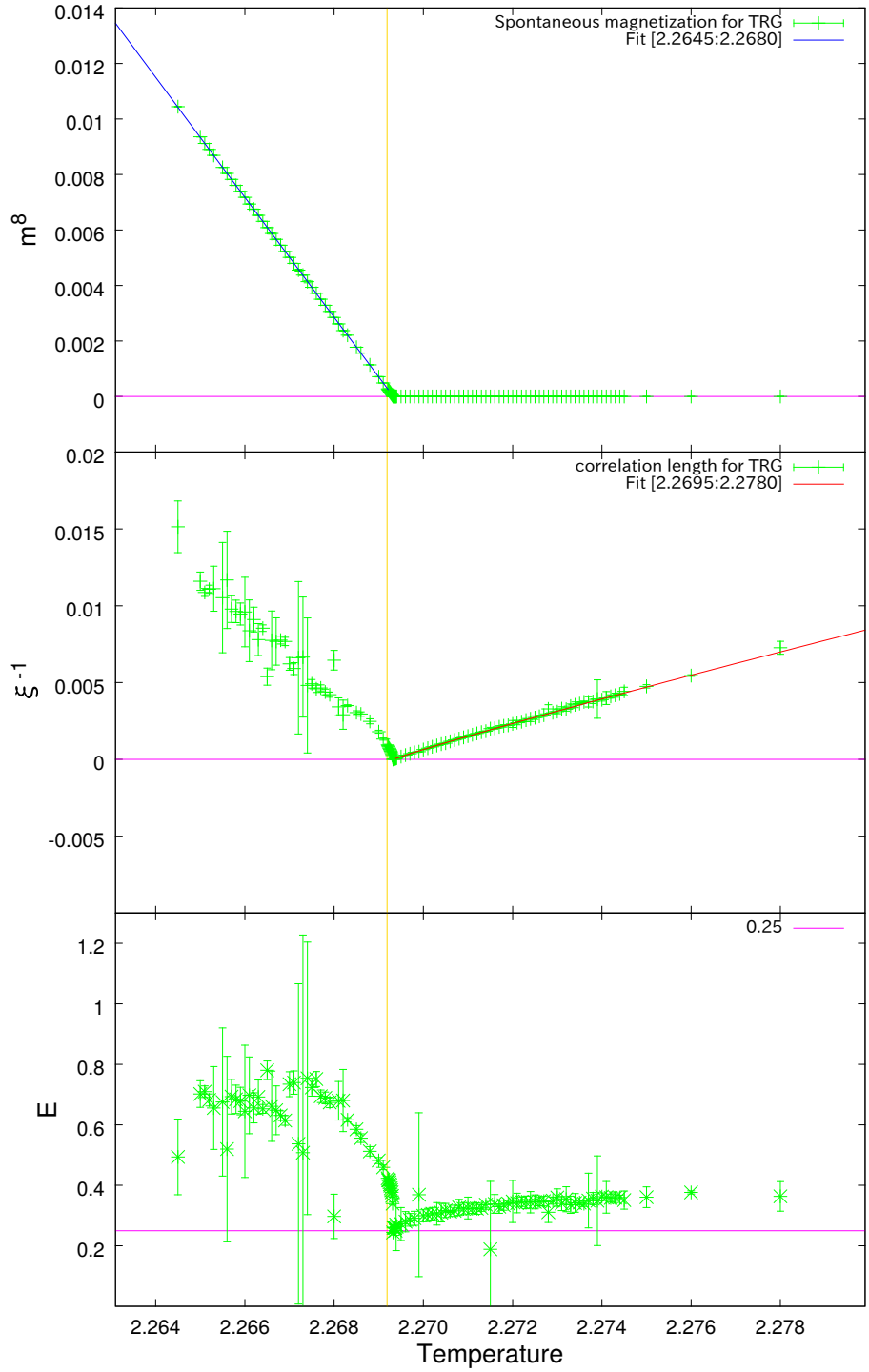

Figure 6: The spontaneous magnetization m, the correlation length ξ and E as a function of temperature around T_c in the 2D Ising model at $D_{\mathrm{cut}} = 32$. The vertical yellow line represents the location of the exact critical temperature T_c. Fitting lines, whose fitting range is given in the legend, are also shown.

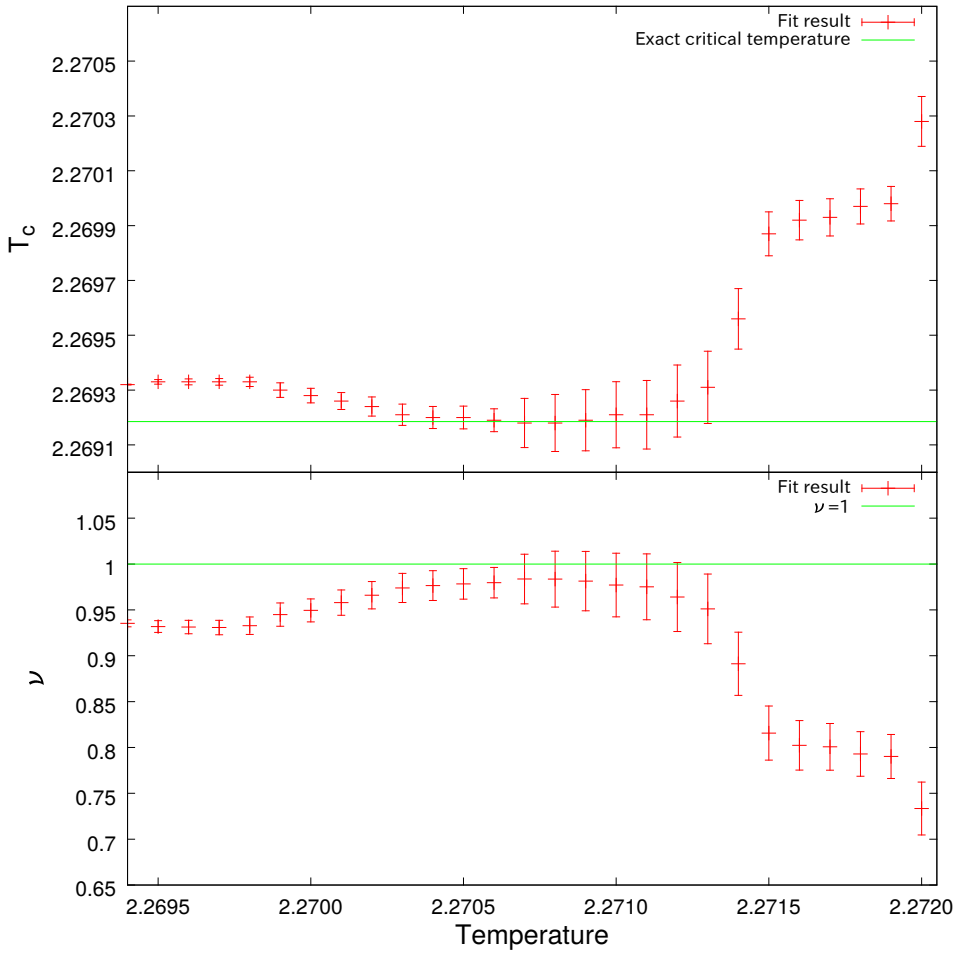

Figure 7: Fitting range dependence of the fitting parameters T_c and ν in the high temperature phase at $D_{\text{cut}} = 32$. The x-axis shows the lowest temperature in a fitting range where the high temperature end is fixed to 2.278.

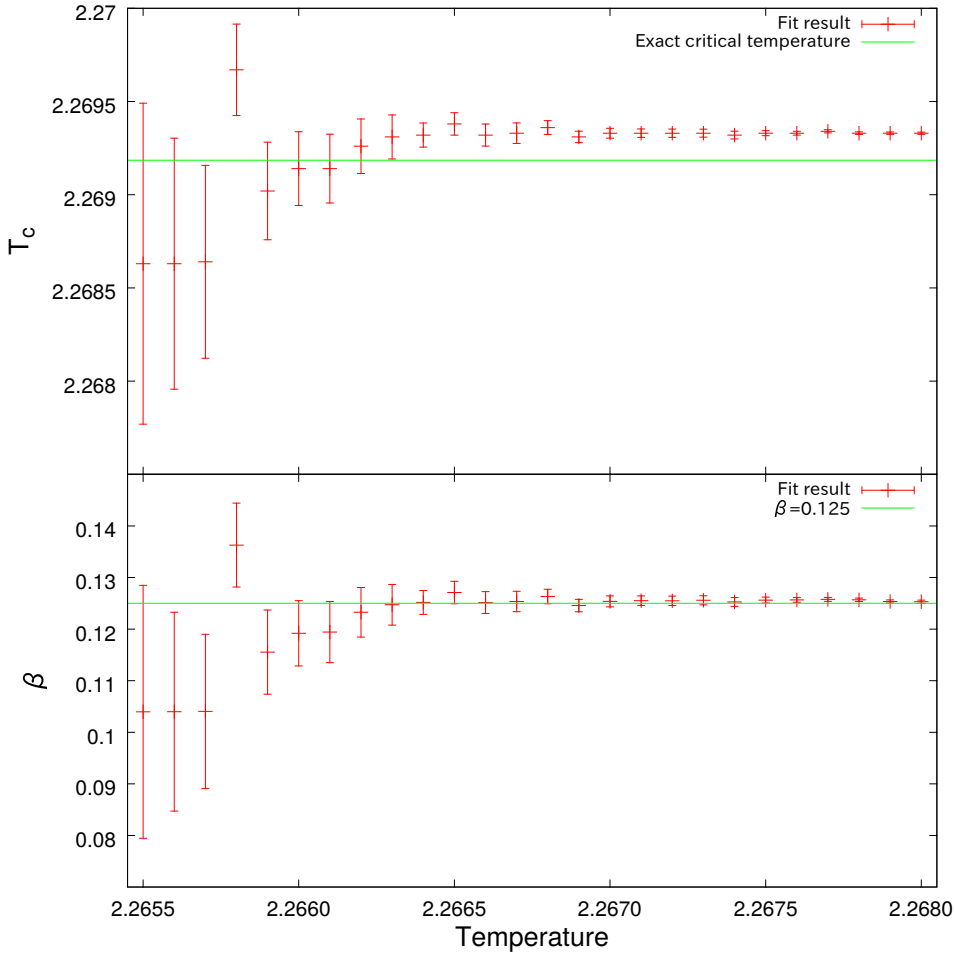

Figure 8: Fitting range dependence of the fitting parameters T_c and β in the low temperature phase at $D_{\text{cut}} = 32$. The x-axis for T_c and β shows the highest temperature in a fitting range where the low temperature end is fixed to 2.2645.

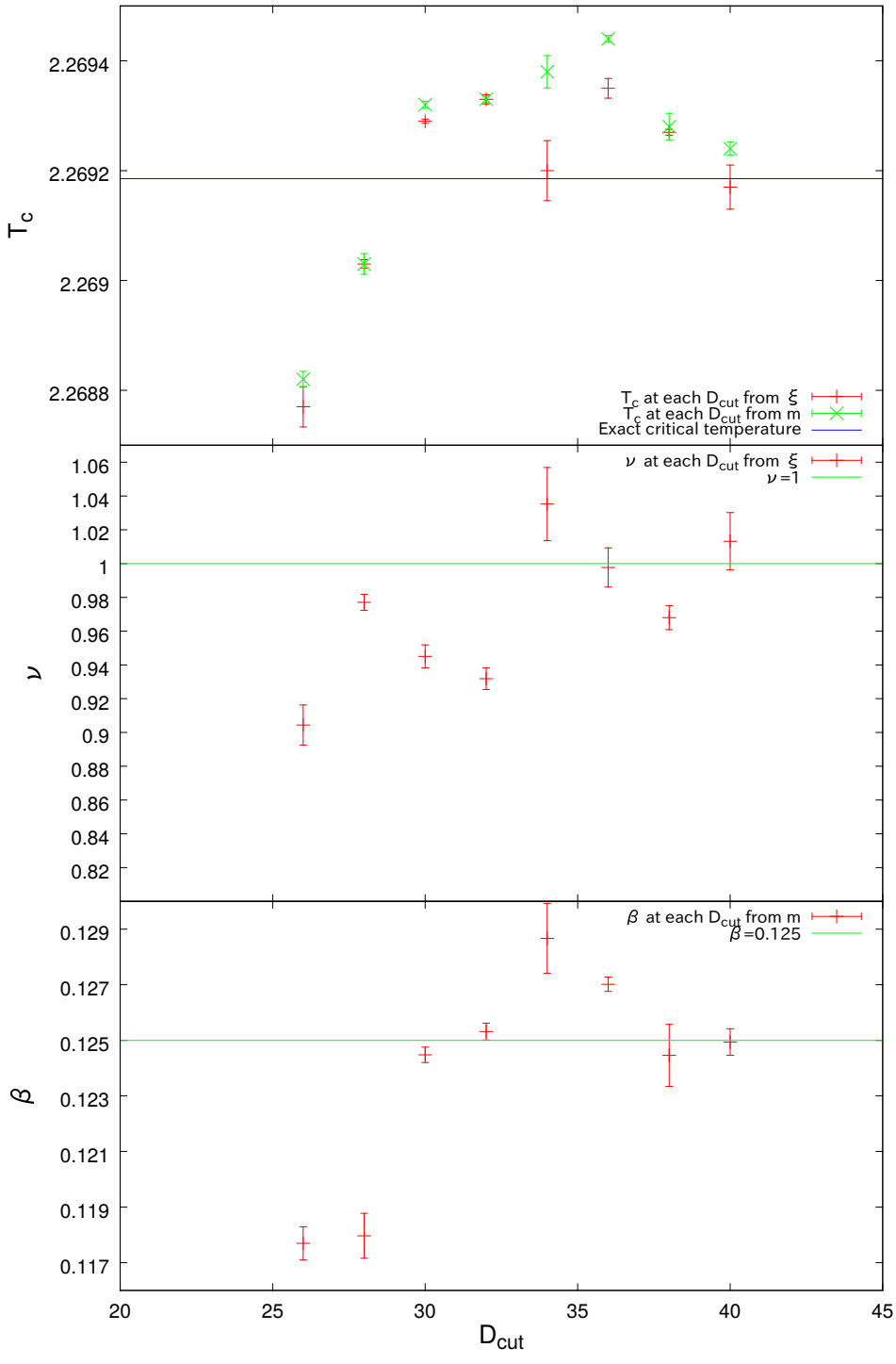

Figure 9: D_{cut}-dependence of T_{c}, ν and β.