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Spectral convergence of Riemannian vector bundles

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Abstract We consider a sequence of Hermitian vector bundles of the same rank endowed with metric connections over compact Riemannian manifolds whose heat kernels have uniform on-diagonal upper bounds, and we prove that there exists a subsequence of the vector bundles and a closed form on a Hilbert space to which the energy forms on the Hilbert spaces of square integrable sections of the vector bundles Mosco-converge; if, in addition, the rank is equal to one, the limit Hilbert space consists of square integrable sections of a continuous Hermitian line bundle over an open subspace in a compact metric space endowed with a Radon measure.

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Key words: Vector bundle, metric connection, energy form, Laplacian, spectral convergence

1 Introduction

A spectral distance on a set of compact Riemannian manifolds is introduced in [11] by means of their heat kernels, and it is proved that a family of compact Riemannian manifolds whose heat kernels uniformly satisfy on-diagonal upper estimates is precompact and further any Cauchy sequence in the family converges to a compact regular Dirichlet space in such a way that the eigenvalues and eigenfunctions of the manifolds tend to those of the limit space. The uniform topology induced from the spectral distance is related to that of the Gromov-Hausdorff distance and also the topology of Mosco-convergence of energy forms. These are the subjects of [9] and [10].

In this note, we are concerned with energy forms of Hermitian vector bundles endowed with metric connections over compact Riemannian manifolds whose heat kernels uniformly satisfy on-diagonal upper estimates. The main result is stated in the following

Theorem 1.1. *Let $\{E_n \to B_n\}$ be a sequence of Hermitian vector bundles of the same rank r endowed with metric connections over compact Riemannian manifolds B_n. Suppose that the heat kernels p_{B_n} of B_n satisfy $p_{B_n}(t,x,y) \leq A/t^{\nu/2}$ for some positive constants A and ν, and for all $t \in (0,1]$ and $x,y \in B_n$. Then there exists a subsequence $\{E_k\}$ of $\{E_n\}$ and a (symmetric) closed form $(\mathscr{F}, D[\mathscr{F}])$ on a Hilbert space to which the energy form on the Hilbert space of L^2-sections of E_k Mosco-converges as $k \to \infty$. Moreover if $r = 1$, then the Hilbert space consists of L^2-sections of a continuous Hermitian line bundle over an open subset in a compact metric space with a Radon measure.*

Here we sketch the proof of the main theorem. Let $\{E_n \to B_n\}$ be as in the theorem. Let M_n be the principal $U(r)$-bundle of unitary frames of E_n. Then passing to a subsequence, we see that M_n Mosco-converges, as $n \to \infty$, to a compact Dirichlet space on which the unitary group keeps to act continuously; the measure and the Dirichlet form are invariant under the action. The limit Hilbert space and the closed form in the main theorem are respectively given by the L^2-closure of the space of continuous functions of the Dirichlet space to $\mathbf{C}^r$ which are equivariant under the action of $U(r)$ and the form restricted to this space. If the vector bundles are of rank one, then the outside of the subset of points at which all equivariant continuous functions to $\mathbf{C}$ vanish is invariant under the action of $U(1)$ and we get a continuous Hermitian line bundle over the quotient space which is associated with the canonical action of $U(1)$ on $\mathbf{C}$. This is the limit bundle mentioned in the main theorem.

The notion of Mosco-convergence on forms is introduced in Kuwae andd Shioya [13]. In section 2, we recall several notions on convergence of forms including Mosco-convergence, and some known results relevant to the above theorem. In section 3, energy forms on vector bundles are discussed. Theorem 1.1 will be verified in section 4.

This note is a revised version of [8].

2 Convergence of compact Dirichlet spaces

In this section, we recall some results in [13] and [15] on convergence of closed forms on Hilbert spaces, and then some in [9] and [11] on that of Dirichlet spaces.

2.1 Let us consider a sequence of separable Hilbert spaces H_n, a separable Hilbert space H and a dense subspace $\mathscr{C} \subset H$, and assume that there exist linear maps Φ_n of $\mathscr{C}$ to H_n satisfying

$$\lim_{n\to\infty} |\Phi_n(u)|_{H_n} = |u|_H, \quad u \in \mathscr{C}.$$

The existence of such linear maps allows us to define several notions on convergence of functions, nonnegative symmetric closed forms, and associated operators. We follow Kuwae and Shioya [13]:

(i) A sequence of $u_n \in H_n$ *strongly converges* to $u \in H$ as $n \to \infty$ if there exists a sequence of $\tilde{u}_k$ in $\mathscr{C}$ tending to u as $k \to \infty$ such that

$$\lim_{k\to\infty} \limsup_{n\to\infty} |\Phi_n(\tilde{u}_k) - u_n|_{H_n} = 0.$$

(ii) A sequence of $u_n \in H_n$ *weakly converges* to $u \in H$ as $n \to \infty$ if

$$\lim_{n\to\infty} \langle u_n, v_n \rangle_{H_n} = \langle u, v \rangle_H$$

for every $v \in H$ and any sequence of $v_n \in H_n$ which strongly converges to v as $n \to \infty$.

(iii) A sequence of bounded linear operators $B_n : H_n \to H_n$ *strongly converges* to a bounded linear operator $B : H \to H$ as $n \to \infty$ if $B_n(u_n)$ strongly converges to $B(u)$ for every $u \in H$ and any sequence of $u_n \in H_n$ which strongly converges to u .

(iv) A sequence of closed forms $\mathscr{E}_n$ on H_n *Mosco-converges*, as $n \to \infty$, to such a form $\mathscr{E}$ on H if the following two conditions hold:

(iv-a) if a sequence of $u_n \in H_n$ weakly converges to $u \in H$, then

$$\mathscr{E}(u,u) \le \liminf_{n\to\infty} \mathscr{E}_n(u_n,u_n) \ (\le +\infty);$$

(iv-b) for any $u \in H$, there exists a sequense of $u_n \in H_n$ which strongly converges to u in such a way that

$$\limsup_{n\to\infty} \mathscr{E}_n(u_n,u_n) \le \mathscr{E}(u,u) \ (\le +\infty).$$

(v) A sequence of closed forms $\mathscr{E}_n$ on H_n is *asymptotically compact* if any sequence of $u_n \in H_n$ with $\sup_n |u_n|^2_{H_n} + \mathscr{E}_n(u_n,u_n) < +\infty$ contains a subsequence which strongly converges (to an element of H).

In case the weak convergence of a sequence of elements $u_n \in H_n$ in condition (iv-a) is replaced by the strong convergence of $\{u_n\}$, we say that the sequence of forms $\mathscr{E}_n$ *Γ-converges* to a form $\mathscr{E}$. Recall that *any sequence of closed forms on H_n admits a subsequence which Γ-converges to a closed form on H* (cf. e.g. [4]).

In the definitions above, (iv) and (v) are due to Mosco [15] for the case where all H_n coincide with H and Φ_n is the identity map of H, and the present ones are due to Kuwae and Shioya [13]; the following results are also proved in [13] and [15].

Theorem 2.1. *Let $\mathscr{E}_n$ and $\mathscr{E}$ be densely defined, closed forms on H_n and H, respectively. Let $G_{n;\zeta}$, $P_{n;t}$, $\mathscr{L}_n$, and E_n respectively be the resolvent, the semigroup, the infinitesimal generator, and the spectral resolution associated with $\mathscr{E}_n$ and also, let G_ζ, P_t, $\mathscr{L}$, and E respectively be those corresponding to $\mathscr{E}$.*

(1) *The following conditions are mutually equivalent.*

(i) *$\mathscr{E}_n$ Mosco-converges to $\mathscr{E}$.*

(ii) *For some $\zeta < 0$, $G_{n;\zeta}$ strongly converges to G_ζ.*

(iii) *For some $t > 0$, $P_{n;t}$ strongly converges to P_t.*

(iv) *If a sequence of continuous functions $\psi_n : [0,+\infty) \to \mathbf{R}$ satisfying $\lim_{x\to\infty} \psi_n(x) = 0$ uniformly converges to a continuous function $\psi : [0,+\infty) \to \mathbf{R}$ satisfying $\lim_{x\to\infty} \psi(x) = 0$, then $\psi_n(\mathscr{L}_n)$ strongly converges to $\psi(\mathscr{L})$.*

(v) *For all α, β ($\alpha < \beta$) which do not belong to the point spectrum of $\mathscr{L}$, $E_n((\alpha,\beta])$ strongly converges to $E((\alpha,\beta])$.*

(2) *If $\{\mathscr{E}_n\}$ is asymptotically compact and Γ-converges to $\mathscr{E}$ as $n \to \infty$, then it Mosco-converges to $\mathscr{E}$.*

2.2 Let X be a second countable, Hausdorff space and μ a positive σ-finite Borel measure with $\mathrm{supp}[\mu] = X$. A Dirichlet form $\mathscr{E}$ on $L^2(X,\mu)$ with domain $D[\mathscr{E}]$ is meant to be a closed symmetric form on $L^2(X,\mu)$ satisfying the unit contraction property that $u \in D[\mathscr{E}]$ implies $v = \min\{\max\{u,0\},1\} \in D[\mathscr{E}]$ and $\mathscr{E}(v,v) \le \mathscr{E}(u,u)$. When X is locally compact, μ is a positive Radon measure, and $D[\mathscr{E}]$ is dense in $L^2(X,\mu)$, we say that a Dirichlet form $\mathscr{E}$ is regular if $\mathscr{E}$ has a core, that is, a subspace $\mathscr{C}$ of $D[\mathscr{E}] \cap C_0(X)$ which is dense in $D[\mathscr{E}]$ with norm $(\mathscr{E}(*,*) + \|*\|_{L^2}^2)^{1/2}$ and dense in $C_0(X)$ with respect to uniform norm. We call two Dirichlet spaces $(X,\mu,\mathscr{E})$ and $(\widetilde{X},\widetilde{\mu},\widetilde{\mathscr{E}})$ equivalent if there is an algebraic isomorphism Φ from $D[\mathscr{E}] \cap L^\infty$ to $D[\widetilde{\mathscr{E}}] \cap L^\infty$ and Φ preserves three kinds of metrics: for $u \in D[\mathscr{E}] \cap L^\infty$, $\|u\|_{L^\infty} = \|\Phi(u)\|_{L^\infty}$, $\|u\|_{L^2} = \|\Phi(u)\|_{L^2}$, and $\mathscr{E}(u,u) = \widetilde{\mathscr{E}}(\Phi(u),\Phi(u))$. Recall the following fundamental results: (i) A Dirichlet space $(X,\mu,\mathscr{E})$ is equivalent to a regular one. (ii) If a regular Dirichlet space $(X,\mu,\mathscr{E})$ is equivalent to another one $(\widetilde{X},\widetilde{\mu},\widetilde{\mathscr{E}})$, then there exists a capacity preserving quasi homeomorphism q from X to $\widetilde{X}$ which has the properties that (ii-a) for $u \in D[\mathscr{E}] \cap L^\infty$, $\Phi(u)(\widetilde{x}) = u(q^{-1}(\widetilde{x}))$, $\widetilde{\mu}$-a.e. $\widetilde{x} \in \widetilde{X}$; (ii-b) q is measure preserving. See [6], Appendix, for these results and details.

Now we consider a regular Dirichlet space $(X,\mu,\mathscr{E})$ satisfying the following property of the semigroup P_t associated with $\mathscr{E}$:

$$\|P_t u\|_{L^\infty} \le \frac{a}{t^{\nu/4}} \|u\|_{L^2}, \quad u \in L^2(X),\ 0 < t \le 1$$

for some positive constants a and ν, or equivalently the property that P_t admits an integral kernel $p(t,x,y)$ satisfying

$$p(t,x,y) \le \frac{A}{t^{\nu/2}}, \quad a.a.\ (x,y) \in X \times X,\ 0 < t \le 1 \qquad [H]_{A,\nu}$$

for some positive constants A and ν. We recall that the property $[H]_{A,\nu}$ is equivalent to the Nash inequality:

$$\|u\|_{L^2}^{2(1+2/\nu)} \le A'(\mathscr{E}_X(u,u)+\|u\|_{L^2}^2)\|u\|_{L^1}^{4/\nu}, \quad u \in D[\mathscr{E}_X],$$

and for $\nu > 2$, it is equivalent to the Sobolev inequality:

$$\|u\|_{L^{2\nu/(\nu-2)}}^2 \le A''(\mathscr{E}_X(u,u)+\|u\|_{L^2}^2), \quad u \in D[\mathscr{E}_X], \qquad [S]_{A'',\nu}$$

where A' and A'' are positive constants depending only on A and ν; furthermore the imbedding of $D[\mathscr{E}]$ into $L^2(X)$ is compact and the spectrum of $\mathscr{E}$ is discrete if $[H]_{A,\nu}$ holds and $\mu(X)$ is finite (see [5]).

Given positive numbers A and ν, we denote by $\mathscr{D}_{A,\nu}$ the set of regular Dirichlet spaces $X=(X,\mu,\mathscr{E})$ satisfying $[H]_{A,\nu}$, $\mu(X)=1$ and $P_t1=1$. Let $\mathscr{D}=\cup_{A,\nu}\mathscr{D}_{A,\nu}$. Given $X=(X,\mu,\mathscr{E})\in\mathscr{D}$, we can define a map of X into a Banach space, using complete orthonormal systems of eigenfunctions of $L^2(X,\mu)$. To be precise, let us denote by $\mathscr{C}_*([0,\infty],\ell^2)$ the set of continuous curves $\gamma(t)$ $(t\in[0,\infty])$ in ℓ^2 such that $\gamma(0)=\gamma(\infty)=0$. Here ℓ^2 stands for the Hilbert space consisting of square summable sequences. The set is considered as a metric space with a distance

$$\Theta(\gamma,\sigma)=\sup\{\|\gamma(t)-\sigma(t)\|_{\ell^2} \mid t\in[0,\infty]\}, \quad \gamma,\sigma\in\mathscr{C}_*([0,\infty],\ell^2).$$

Let $\{\phi_i \mid i=0,1,2,\ldots\}$ be a complete orthonormal system of eigenfunctions of $\mathscr{E}$ with $\phi_0=1$. The eigenfunction ϕ_i has the i-th eigenvalue λ_i of $\mathscr{E}$ and it is assumed to be quasi continuous (cf. [6], Chap.2). Then we can define a Borel measurable map I of X into $\mathscr{C}_*([0,\infty],\ell^2)$ by

$$I[x](t)=(e^{-(t+1/t)/2}e^{-\lambda_i t/2}\phi_i(x))_{i=0,1,2,\ldots}, \quad x\in X,\ 0\le t\le+\infty.$$

The image $I[X]$ is included in a compact subset K of $\mathscr{C}_*([0,\infty],\ell^2)$, which can be chosen in such a way that $I[X]\subset K$ for all $X\in\mathscr{D}_{A,\nu}$, and there exists a sequence of continuous functions ψ_i on the closure $\overline{I[X]}$ of $I[X]$ such that $I^*\psi_i=\phi_i$ $(i=0,1,2,\ldots)$, each element γ of $\overline{I[X]}$ can be written as

$$\gamma(t)=(e^{-(t+1/t)/2}e^{-\lambda_i t/2}\psi_i(\gamma))_{i=0,1,2,\ldots},$$

and $\{\psi_i\}$ becomes a complete orthonormal system in $L^2(\tilde{X},\tilde{\mu})$, where $\tilde{\mu}$ and $\tilde{X}$ are respectively the image measure $I_*\mu$ and its support. Then the quadratic form $\tilde{\mathscr{E}}$ on $L^2(\tilde{X},\tilde{\mu})$ obtained by the system $\{\psi_i\}$ coupled with the eigenvalues $\{\lambda_i\}$ gives a regular representaion of X satisfying the following three properties: (i) the state space $\tilde{X}$ is compact, (ii) the semigroup $\tilde{P}_t$ of $\tilde{\mathscr{E}}$ admits a continuous kernel $\tilde{p}(t,x,y)$ $(t>0, x,y\in\tilde{X})$, and (iii) if we set

$$d^{spec}(x,y) = \left(\sup_{t>0} e^{-(t+1/t)}(\tilde{p}(t,x,x) + \tilde{p}(t,y,y) - 2\tilde{p}(t,x,y))\right)^{1/2}, \quad x,y \in \tilde{X},$$

then d^{spec} becomes a distance on $\tilde{X}$ which induces the same topology of $\tilde{X}$.

The Borel measurable map I described above gives rise to the equivalence between X and $\tilde{X}$, and moreover such a regular representation of $X = (X,\mu,\mathscr{E}) \in \mathscr{D}$ is unique in the sense that if $\tilde{X}' = (\tilde{X}',\tilde{\mu}',\tilde{\mathscr{E}}')$ is another one satisfying the same properties (i), (ii), and (iii) as above, then there exists a homeomorphism $f; \tilde{X} \to \tilde{X}'$ between $\tilde{X}$ and $\tilde{X}'$ such that f preserves the kernel functions, i.e., $\tilde{p}'(t,f(x),f(y)) = \tilde{p}(t,x,y)$ $(x,y \in \tilde{X})$ and the measures, i.e., $f_*\tilde{\mu} = \tilde{\mu}'$.

An element $X = (X,\mu,\mathscr{E}) \in \mathscr{D}$ is assumed, unless otherwise is stated, to possess the properties (i), (ii) and (iii) described above and we denote by μ_X, $\mathscr{E}_X$, $P_{X;t}$, and p_X respectively the measure, the Dirichlet form, the semigroup, and the kernel function of X. Since p_X is continuous, it holds that $P_{X;t}(L^2(X,\mu)) \subset C(X)$.

We observe that X is decomposed into the irreducible components $\{X_i\}_{i=1}^k$, where a pair of points $x,y \in X$ belongs to the same component if and only if $p_X(t,x,y) > 0$ for some $t > 0$ (and hence for all $t > 0$), and that if $X \in \mathscr{D}_{A,\nu}$, then the number of the components is bounded by a constant depending only on the given constants A and ν; in addition, each component is open and closed in X and a function u with $\mathscr{E}(u,u) = 0$ is constant there.

2.3 Given X,Y in $\mathscr{D}$ and a positive number ε, a Borel measurable map $f : X \to Y$ is called an *ε-spectral approximating map* if it satisfies

$$e^{-(t+1/t)}|p_X(t,x,x') - p_Y(t,f(x),f(x'))| < \varepsilon, \quad t > 0,\ x,x' \in X.$$

The *spectral distance* $SD(X,Y)$ between X and Y is by definition the greatest lower bound for positive numbers ε such that there exist ε-spectral approximating maps $f : X \to Y$ and $h : Y \to X$. The spectral distance SD gives a uniform topology on $\mathscr{D}$. We note that an ε-spectral approximating map $f : X \to Y$ induces a 4ε-Hausdorff approximating map between the metric spaces (X, d_X^{spec}) and (Y, d_Y^{spec}).

The following is proved in [11].

Theorem 2.2. *Given positive numbers A and ν, the metric space $(\mathscr{D}_{A,\nu}, SD)$ is precompact, that is, any sequence in $\mathscr{D}_{A,\nu}$ contains SD-Cauchy subsequences. Moreover let $\{X_n = (X_n,\mu_{X_n},\mathscr{E}_{X_n})\}$ be an SD-Cauchy sequence in $\mathscr{D}_{A,\nu}$. Then there exists a compact metric space $(\hat{X}, \hat{d}_X^{spec})$, a nonnegative Radon measure μ_X on $\hat{X}$, a nonnegative continuous function $p_X(t,x,y)$ on $(0,\infty) \times \hat{X} \times \hat{X}$, Borel measurable maps $f_n : X_n \to \hat{X}$,*

$h_n : \hat{X} \to X_n$, and a sequence of positive numbers $\{\varepsilon_n\}$ tending to zero as $n \to \infty$, which satisfy the following properties:

(1) *The function $p_X(t,x,y)$ is the kernel of a strongly continuous semigroup $\{P_t : t > 0\}$ on $L^2(X,\mu_X)$ associated with a regular Dirichlet form $\mathscr{E}_X$ on $L^2(X,\mu_X)$, where X denotes the support of μ_X.*

(2) *The regular Dirichlet space $(X,\mu_X,\mathscr{E}_X)$ belongs to the same classe $\mathscr{D}_{A,\nu}$.*

(3) *The distance $\hat{d}_X^{spec}$ is given by*

$$\hat{d}_X^{spec}(x,y) = \left(\sup_{t>0} e^{-(t+1/t)}(p_X(t,x,x) + p_X(t,y,y) - 2p_X(t,x,y)\right)^{1/2}, \ x,y \in \hat{X}.$$

(4) *The push-forward $f_{n*}\mu_n$ of the measure μ_n by f_n converges to the measure μ_X with respect to the vague topology.*

(5) *The maps f_n and h_n are ε_n-spectral approximating maps in the sense that*

$$\sup_{t>0} e^{-(t+1/t)}|p_{X_n}(t,x,y) - p_X(t,f_n(x),f_n(y))| < \varepsilon_n; \quad x,y \in X_n$$

$$\sup_{t>0} e^{-(t+1/t)}|p_{X_n}(t,h_n(x),h_n(y)) - p_X(t,x,y)| < \varepsilon_n, \quad x,y \in \hat{X},$$

and furthermore

$$\hat{d}_X^{spec}(f_n \circ h_n(x),x) < \varepsilon_n, \quad x \in \hat{X}.$$

(6) *The i-th eigenvalue $\lambda_i^{(n)}$ of $\mathscr{E}_{X_n}$ for each $i = 0,1,2,\ldots$ converges to the i-th eigenvalue λ_i of $\mathscr{E}_X$ as $n \to \infty$, and further letting a positive integer i be fixed, for each eigenfunction u of $\mathscr{E}_n$ with eigenvalue $\lambda_i^{(n)}$ and unit L^2-norm, there exists a continuous function v on $\hat{X}$ which is an eigenfunction of eigenvalue λ_i on X, such that*

$$\sup_{x\in X_n} |u(x) - v(f_n(x))| < \varepsilon_{i;n} \ ; \ \sup_{x\in \hat{X}} |u(h_n(x)) - v(x)| < \varepsilon_{i;n},$$

where $\{\varepsilon_{i;n}\}$ is a sequence of positive numbers depending only on A, ν, and i, and tending to zero as $n \to \infty$.

The property (5) above implies that $f_n : X_n \to \hat{X}$ and $h_n : \hat{X} \to X_n$ are $3\varepsilon_n$-Hausdorff approximating maps between the metric spaces $(X_n, d_{X_n}^{spec})$ and $(\hat{X}, \hat{d}_X^{spec})$, and hence the sequence $\{(X_n, d_{X_n}^{spec})\}$ converges to $(\hat{X}, \hat{d}_X^{spec})$ as $n \to \infty$ with respect to the Gromov-Hausdorff distance. We remark that if X_n (resp. X) has k_n (resp. k) irreducible components, then $\limsup_{n\to\infty} k_n \le k$; the equality does not hold true in general as simple examples show.

Let $X_n = (X_n, \mu_{X_n}, \mathscr{E}_{X_n})$, $(\hat{X}, \hat{d}_X^{spec})$, $X = (X, \mu_X, \mathscr{E}_X)$, $f_n : X_n \to \hat{X}$ be as in Theorem 2.2. Let $\mathscr{C}$ be the algebra generated by the eigenfunctions of $\mathscr{E}_X$. Then $\mathscr{C}$ is a subalgebra of $C(\hat{X})\cap D[\mathscr{E}_X]$ and becomes a core of $D[\mathscr{E}_X]$. In fact, $\mathscr{C}$ is dense both in $C(\hat{X})$ and in $D[\mathscr{E}_X]$. Let us define a linear map $\Phi_n : \mathscr{C}\to L^2(X_n)$ by

$$\Phi_n(u) = {f_n}^* u, \quad u\in\mathscr{C}.$$

Then Theorem 3.2 in [9] is restated in the following

Theorem 2.3. *The sequence of forms $\{\mathscr{E}_{X_n}\}$ is asymptotically compact and Mosco-converges to the form $\mathscr{E}$*

Now we would like to introduce an intrinsic pseudo distance on $X\in\mathscr{D}$. For any $u\in D[\mathscr{E}_X]\cap L^\infty$, the energy measure μ_u is defined by

$$\mu_u(\phi) = \mathscr{E}_X(u,\phi u) - \frac{1}{2}\mathscr{E}_X(u^2,\phi), \quad \phi\in D[\mathscr{E}_X]\cap C(X).$$

We assume that $\mathscr{E}_X$ is strongly local. Letting $\mathscr{A}[\mathscr{E}] = \{u\in D[\mathscr{E}_X] \mid \mu_u = \gamma(u)\mu_X, \gamma(u)\in L^1(X,\mu_X)\}$, we introduce a pseudo distance, called the intrinsic pseudo distance of $\mathscr{E}_X$, by

$$d_X(x,y) = \sup\{|u(x)-u(y)| \mid u\in\mathscr{A}[\mathscr{E}_X]\cap C(X), \gamma(u)\le 1\ \mu_X - a.e.\},\ x,y\in X.$$

Then the on-diagonal estimate of the kernel function, $[H]_{A,\nu}$, yields the following:

$$d_X(x,y)\le\psi(\hat{d}_X^{spec}(x,y)), \quad x,y\in X$$

and

$$|d_X(x,y)-d_X(x',y')|\le\psi(\hat{d}_X^{spec}(x,y)+\hat{d}_X^{spec}(x',y')), \quad x,x',y,y'\in X,$$

where $\psi(t)$ is a monotone increasing continuous function on $[0,+\infty)$ with $\psi(0)=0$ depending only on A and ν. This suggests that the approximating maps f_n in Theorem 2.2 is also such a map for the intrinsic pseudo distance of X_n. In fact, the following is shown in [9].

Theorem 2.4. *Let $X_n = (X_n,\mu_{X_n},\mathscr{E}_{X_n})$, $(\hat{X},\hat{d}_X^{spec})$, $X = (X,\mu_X,\mathscr{E}_X)$, and $f_n : X_n\to\hat{X}$ be as in Theorem 2.2 and assume that each X_n is strongly local and the intrinsic pseudo distance d_{X_n} defines a distance which induces the same topology of X_n. Then the following assertions hold:*

(1) *There exist a subsequence $\{X_m\}$, a sequence of positive numbers $\{\varepsilon_m\}$ tending to zero as $m\to\infty$, and a continuous pseudo distance δ on $\hat{X}$ such that the maps $f_m : X_m\to\hat{X}$ are ε_m-Hausdorff approximating maps between (X_m,d_{X_m}) and $(\hat{X},\delta)$, that is,*

$f_m(X_m)$ is ε_m-dense in $(\hat{X},\delta)$ and

$$|d_{X_m}(x,y) - \delta(f_m(x), f_m(y))| < \varepsilon_m, \quad x,y \in X_m.$$

(2) *The kernel function p_X of X has an off-diagonal upper bound as follows:*

$$p_X(t,x,y) \le \frac{A(\alpha)}{t^{\nu/2}} \exp\left(-\frac{\delta(x,y)^2}{(4+\alpha)t}\right), \quad t \in (0,1],\ x,y \in \hat{X}, \tag{2.1}$$

where α is any positive constant and $A(\alpha)$ is a positive constant depending only on A and α.

(3) *Let δ be a continuous pseudo distance on $\hat{X}$ obtained in the first assertion and $C^{0,1}(\hat{X},\delta)$ the space of functions on $\hat{X}$ which are Lipschitz continuous with respect to δ. Then $C^{0,1}(\hat{X},\delta) \subset D[\mathscr{E}_X] \cap C(\hat{X})$ and for $u \in C^{0,1}(\hat{X},\delta)$ and $v \in D[\mathscr{E}_X]$, $\mathscr{E}_X(u,v) = 0$ if the support of u does not intersect that of v. Moreover the energy measure μ_u of $u \in C^{0,1}(\hat{X},\delta)$ is absolutely continuous with respect to the measure μ_X and the Radon-Nikodym derivative $\gamma(u) = d\mu_u/d\mu_X$ satisfies*

$$\gamma(u)(x)^{1/2} \le \mathrm{dil}_\delta u(x), \quad a.a.\ x \in X.$$

The local dilatation of a Lipschitz function in this theorem is defined as follows: Given a Lipschitz function u on a subspace A of (X,δ), the dilatation of u on A, that is the infimal number λ satisfying $|u(x) - u(y)| \le \lambda\delta(x,y)$ for all $x,y \in A$, is denoted by $\mathbf{dil}_\delta(u)$, and for a Lipschitz function u on X, the local dilatation of u at a point x is the number

$$\mathrm{dil}_\delta u(x) = \lim_{r\to 0} \mathbf{dil}_\delta(u_{|B_\delta(x,r)}),$$

where $B_\delta(x,r)$ stands for the metric ball around x of radius r with respect to the pseudo distance δ.

Let us denote by $\hat{X}_\delta$ and $\pi_\delta;\hat{X} \to \hat{X}_\delta$ respectively the metric space obtained by identifying points x,y of $\hat{X}$ with $\delta(x,y) = 0$ and the projection from $\hat{X}$ onto $\hat{X}_\delta$. The first assertion of Theorem 2.4 says that (X_m, d_{X_m}) converges to the metric space $(\hat{X}_\delta,\delta)$ with respect to the Gromov-Hausdorff distance via the approximating maps $\pi_\delta \circ f_m : X_m \to \hat{X}_\delta$.

By virtue of the off-diagonal estimate (2.1) for $p_X(t,x,y)$, we can deduce the following assertions:

(i) For any $x \in \hat{X}$, $\delta(x,X) = 0$.

(ii) For any continuous function u on $\hat{X}$ which comes from the quotient space $\hat{X}_\delta$, i.e., $u \in \pi_\delta^* C(\hat{X}_\delta)$, we have

$$\lim_{t\to 0} \sup_{x\in\hat{X}} \left| \int_X p_X(t,x,y)u(y)\, d\mu(y) - u(x) \right| = 0. \qquad [SC]$$

(iii) If δ is nondegenerate in an open subset Ω of $\hat{X}$, i.e., $\delta(x,y) > 0$ for all $x,y \in \Omega$ with $x \neq y$, then [SC] holds true in each compact subset of Ω for any $u \in C(\Omega)$. Note that there are examples such that δ degenerates somewhere, but $X = \hat{X}$ and [SC] holds true in X for any $u \in C(X)$ (cf. [12], [16]).

(iv) If δ is nondegenerate in Ω, then $\mathscr{E}_X(u,v) = 0$ for all $u,v \in D[\mathscr{E}_X]$ such that u is supported in Ω and v is constant on the support of u. Notice again that our limit spaces admit non-local forms. See [9] for elementary examples, and also [16] for constructing non-local forms as limits of those of local type.

3 Energy forms on vector bundles

Let B be a smooth manifold and G a compact Lie group. We consider a principal bundle $M = (M, B, \pi : M \to B)$ over B with structure group G. Let V be a finite dimensional complex vector space endowed with a Hermitian inner product $(\ ,\ )_V$. Associated to a finite dimensional unitary representation $\rho : G \to U(V)$ of G, we have a Hermitian vector bundle $E_\rho = (M \times_\rho V, B, \pi_\rho : M \times_\rho V \to B)$. Let $C^\infty(M,V)^\rho$ denote the space of ρ-equivariant smooth maps f of M to V, that is, smooth maps $f : M \to V$ satisfying

$$f(xa) = \rho(a^{-1})f(x), \quad x \in M,\ a \in G.$$

Then there is a canonical correspondence between $C^\infty(M,V)^\rho$ and the space of smooth sections $\Gamma(B,E_\rho)$ of the vector bundle E_ρ on B: given $f \in C^\infty(M,V)^\rho$, we define $\sigma(f) \in \Gamma(B,E_\rho)$ by

$$\sigma(f)(y) = [x, f(x)], \quad y \in B,$$

where $x \in \pi^{-1}(y)$ and $[x, f(x)]$ stands for the equivalence class containing $(x, f(x))$, $\{(xa, \rho(a^-)f(x)) \in M \times V \mid a \in G\}$.

Let $\mathscr{L}(G)$ be the Lie algebra of G. Given a principal connection $\omega : TM \to \mathscr{L}(G)$ on the principal bundle M, we have a linear connection ∇ on E_ρ, which is compatible to the Hermitian inner product, in such a way that

$$\nabla_X \sigma(f)(y) = [x, df(X_H) + \rho(\omega(X_H))f(x)] = [x, X_H(f)(x)], \quad X \in T_yB,\ y \in B,$$

where $x \in \pi^{-1}(y)$ and X_H denotes the horizontal lift of a tangent vector X of B at $y \in B$ to T_xM.

In what follows, we assume that $B = (B, g_B)$ is a compact Riemannian manifold and G is endowed with the bi-invariant metric $< \, , \, >$ normalized in such a way that G has unit volume, $\mathrm{Vol}(G) = 1$. Then given a connection form $\omega : TM \to \mathscr{L}(G)$, we define a Riemannian metric g_ω on the total space M by

$$g_\omega = \pi^* g_B + < \omega, \omega > .$$

The metric g_ω is G-invariant and the projection $\pi : (M, g_\omega) \to (B, g_B)$ is a Riemannian submersion with totally geodesic fibers isometric to G. Suppose that the representation $\rho : G \to U(V)$ is irreducible. So far as the bijective correspondence between $C^\infty(M,V)^\rho$ and $\Gamma(B, E_\rho)$ is concerned, we have

$$\int_M |f|^2 d\mu_M = \int_B |\sigma(f)|^2 d\mu_B, \quad f \in C^\infty(M,V)^\rho;$$

$$\int_M |df|^2_{g_\omega} d\mu_M = \int_B |\nabla\sigma(f)|^2 + C_\rho |\sigma(f)|^2 d\mu_B, \quad f \in C^\infty(M,V)^\rho,$$

where μ_M and μ_B respectively denote the normalized volume elements of $M = (M, g_\omega)$ and $B = (B, g_B)$, and C_ρ stands for the eigenvalue of the Casimir operator (or the Laplace operator of G) of the irreducible representation ρ, namely $C_\rho id_V = -\sum_{\alpha=1}^{\dim\rho} \rho(e_\alpha)^2$ (cf. e.g., [17]). It follows that $f \in C^\infty(M,V)^\rho$ is an eigenfunction of the Laplacian of M with eigenvalue $\lambda + C_\rho$ if and only if $\sigma(f)$ is an eigensection of the rough Laplacian $-\nabla^*\nabla$ of the vector bundle E_ρ with eigenvalue λ:

$$\Delta_M f + (\lambda + C_\rho) f = 0 \Longleftrightarrow -\nabla^*\nabla\sigma(f) + \lambda\sigma(f) = 0.$$

Now we define a closed (quadratic) form $(\mathscr{F}_B^{(\rho)}, D[\mathscr{F}_B^{(\rho)}])$ on the Hilbert space $L^2(B, E_\rho)$ of square integrable sections of E_ρ by

$$\mathscr{F}_B^{(\rho)}(\sigma) = \int_B |\nabla\sigma|^2 d\mu_B, \quad \sigma \in D[\mathscr{F}_B^{(\rho)}].$$

Then Kato's inequality in [7] (cf. also [1]) reads as follows:

$$\mathscr{E}_B(|\sigma|) \leq \mathscr{F}_B^{(\rho)}(\sigma), \quad \sigma \in D[\mathscr{F}_B^{(\rho)}]. \tag{3.1}$$

Let R be a continuous section of the endomorphism algebra bundle $\mathrm{Hom}(E_\rho, E_\rho)$ which is symmetric with respect to the inner product. Then a closed form $(\mathscr{F}_{B;R}^{(\rho)}, D[\mathscr{F}_{B;R}^{(\rho)}])$ on $L^2(B, E_\rho)$ is given by

$$\mathscr{F}_{B;R}^{(\rho)}(\sigma) = \int_B |\nabla\sigma|^2 d\mu_B + \int_B (R(\sigma), \sigma)_{E_\rho} d\mu_B.$$

Note that the domain $D[\mathscr{F}_{B;R}^{(\rho)}]$ of $\mathscr{F}_{B;R}^{(\rho)}$ coincides with that of $\mathscr{F}_{B}^{(\rho)}$, because R is assumed to be continuous.

Lemma 3.1. *Let $r(x)$ be the least eigenvalue of R_x ($x \in B$) and let $r^-(x) = \max\{-r(x), 0\}$.*

(i) Suppose that the base Riemannian manifold B satisfies the Sobolev inequality $[S]_{\nu,A''}$ for $\nu > 2$, $A'' > 0$, and

$$\int_B |r^-|^p d\mu_B \leq K^p$$

for some $p > \nu/2$ and $K > 0$. Then for any $\alpha \in (0,1)$, one has

$$\mathscr{F}_{B;R}^{(\rho)}(\sigma) \geq (1-\alpha)\mathscr{F}_{B}^{(\rho)}(\sigma) - \beta \int_B |\sigma|^2 d\mu_B, \quad \sigma \in D[\mathscr{F}_{B;R}^{(\rho)}]$$

where β is a positive constant depending only on ν, A'', p, K and α.

(ii) Suppose that the base Riemannian manifold B satisfies the Sobolev inequality $[S]_{\nu,A}$ for $\nu > 2$, $A'' > 0$, and

$$\int_B |r^-|^{\nu/2} d\mu_B \leq \frac{\alpha}{A''}$$

for some $\alpha \in (0,1)$. Then one has

$$\mathscr{F}_{B;R}^{(\rho)}(\sigma) \geq (1-\alpha)\mathscr{F}_{B}^{(\rho)}(\sigma) - \beta \int_B |\sigma|^2 d\mu_B, \quad \sigma \in D[\mathscr{F}_{B;R}^{(\rho)}],$$

where β is a positive constant depending only on ν, A'' and α.

Proof. Let $L(s) = \{x \in B \mid r^-(x) \geq s\}$ for $s \geq 0$. Then we have

$$\begin{aligned}
&\int_B r^- |\sigma|^2 d\mu_B \\
&\quad = \int_{B \setminus L(s)} r^- |\sigma|^2 d\mu_B + \int_{L(s)} r^- |\sigma|^2 d\mu_B \\
&\quad \leq s \int_B |\sigma|^2 d\mu_B + \left(\int_{L(s)} (r^-)^{\nu/2} d\mu_B \right)^{2/\nu} \left(\int_B |\sigma|^{2\nu/(\nu-2)} d\mu_B \right)^{(\nu-2)/\nu} \\
&\quad \leq s \int_B |\sigma|^2 d\mu_B + s^{(\nu-2p)/\nu} \left(\int_{L(s)} (r^-)^p d\mu_B \right)^{2/\nu} \left(\int_B |\sigma|^{2\nu/(\nu-2)} d\mu_B \right)^{(\nu-2)/\nu} \\
&\quad \leq s \int_B |\sigma|^2 d\mu_B + s^{(\nu-2p)/\nu} K^{2p/\nu} \left(\int_B |\sigma|^{2\nu/(\nu-2)} d\mu_B \right)^{(\nu-2)/\nu},
\end{aligned}$$

and then applying the Sobolev inequality , we get

$$\int_B r^-|\sigma|^2 d\mu_B \le s\int_B |\sigma|^2 d\mu_B + s^{(\nu-2p)/\nu}K^{2p/\nu}A''\left(\mathscr{E}_B(|\sigma|)+\int_B|\sigma|^2 d\mu_B\right).$$

Take s so that $s^{(\nu-2p)/\nu}K^{2p/\nu}A'' = \alpha$. Then using Kato's inequality, we obtain

$$\int_B r^-|\sigma|^2 d\mu_B \le \alpha\mathscr{F}_B(\sigma) + \beta\int_B|\sigma|^2 d\mu_B,$$

where we put $\beta = \alpha + s$. Using this, we get

$$\begin{aligned}
\mathscr{F}^{(\rho)}_{B;R}(\sigma) &= \mathscr{F}^{(\rho)}_B(\sigma) + \int_B (R(\sigma),\sigma)_{E_{|rho}} d\mu_B \\
&\ge \mathscr{F}^{(\rho)}_B(\sigma) - \int_B r^-|\sigma|^2 d\mu_B \\
&\ge (1-\alpha)\mathscr{F}^{(\rho)}_B(\sigma) - \beta\int_B |\sigma|^2 d\mu_B.
\end{aligned}$$

This shows the assertion (i). If we take $s = 0$ and $p = \nu/2$ as in the above arguments we can prove the second assertion (ii). This completes the proof of Lemma 3.1. □

Now let $\hat{G}$ denote the set of irreducible unitary representations of G. Given $\rho \in \hat{G}$, $\rho : G \to U(V)$, we consider smooth functions $(f,v)_V$ on M, where $f \in C^\infty(M,V)^\rho$ and $v \in V$, and denote by $C^\infty(M,\rho)$ the vector space spanned by such functions. This is canonically isomorphic to the vector space $C^\infty(M,V)^\rho \otimes V^*$, and invariant by the action of G. Then Schur's orthogonal relation shows that for ρ, $\rho' \in \hat{G}$ with $\rho \ne \rho'$, $C^\infty(M,\rho)$ and $C^\infty(M,\rho')$ are orthogonal in $L^2(M,\mu_M)$, that is,

$$\int_M (f,v)_V(f',v')_V d\mu_M = \begin{cases} \int_M (f,f')_V d\mu_M (v,v')_V, & \text{if } \rho = \rho', \\ 0, & \text{if } \rho \ne \rho'. \end{cases}$$

Let $L^2(M,\rho)$ denote the closure of the subspace $C^\infty(M,\rho)$ in $L^2(M,\mu_M)$. Then according to Peter-Weyl 's theorem, we have the orthogonal decomposition of $L^2(M,\mu_M)$ as follows:

$$L^2(M,\mu_M) = \sum_{\rho\in\hat{G}} L^2(M,\rho).$$

We remark that for the trivial representation $\rho = 1$, the space $L^2(M,1)$ can be identified with $L^2(B,\mu_B)$ via the projection $\pi : M \to B$, that is, $L^2(M,1) = \{\pi^*u \mid u \in L^2(B,\mu_B)\}$, and it holds that

$$\int_M |\pi^*u|^2 d\mu_M = \int_B |u|^2 d\mu_B,\ \mathscr{E}_M(\pi^*u,\pi^*u) = \mathscr{E}_B(u,u), \Delta_M\pi^*u = \pi^*\Delta_B u.$$

For $\rho \in \hat{G}$, let $\{\lambda_i^{(\rho)}\}$ be the eigenvalues of the rough Laplacian $-\nabla^*\nabla$ acting on $L^2(B,E_\rho)$ and take a complete orthonormal system of eigenfunctions $\{\phi_i^{(\rho)}\}$ in $C(M,V)^\rho$ such that $\phi_i^{(\rho)}$ has eigenvalue $\lambda_i^{(\rho)}+C_\rho$. Let $\{e_1,e_2,\dots,e_{\dim\rho}\}$ be an orthonormal basis of V. Then $\{\sqrt{\dim\rho}(\phi_i^{(\rho)},e_\alpha)_V \mid i=1,2,\dots,\alpha=1,2,\dots,\dim\rho\}$ is a complete orthonormal system of eigenfunctions of $L^2(M,\rho)$. Set

$$p^{(\rho)}(t,x,y)=\frac{\dim\rho}{2}e^{-C_\rho t}\left(\sum_{i=1}^{\infty}e^{-\lambda_i^{(\rho)}t}\{(\phi_i^{(\rho)}(x),\phi_i^{(\rho)}(y))_V+(\phi_i^{(\rho)}(y),\phi_i^{(\rho)}(x))_V\}\right),$$

where $t>0$ and $x,y\in M$. Then the heat kernel p_M of M is decomposed into the sum of $p^{(\rho)}$ $(\rho\in\hat{G})$ as follows:

$$p_M(t,x,y)=\sum_{\rho\in\hat{G}}p^{(\rho)}(t,x,y),\quad t>0,\ x,y\in M. \tag{3.2}$$

For $\rho=1$, we have

$$p^{(1)}(t,x,y)=p_B(t,\pi(x),\pi(y)),\quad t>0,\ x,y\in M.$$

For $\rho\in\hat{G}$, $\rho\neq 1$, we can apply a result by [7] to get the following estimate:

$$p^{(\rho)}(t,x,y)\le \dim\rho\ e^{-C_\rho t}p_B(t,\pi(x),\pi(y)),\quad t>0,\ x,y\in M. \tag{3.3}$$

This implies in particular that

$$\begin{aligned} p_M(t,x,y) &\le \left(\sum_{\rho\in\hat{G}}\dim\rho\ e^{-C_\rho t}\right)p_B(t,\pi(x),\pi(y)) \\ &= \int_G p_G(t,a,a)d\mu_G(a)\ p_B(t,\pi(x),\pi(y)),\quad x,y\in M \end{aligned}$$

(cf. [3]).

Lemma 3.2. *There exists a positive constant C depending only on the dimension d and the diameter* $\operatorname{diam}(G)$ *of G such that*

$$\begin{aligned} &0\le p_M(t,x,x)-p_M(t,x,xa) \\ &\quad\le C\left(\sum_{\rho\in\hat{G}}(\dim\rho)^2C_\rho^{(1+d)/2}e^{-C_\rho t}\right)|a|_G\ p_B(t,\pi(x),\pi(x)),\ x\in M,\ a\in G, \end{aligned}$$

where $|a|_G$ stands for the distance between a and the unit element e of G.

Proof. Let $\rho \in \hat{G}$ and $\rho : G \to U(V)$. We observe first that

$$(\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(x))_V - \frac{1}{2}\{(\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(xa))_V + (\phi_i^{(\rho)}(xa), \phi_i^{(\rho)}(x))_V\} =$$

$$\frac{1}{2}\{(\phi_i^{(\rho)}(x), (1-\rho(a^{-1}))\phi_i^{(\rho)}))_V + ((1-\rho(a^{-1}))\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(x))_V\}.$$

Let $\{e_\alpha(a) \mid \alpha = 1, \ldots, \dim\rho\}$ be an orthonormal basis of V consisting of eigenvectors $e_\alpha(a)$ of $\rho(a^{-1})$ with eigenvalues $\exp\sqrt{-1}\theta_\alpha(a)$. Then the right-hand side of the above identity is equal to $\sum_{\alpha=1}^{\dim\rho}(1-\cos\theta_\alpha(a))|(\phi_i^{(\rho)}(x), e_\alpha(a))_V|^2$, and hence we have

$$(\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(x))_V - \frac{1}{2}\{(\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(xa))_V + (\phi_i^{(\rho)}(xa), \phi_i^{(\rho)}(x))_V\}$$

$$= \sum_{\alpha=1}^{\dim\rho}(1-\cos\theta_\alpha(a))|(\phi_i^{(\rho)}(x), e_\alpha(a))_V|^2. \tag{3.4}$$

Let $\chi_\rho : G \to \mathbf{C}$ be the character of ρ, that is, $\chi_\rho = \operatorname{trace}\rho$. Then we have

$$\begin{aligned} \sum_{\alpha=1}^{\dim\rho}(1-\cos\theta_\alpha(a))|(\phi_i^{(\rho)}(x), e_\alpha(a))_V|^2 &\le \left(\sum_{\alpha=1}^{\dim\rho}(1-\cos\theta_\alpha(a))\right)|\phi_i^{(\rho)}(x)|_V^2 \\ &\le |\chi_\rho(e) - \chi_\rho(a)||\phi_i^{(\rho)}(x)|_V^2, \end{aligned}$$

and thus by (3.3) and (3.4)), we get

$$\begin{aligned} p^{(\rho)}(t,x,x) - p^{(\rho)}(t,x,xa) &\le e^{-C_\rho t}|\chi_\rho(e) - \chi_\rho(a)|\sum_{i=1}^{\infty} e^{-\lambda_i^{(\rho)}t}|\phi_i^{(\rho)}(x)|_V^2 \\ &\le \dim\rho\; e^{-C_\rho t}|\chi_\rho(e) - \chi_\rho(a)|p_B(t,\pi(x),\pi(x)) \end{aligned} \tag{3.5}$$

Here since G has nonnegative sectional curvature and χ_ρ is an eigenfunction of eigenvalue C_ρ, we apply a well known estimate due to Li-Yau [14] to χ_ρ, and obtain

$$|\chi_\rho(e) - \chi_\rho(a)| \le C\dim\rho\; C_\rho^{(1+d)/2}|a|_G, \tag{3.6}$$

where C is a positive constant depending only on d and $\operatorname{diam}(G)$. Then in view of (3.2), (3.5) and (3.6), we arrive at the inequality of Lemma 3.2. □

Let d_M and d_B be, respectively, the Riemannian distances of M and B. Then d_M is G-invariant and the projection $\pi : M \to B$ is contractive, that is, $d_B(\pi(x), \pi(y)) \le d_M(x,y),\;\; x,y \in M$. Moreover since the fibers are totally geodesic and isometric to G, for any $x,y \in M$, $d_M(x,ya) = d_B(\pi(x),\pi(y))$ for some $a \in G$, and it holds that

$|d_M(x,y) - d_M(xa,yb)| \leq |ab^{-1}|_G, \quad x,y \in M,\ a,b \in G.$

Let d_M^{spec} and d_B^{spec} be, respectively, the distances on M and B determined by their heat kernels. It is clear that d_M^{spec} is G-invariant and the projection $\pi : M \to B$ is a contraction map with respect to these distances, that is,

$$d_B^{spec}(\pi(x),\pi(y)) \leq d_M^{spec}(x,y), \quad x,y \in M.$$

Lemma 3.3. *Suppose that B satisfies $[H]_{\nu,A}$ and the diameter of G is bounded from above by a constant D. Then there exists a positive constant C depending only on ν, A, D, and d such that*

$$|d_M^{spec}(x,y) - d_M^{spec}(xa,yb)| \leq C|ab^{-1}|_G^{1/2}, \quad x,y \in M,\ a,b \in G$$

Proof. It follows from Lemma 3.2 that

$$\begin{aligned} d_M^{spec}(x,xa)^2 &= \sup_{t>0} e^{-(t+1/t)}(p_M(t,x,x) + p_M(t,xa,xa) - 2p_M(t,x,xa)) \\ &= 2\sup_{t>0} e^{-(t+1/t)}(p_M(t,x,x) - p_M(t,x,xa)) \\ &\leq 2C'|a|_G, \end{aligned}$$

where C' is a positive consatnt depending only on ν, A, D and d. Therefore we have

$$\begin{aligned} d_M^{spec}(xa,yb) &= d_M^{spec}(xab^{-1},y) \\ &= d_M^{spec}(xab^{-1},y) - d_M^{spec}(xab^{-1},x) + d_M^{spec}(xab^{-1},x) \\ &\leq d_M^{spec}(x,y) + (2C')^{1/2}|ab^{-1}|_G^{1/2}. \end{aligned}$$

Similarly we have

$$d_M^{spec}(x,y) = d_M^{spec}(xaa^{-1},ybb^{-1}) \leq d_M^{spec}(xa,yb) + (2C')^{1/2}|ab^{-1}|_G^{1/2}.$$

This completes the proof of Lemma 3.3. □

Remark If B satisfies $[H]_{A,\nu}$, then M satisfies $[H]_{A',\nu+d}$ for some constant A' depending only on A, the dimension and the diameter of G. If the sectional curvature of B and the curvature of a principal connection on M have double-sides bounds, then the sectional curvature of M is also bounded from both sides (cf. [2], Chap.9).

4 Convergence of Riemannian vector bundles

In this section, we consider a sequence of principal G-bundles $M_n = (M_n, B_n, \pi_n : M_n \to B_n)$, where G is a compact Lie group of dimension d endowed with the normalized bi-invariant metric, the total space M_n has the Riemannian metric g_{ω_n} associated to a principal connection ω_n, and the base space B_n is a compact Riemannian manifold. We assume that all B_n satisfy $[H]_{A,\nu}$ for some A and ν, and hence all M_n also satisfy $[H]_{A',\nu+d}$. Then in view of Theorem 2.2, we see by passing to a subsequence that there exists a compact metric space $(\hat{X}, \hat{d}_X^{spec})$, a nonnegative Radon measure μ_X on $\hat{X}$, a nonnegative continuous function $p_X(t,x,y)$ on $(0,\infty) \times \hat{X} \times \hat{X}$, Borel measurable maps $f_n : B_n \to \hat{X}$, $h_n : \hat{X} \to B_n$, and a sequence of positive numbers $\{\varepsilon_n\}$ tending to zero as $n \to \infty$, which satisfy the following properties: (1) $p_X(t,x,y)$ is the kernel of a strongly continuous semigroup on $L^2(X, \mu_X)$ associated with a regular Dirichlet form $\mathscr{E}_X$ on $L^2(X, \mu_X)$, where X denotes the support of μ; (2) the regular Dirichlet space $(X, \mu_X, \mathscr{E}_X)$ belongs to the same classe $\mathscr{D}_{A,\nu}$; (3) the distance $\hat{d}_X^{spec}$ is given by $\hat{d}_X^{spec}(x,y) = \big(\sup_{t>0} e^{-(t+1/t)}(p_X(t,x,x) + p_X(t,y,y) - 2p_X(t,x,y))\big)^{1/2}$ for $x, y \in \hat{X}$; (4) the push-forward $f_{n*}\mu_{B_n}$ of the measure μ_{B_n} by f_n converges to the measure μ_X with respect to the vague topology; (5) the maps f_n and h_n are ε_n-spectral approximating maps in the sense that $\sup_{t>0,x,y\in B_n} e^{-(t+1/t)}|p_{X_n}(t,x,y) - p_X(t, f_n(x), f_n(y))| < \varepsilon_n$, $\sup_{t>0,x,y\in\hat{X}} e^{-(t+1/t)}|p_{X_n}(t, h_n(x), h_n(y)) - p_X(t,x,y)| < \varepsilon_n$, and furthermore $\sup_{x\in\hat{X}} \hat{d}_X^{spec}(f_n \circ h_n(x), x) < \varepsilon_n$.

Similarly, passing to a subsequence, we see that there exists a compact metric space $(\hat{Y}, \hat{d}_Y^{spec})$, a nonnegative Radon measure μ_Y on $\hat{Y}$, a nonnegative continuous function $p_Y(t,x,y)$ on $(0,\infty) \times \hat{Y} \times \hat{Y}$, Borel measurable maps $F_n : M_n \to \hat{Y}$, $H_n : \hat{Y} \to M_n$, and a sequence of positive numbers $\{\varepsilon_n\}$ tending to zero as $n \to \infty$, which satisfy the following properties:(1) $p_Y(t,x,y)$ is the kernel of a strongly continuous semigroup on $L^2(Y, \mu)$ associated with a regular Dirichlet form $\mathscr{E}_Y$ on $L^2(Y, \mu_Y)$, where Y denotes the support of μ_Y; (2) the regular Dirichlet space $(Y, \mu_Y, \mathscr{E}_Y)$ belongs to the same classe $\mathscr{D}_{A',\nu+d}$; (3) the distance $\hat{d}_Y^{spec}$ is given by $\hat{d}_Y^{spec}(x,y) = (\sup_{t>0} e^{-(t+1/t)}(p_Y(t,x,x) + p_Y(t,y,y) - 2p_Y(t,x,y))^{1/2}$ for $x, y \in \hat{Y}$; (4) the push-forward $F_{n*}\mu_{M_n}$ of the measure μ_{M_n} by F_n converges to the measure μ_Y with respect to the vague topology; (5) the maps F_n and H_n are ε_n-spectral approximating maps in the sense that
$\sup_{t>0,x,y\in M_n} e^{-(t+1/t)}|p_{M_n}(t,x,y) - p_Y(t, F_n(x), F_n(y))| < \varepsilon_n$,
$\sup_{t>0,x,y\in\hat{Y}} e^{-(t+1/t)}|p_{M_n}(t, H_n(x), H_n(y)) - p_Y(t,x,y)| < \varepsilon_n$, and furthermore
$\sup_{x\in\hat{Y}} \hat{d}_Y^{spec}(F_n \circ H_n(x), x) < \varepsilon_n$.

Let $\hat{\pi}_n$ be a Borel measurable map from $\hat{Y}$ to $\hat{X}$ defined by $\hat{\pi}_n = f_n \circ \pi_n \circ H_n$. Passing to a subsequence, we will see that $\hat{\pi}_n$ converges uniformly to a continuous map $\hat{\pi}_\infty : \hat{Y} \to \hat{X}$ as $n \to \infty$. In fact, taking an increasing family of finite subsets of $\hat{Y}$ whose union S is dense in $\hat{Y}$, and applying the diagonal argument, we get a subsequence of

π_n, denoted by the same letters, and a map π_∞ from S to X to which π_n converges pointwise as $n \to \infty$. Since we have

$$\begin{aligned} \hat{d}_X^{spec}(\hat{\pi}_n(y), \hat{\pi}_n(y')) &\leq d_{B_n}^{spec}(\phi_k \circ H_n(y), \pi_k \circ H_n(y')) + \varepsilon_k \\ &\leq d_{M_n}^{spec}(y, y') + 2\varepsilon_n, \quad y, y' \in S, \end{aligned}$$

letting $n \to \infty$, we obtain

$$\hat{d}_X^{spec}(\hat{\pi}_\infty(y), \hat{\pi}_\infty(y')) \leq \hat{d}_Y^{spec}(y, y'), \quad y, y' \in S.$$

Since S is dense in $\hat{Y}$, we may assume that $\hat{\pi}_\infty$ is a contraction map from $\hat{Y}$ to $\hat{X}$. It is easy to verify that $\hat{\pi}_n$ converges uniformly to $\hat{\pi}_\infty$ as $n \to \infty$, since F_n, H_n and f_n are Gromov-Hausdorff approximations with respect to the metrics under consideration. In addition, since $\hat{\pi}_n(\hat{Y})$ is $2\varepsilon_n$-dense in $\hat{X} = (\hat{X}, \hat{d}_X^{spec})$, we have $\hat{\pi}_\infty(\hat{Y}) = \hat{X}$. The measure μ_Y and μ_X are respectively the vague limits of the image measures $F_{n*}\mu_{M_n}$ and $f_{n*}\mu_{B_n}$ as $n \to \infty$, and $\pi_{n*}\mu_{M_n} = \mu_{B_n}$. Hence we get $\hat{\pi}_{\infty*}\mu_Y = \mu_X$.

Now we want to show that a sequence of Borel measurable maps $\theta_n : Y \times G \to Y$ defined by $\theta_n(y,a) = F_n(H_n(y)a)$, $y \in Y, a \in G$, converges uniformly to a continuous map $\theta : \hat{Y} \times G \to \hat{Y}$ which gives an action of G on $\hat{Y}$. To see this, we have by Lemma 3.3

$$\begin{aligned} \hat{d}_Y^{spec}(\theta_n(y,a), \theta_n(y',b)) &\leq d_{M_n}^{spec}(H_n(y)a, H_n(y')b) + \varepsilon_n \\ &\leq d_{M_n}^{spec}(H_n(y), H_n(y')) + C|ab^{-1}|_G + \varepsilon_n \\ &\leq \hat{d}_Y^{spec}(y, y') + C|ab^{-1}|_G + 2\varepsilon_n, \\ &\qquad y, y' \in \hat{Y},\ a, b \in G. \end{aligned} \tag{4.1}$$

Passing to a subsequence of $\{\theta_n\}$, denoted by the same letters, we see that θ_n converges to a map $\theta : S \times T \to \hat{Y}$, where T is a countably infinite, dense subset of G. For simplicity, we write ya for $\theta(y,a)$, $y \in S,\ a \in T$. Then it follows from (4.1) that

$$\hat{d}_Y^{spec}(ya, y'b) \leq \hat{d}_Y^{spec}(y, y') + C|ab^{-1}|_G^{1/2}, \quad y, y' \in S,\ a, b \in T.$$

In the same way, we obtain

$$\hat{d}_Y^{spec}(y, y') \leq \hat{d}_Y^{spec}(ya, y'b) + C|ab^{-1}|_G^{1/2}, \quad y, y' \in S,\ a, b \in T.$$

Thus we have

$$|\hat{d}_Y^{spec}(ya, y'b) - \hat{d}_Y^{spec}(y, y')| \leq C|ab^{-1}|_G^{1/2}, \quad y, y' \in S,\ a, b \in T. \tag{4.2}$$

Then we can deduce that (4.2) holds on $\hat{Y} \times G$ and G acts continuously on $\hat{Y}$. With respect to this action, the approximating maps $F_n : M_n \to \hat{Y}$ and $H_n : \hat{Y} \to M_n$ are $\bar{\varepsilon}_n$-equivariant in the sense that

$$\begin{cases} d_{\hat{Y}}^{spec}(F_n(za), F_n(z)a) & \leq \bar{\varepsilon}_n, \quad z \in M_n, \quad a \in G \\ d_{M_n}^{spec}(H_n(ya), H_n(y)a)) & \leq \bar{\varepsilon}_n, \quad y \in \hat{Y}, \quad a \in G, \end{cases}$$

where $\{\bar{\varepsilon}_n\}$ is a sequence of positive numbers tending to 0 as $n \to \infty$. Moreover the Borel measure μ_Y and the kernel function p_Y of $\hat{Y}$ are invariant under the action of G on $\hat{Y}$, that is

$$\begin{aligned} \int_Y u(ya)d\mu_Y(y) &= \int_Y u(y)d\mu_Y(y), \quad u \in C(\hat{Y}),\ a \in G \\ p_Y(t, ya, y'a)) &= p_Y(t, y, y'), \quad t > 0,\ y, y' \in \hat{Y}, a \in G \end{aligned}$$

We denote the support of μ_Y by Y.

As is done in the case of $L^2(M_n, \mu_{M_n})$, we decompose the Hilbert space $L^2(Y, \mu_Y)$ into the sum of the subspaces corresponding to the irreducible unitary representations. For $\rho \in \hat{G}$, $\rho : G \to U(V)$, we denote by $C(\hat{Y}, V)^\rho$ the space of ρ-equivariant continuous maps of $\hat{Y}$ to V and $C(\hat{Y}, \rho)$ the linear space spanned by functions $(f, v)_V$, $f \in C(\hat{Y}, V)^\rho$, $v \in V$. Here we notice that

$$\pi_\infty^* C(\hat{X}) \subset C(\hat{Y}, 1) = \pi_{\infty;G}{}^*(C(\hat{Y}/G)),$$

where $\hat{\pi}_{\infty;G} : \hat{Y} \to \hat{Y}/G$ stands for the projection of $\hat{Y}$ onto $\hat{Y}/G$. We also remark that $C(\hat{Y}, \rho)$ and $C(\hat{Y}, \rho')$ are orthogonal in $L^2(Y, \mu_Y)$ if $\rho \neq \rho'$. In fact, for $(f, v)_V \in C(\hat{Y}, \rho)$ and $(f', v') \in C(\hat{Y}, \rho')$, we have

$$\int_Y (f, v)_V (f', v')_{V'} d\mu_Y = \begin{cases} \int_Y (f, f')_V d\mu_Y (v, v')_V, & \rho = \rho', \\ 0, & \rho \neq \rho'. \end{cases}$$

Let $L^2(Y, \rho)$ be the closure of $C(\hat{Y}, \rho)$ in $L^2(Y, \mu_Y)$. Then $L^2(Y, \mu_Y)$ is decomposed into the direct sum of $L^2(Y, \rho)$, $\rho \in \hat{G}$:

$$L^2(Y, \mu_Y) = \sum_{\rho \in \hat{G}} L^2(Y, \rho),$$

and a function $u \in L^2(Y, \mu_Y)$ is written as

$$u = \sum_{\rho \in \hat{G}} u^{(\rho)}, \quad u^{(\rho)} \in L^2(Y, \rho).$$

According to this decomposition, the regular Dirichlet form $\mathscr{E}_Y = (\mathscr{E}_Y, D[\mathscr{E}_Y])$ on $L^2(Y, \mu_Y)$ is also decomposed as follows:

$$\mathscr{E}_Y(u,u) = \sum_{\rho \in \hat{G}} \mathscr{E}_Y^{(\rho)}(u^{(\rho)}, u^{(\rho)}), \quad u \in D[\mathscr{E}_Y].$$

We remark that

$$L^2(Y,1) = \overline{C(Y,1)} = \overline{\pi_{\infty;G}{}^* C(X)}.$$

Given $\rho \in \hat{G}$, let $\{\sqrt{\dim \rho}(\phi_{n;i}^{(\rho)}, e_\alpha)_V \mid i = 1,2,\ldots,\ \alpha = 1,\ldots,\dim \rho\}$ be a complete orthonormal system of $L^2(M_n, \rho)$ which consists of eigenfunctions $(\phi_{n;i}^{(\rho)}, e_\alpha)_V$ with eigenvalues $\lambda_{n;i}^{(\rho)} + C_\rho$, where $\{e_\alpha \mid \alpha = 1,\ldots,\dim \rho\}$ is an orthonormal basis of V. Then we have the following assertions:

(i) For each $i = 1,2,\ldots$, $\lambda_{n;i}^{(\rho)} + C_\rho$ converges, as $n \to \infty$, to $\lambda_i^{(\rho)} + C_\rho \in [C_\rho, +\infty]$.

(ii) There exists a complete orthonormal system $\{\sqrt{\dim \rho}(\phi_i^{(\rho)}, e_\alpha)_V \mid 1 \le i < N \le +\infty, \alpha = 1,\ldots,\dim \rho\}$ of $L^2(Y, \rho)$, which consists of eigenfunctions $(\phi_i^{(\rho)}, e_\alpha)_V (\in C(\hat{Y}, \rho))$ with eigenvalues $\lambda_i^{(\rho)} + C_\rho$, and $\phi_{n;i}^{(\rho)}$ converges uniformly to $\phi_i^{(\rho)}$ as $n \to \infty$ via the spectral approximating maps $F_n : M_n \to \hat{Y}$ and $H_n : \hat{Y} \to M_n$ in such a way that

$$|\phi_{n;i}^{(\rho)}(H_n(y)) - \phi_i^{(\rho)}(y)| \le \varepsilon_n(\rho; i), \quad y \in \hat{Y};$$

$$|\phi_{n;i}^{(\rho)}(z) - \phi_i^{(\rho)}(F_n(z))| \le \varepsilon_n(\rho; i), \quad z \in M_n,$$

where $\lim_{n\to\infty} \varepsilon_n(\rho; i) = 0$.

(iii) Let

$$p_Y^{(\rho)}(t,x,y) = \frac{\dim \rho}{2} e^{-C_\rho t} \left(\sum_{i=1}^{\infty} e^{-\lambda_i^{(\rho)} t} \{ (\phi_i^{(\rho)}(x), \phi_i^{(\rho)}(y))_V + (\phi_i^{(\rho)}(y), \phi_i^{(\rho)}(x))_V \} \right).$$

Then we have

$$\sup_{t>0; x,y \in \hat{Y}; z,w \in M_n} e^{-(t+1/t)} |p_{M_n}^{(\rho)}(t, H_n(x), H_n(y)) - p_Y^{(\rho)}(t,x,y)| \le \varepsilon_n(\rho);$$

$$\sup_{t>0; x,y \in \hat{Y}; z,w \in M_n} e^{-(t+1/t)} |p_{M_n}^{(\rho)}(t,z,w) - p_Y^{(\rho)}(t, F_n(z), F_n(w))| \le \varepsilon_n(\rho),$$

where $\lim_{n\to\infty} \varepsilon_n(\rho) = 0$.

Now we want to discuss a convergence of Hermitian vector bundles associated to principal fiber bundles. For any $\rho \in \hat{G}$, we have the Hermitian vector bundle $E_{n;\rho}$

over B_n and a bijective linear correspondence σ_n between $C(M_n,V)^\rho$ and the space of continuous sections $\Gamma(B_n,E_{n;\rho})$ of $E_{n;\rho}$. However in the case of the limit space $(\hat{Y},\hat{X},\hat{\pi}_\infty:\hat{Y}\to\hat{X})$, the continuous action of G may have nontrivial isotropy groups, and thus this process does not work well.

In what follows, we consider the case where G is the unit circle $S^1=\{e^{2\pi\sqrt{-1}\theta}\mid\theta\in\mathbf{R}\}$ and $V=\mathbf{C}$. Let

$$\bar{\mathscr{S}_\rho}=\{y\in\hat{Y}\mid f(y)=0,\forall f\in C(\hat{Y},\mathbf{C})^\rho\}=\{y\in\hat{Y}\mid u(y)=0,\forall u\in C(\hat{Y},\rho)\}.$$

Then $\bar{\mathscr{S}_\rho}$ is an S^1-invariant closed subspace of $\hat{Y}$. In view of (3.4), we observe that for $y\in\hat{Y}\setminus\bar{\mathscr{S}_\rho}$ and $a\in S^1$, $\rho(a)=1$ if $ya=y$. This implies that we have a continuous Hermitian line bundle $E_{\infty;\rho}=(\hat{Y}\setminus\bar{\mathscr{S}_\rho})\times_{S^1}\mathbf{C}$ over $(\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1$ and an element of $C(\hat{Y},\mathbf{C})^\rho$ corresponds to a continuous section $\sigma:(\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1\to E_{\infty;\rho}$ such that $|\sigma(x)|_{E_{\infty;\rho}}$ tends to zero as $x\in(\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1$ tends to $\bar{\mathscr{S}_\rho}/S^1$. We denote by $C_*((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$ the space of such sections, and by $L^2((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$ the closure of $C_*((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$ in $L^2((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$, where the push-forward measure $\hat{\pi}_{\infty;S^1*}\mu_Y$ is taken. In this way, we obtain a densely defined closed form $(\mathscr{F}^{(\rho)},D[\mathscr{F}^{(\rho)}])$ on $L^2((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$ which corresponds to the closed form $\mathscr{E}_Y^{(\rho)}$ on $L^2(\hat{Y},\rho)$. We note that an eigensection of $\mathscr{F}^{(\rho)}$is continuous and belongs to $C_*((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$.

Now we define a linear map $\Phi_n^{(\rho)}:C(\hat{Y},\rho)\to L^2(M_n,\rho)$ by

$$\Phi_n^{(\rho)}(u)=(u\circ F_n)^{(\rho)},\quad u\in C(\hat{Y},\rho).$$

Then we have

$$\lim_{n\to\infty}\|\Phi_n^{(\rho)}(u)\|_{L^2}=\|u\|_{L^2},\quad u\in C(\hat{Y},\rho),$$

and we can deduce from the above arguments

Assertion 1: *a sequence of the closed forms $\mathscr{E}_{M_n}^{(\rho)}$ on $L^2(M_n,\rho)$ is asymptotically compact and Mosco-converges to the closed form $\mathscr{E}_Y^{(\rho)}$ on $L^2(Y,\rho)$ as $n\to\infty$; moreover if $G=S^1$ and $V=\mathbf{C}$, then a sequence of the closed form $\mathscr{F}_{B_n}^{(\rho)}$ on $L^2(B_n,E_{n;\rho})$ is asymptotically compact and Mosco-converges to the closed form $\mathscr{F}^{(\rho)}$ on $L^2((\hat{Y}\setminus\bar{\mathscr{S}_\rho})/S^1,E_{\infty;\rho})$ as $n\to\infty$.*

Now a continuous symmetric section R_n of the vector bundle $\mathrm{Hom}(E_n,E_n)$ over B_n is given for each n. Then a closed form $\mathscr{F}_{R_n}^{(\rho)}$ on $L^2(B_n,E_\rho)$ is defined by

$$\mathscr{F}_{R_n}^{(\rho)}(\sigma)=\int_{B_n}|\nabla\sigma|^2+(R_n(\sigma),\sigma)_{E_\rho}d\mu_{B_n},\quad \sigma\in D[\mathscr{F}_{R_n}^{(\rho)}]=D[\mathscr{F}_{B_n}^{(\rho)}].$$

Suppose that for some $p > \nu/2$ and $K > 0$, and for all n,

$$\|r_n^-\|_{L^p} \le K,$$

or for some $\alpha \in (0,1)$ and for all n,

$$\|r_n^-\|_{L^{\nu/2}} \le \frac{\alpha}{A''}.$$

Then we can deduce from Lemma 3.1

Assertion 2: *the sequence $\{\mathscr{F}_{R_n}^{(\rho)}\}$ is asymptotically compact, and further it contains a Γ-convergent subsequence (cf. 2.1), which in fact Mosco-converges to a closed form $\mathscr{G}^{(\rho)}$ on $L^2(Y,\rho)$ with $D[\mathscr{G}^{(\rho)}] \subset D[\mathscr{F}^{(\rho)}]$.*

We make a remark. Let d_{M_n} and d_{B_n} be respectively the Riemannian distances of M_n and B_n. Then we can apply Theorem 2.4 to $F_n : M_n \to \hat{Y}$ and $f_n : B_n \to \hat{X}$, and passing to a subsequence denoted by the same letters, we get continuous pseudo distances δ_Y and δ_X on $\hat{Y}$ and $\hat{X}$, respectively, so that $F_n : (M_n, d_{M_n}) \to (\hat{Y}, \delta_Y)$ and $f_n : (B_n, d_{B_n}) \to (\hat{X}, \delta_X)$ are, respectively, ε_n-Hausdorff approximating maps with $\lim_{n\to\infty} \varepsilon_n = 0$. Note that δ_Y is G-invariant. Let $\hat{Y}_\delta$ (resp. $\hat{X}_\delta$) be the geodesic space obtained by identifying points $x, y \in \hat{Y}$ if $\delta_Y(x,y) = 0$ (resp. points $z, w \in \hat{X}$ if $\delta_X(z,w) = 0$). Then the action of G on $\hat{Y}$ descends to that on $\hat{Y}_\delta$ so that the quotient space $\hat{Y}_\delta/G$ coincides with $\hat{X}_\delta$.

Proof of Theorem 1.1. Let $\{E_n \to B_n\}$ be as in Theorem 1.1. Let M_n be the unitary frame bundle of E_n over B_n and consider E_n as the associated Hermitian vector bundle. Passing to a subsequence, we assume that $(M_n, d_{M_n}^{spec})$ converges to a compact metric space $(\hat{Y}, d_Y^{spec})$ endowed with a Radon measure and a Dirichlet form on the Hilbert space of square summable functions relative to the measure; moreover the unitary group $U(r)$ isometrically acts on $\hat{Y}$ in such a way that the measure and the Dirichlet form are invariant. We write ρ_0 for the canonical action of $U(r)$ on $\mathbf{C}^r$, and use the same notations as in the above arguments, such as $C(M_n, \mathbf{C}^r)^{\rho_0}$, $\sigma_n : L^2(M_n, \mathbf{C}^r)^{\rho_0} \to L^2(B_n, E_n)$, and $C(\hat{Y}, \mathbf{C}^r)^{\rho_0}$. Then we have a linear map $\Psi_n : C(\hat{Y}, \mathbf{C}^r)^{\rho_0}$ to $L^2(B_n, E_n)$ defined by $\Psi_n(u) = \sigma_n((u \circ F_n)^{(\rho_0)}), \quad u \in C(\hat{Y}, \mathbf{C}^r)^{\rho_0}$, where $F_n : M_n \to \hat{Y}$ is an ε_n-approximating map with $\lim_{n\to\infty} \varepsilon_n = 0$. Then Ψ_n satisfies that $\lim_{n\to\infty} \|\Phi_n(u)\|_{L^2} = \|u\|_{L^2}$ for all $u \in C(\hat{Y}, \mathbf{C}^r)^{\rho_0}$. Thus Assertion 1 shows the theorem. □

Example. Here we provide a simple example to illustrate Theorem 1.1. Let $M = (M, B, \pi : M \to B)$ be a principal bundle over a compact Riemannian manifold $B = (B, g_B)$ with structure group $S^1 = \{e^{2\pi\sqrt{-1}\theta} \mid \theta \in \mathbf{R}\}$. Given a connection form $\omega : TM \to \mathscr{L}(S^1) = \sqrt{-1}\mathbf{R}$, we have a Riemannian metric g_ω on M defined by $g_\omega = \pi^* g_B + \omega \otimes \overline{\omega}$. Let $\sigma : U \to M$ be a smooth section of M over an open subset U of B

and define a local trivialization $\eta_\sigma((x,a)) : U \times S^1 \to \pi^{-1}(U)$ by $\eta_\sigma((x,a)) = \sigma(x)a$, $(x,a) \in U \times S^1$. Then the pull-back $\eta_\sigma{}^*\omega$ is expressed by $\eta_\sigma{}^*\omega = \sqrt{-1}(d\theta + \Gamma)$, where Γ is a differential one-form on U.

Given an integer ℓ, ρ_ℓ stands for the representation $\rho_\ell(z) = z^\ell$, $z \in S^1$. Let u be a smooth function in $C(M,\rho_\ell)$. Then we have

$$\eta_\sigma{}^* u(x, e^{2\pi\sqrt{-1}\theta}) = \tilde{u}(x)e^{-2\pi\ell\sqrt{-1}\theta}, \quad (x, e^{2\pi\sqrt{-1}\theta}) \in U \times S^1,$$

where $\tilde{u}$ is a smooth function on U, and

$$\eta_\sigma{}^*(\Delta_M u)(x, e^{2\pi\sqrt{-1}\theta}) = \Big(\Delta_B \tilde{u} + 4\pi^2(1+|\Gamma|^2)\tilde{u} - 2\pi\ell\sqrt{-1}((\mathrm{div}\,\Gamma^\sharp)\tilde{u} + 2(\Gamma^\sharp, \nabla\tilde{u})_{g_B})\Big)(x)e^{-2\pi\ell\sqrt{-1}\theta}.$$

Let us now consider a sequence of connection forms ω_n and assume that $(M, g_{\omega_n}, B, \pi : M \to B)$ Mosco-converges to a compact Dirichlet space $(\hat{Y}, \mu_Y, \mathscr{E}_Y, B, \hat{\pi}_\infty : \hat{Y} \to B)$ as $n \to \infty$ in such a manner described as above. We suppose that as $n \to \infty$, ω_n converges to a connection form ω_∞ defined on an open subset $\pi^{-1}(B \setminus K)$, uniformly on a compact subset of $\pi^{-1}(B\setminus K)$, where K is a compact submanifold of co-dimension ≥ 2. Let $\mathscr{E}_\infty$ and p_∞ denote the energy form on the Sobolev space $W^{1,2}(\pi^{-1}(B \setminus K))$ in which $C_0^\infty(\pi^{-1}(B \setminus K))$ is densely embedded. Then $(\pi^{-1}(B \setminus K), g_{\omega_\infty})$ can be densely embedded in $Y = \mathrm{supp}\ \mu_Y$ in such a way that $p_\infty(t,x,y) = p_Y(t,x,y), \quad t > 0, x, y \in \hat{\pi}_\infty^{-1}(B \setminus K)$.

Now we take the base manifold B to be the unit Euclidean sphere $S^2 = (S^2, g_0)$ and consider for simplicity the case where M is just the product space of S^2 and S^1. Let (r,ξ) $(0 < r < 2\pi, 0 \leq \xi \leq 2\pi)$ be the polar coordinates around the north pole p_*. We write the metric g_0 as $g_0 = dr^2 + (\sin r)^2 d\xi^2$ on $S^2 \setminus \{p_*, q_*\}$, where q_* denotes the south pole, and express the metric g_{ω_n} as $g_{\omega_n} = dr^2 + (\sin r)^2 d\xi^2 + (d\theta + \Gamma_n)^2$, where Γ_n is a one-form on S^2. Suppose that as $n \to \infty$, Γ_n converges to $\tau d\xi$, uniformly on a compact subset of $S^2 \setminus \{p_*, q_*\}$. Here τ is a constant. For a fixed interger ℓ, let $\phi^{(\ell)}$ be an eigenfunction in $C(\hat{Y}, \rho_\ell)$ with eigenvalue $\lambda^{(\ell)}$ and set $\tilde{\phi}^{(\ell)}(x) = \phi^{(\ell)}(x, e^{2\pi\sqrt{-1}\theta})e^{2\pi\ell\sqrt{-1}\theta}$ on $S^2 \setminus \{p*, q_*\}$. Then $\tilde{\phi}^{(\ell)}(x)$ satisfies

$$-\Delta_{g_{std}}\tilde{\phi}^{(\ell)} + 4\pi^2\ell^2\left(1 + \frac{\tau^2}{4\pi^2(\sin r)^2}\right)\tilde{\phi}^{(\ell)} - \sqrt{-1}\frac{2\ell\tau}{(\sin r)^2}\frac{\partial\tilde{\phi}^{(\ell)}}{\partial\xi} = \lambda^{(\ell)}\tilde{\phi}^{(\ell)}.$$

Let

$$u_{\ell,k}(r) = \int_0^{2\pi} \tilde{\phi}^{(\ell)}(r,\xi)e^{-k\sqrt{-1}\xi}d\xi, \quad k = 1, 2, \ldots$$

Then $u_{\ell,k}(r)$ satisfies

$$u''_{\ell,k}(r)+\frac{\cos r}{\sin r}u'_{\ell,k}(r)-\frac{(k+\tau\ell)^2}{(\sin r)^2}u_{\ell,k}(r)=(\lambda^{(\ell)}-4\pi^2\ell^2)u_{\ell,k}(r)$$

on $(0,2\pi)$. In view of this equation, we see that $\lim_{r\to 0}u_{\ell,k}(r)=\lim_{r\to 2\pi}u_{\ell,k}(r)=0$ if $k+\tau\ell\neq 0$; in particular, $\tilde{\phi}^{(\ell)}(r,\xi)$ goes to zero as $r\to 0$ or $r\to 2\pi$ if τ is an irrational number. We consider the case that τ is a rational number and write $\tau=a/b$ with integers a and $b(\geq 1)$ which are relatively prime. Then for $\ell=bm$ $(m\in\mathbf{Z})$, $\psi^{(\ell)}(r,\xi)=\tilde{\phi}^{(\ell)}(r,\xi)e^{am\sqrt{-1}\xi}$ satisfies

$$-\Delta_{g_0}\psi^{(\ell)}=(\lambda^{(\ell)}-4\pi^2\ell^2)\psi^{(\ell)}.$$

That is, $\psi^{(\ell)}$ is an eigenfunction of S^2 with eigenvalue $\lambda^{(\ell)}-4\pi^2\ell^2$. In other words, for an eigenfunction ψ on S^2 with eigenvalue λ, the function $\phi^{(\ell)}(r,\xi,e^{2\pi\sqrt{-1}\theta})=\psi(r,\xi)e^{-2m\pi\sqrt{-1}(b\theta+a\xi/2\pi)}$ is an eigenfunction of eigenvalue $\lambda+4\pi^2\ell^2$ relative to the Laplacian of the metric $g_\infty=dr^2+(\sin r)^2d\xi^2+(d\theta+\tau d\xi/2\pi)^2$ on $S^2\setminus\{p_*,q_*\}\times S^1$. In this way, we see that (i) when τ is an irrational number, $\hat{\pi}_\infty^{-1}(p_*)\cap Y$ consists of a single point and so does $\hat{\pi}_\infty^{-1}(q_*)\cap Y$, and further $\hat{\pi}_\infty^{-1}(\{p_*,q_*\})\cap Y=\bar{\mathscr{S}}_{\rho_\ell}\cap Y$ for any ℓ; (ii) when τ is a rational number a/b $(b\geq 1)$, the isotropy group at a point in $\hat{\pi}_\infty^{-1}(\{p_*,q_*\})$ is given by $\{e^{2m\pi\sqrt{-1}/b}\mid m=0,1,\ldots,m-1\}$, and further if b does not divide ℓ , then $\bar{\mathscr{S}}_{\rho_\ell}\cap Y=\hat{\pi}_\infty^{-1}(\{p_*,q_*\})\cap Y$ and if b divides ℓ, then $\bar{\mathscr{S}}_{\rho_\ell}\cap Y=\emptyset$.

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