

On the Generalized Divisor Problem in Arithmetic Progressions

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On the Generalized Divisor Problem in Arithmetic Progressions

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Abstract. In this paper we investigate the asymptotic behavior of the summatory function $D_z(x, q, l)$ and $\pi_k(x, q, l)$, and its relation to the Riemann hypothesis for the Dirichlet L -function.

Introduction

One of the classical problems in analytic number theory which is now called "the Dirichlet divisor problem" is concerned with investigating the asymptotic behavior of $D_k(x) \equiv \sum_{n \leq x} d_k(n)$ where $d_k(n)$ means the number of ways of expressing n as a product of k natural numbers. Namely, $d_k(n)$ is a multiplicative arithmetical function such that

$$d_k(p^m) = \binom{k+m-1}{m} \equiv \frac{k(k+1) \cdots (k+m-1)}{m!}.$$

It is well-known that $D_k(x)$ has an expression in the form

$$D_k(x) = xP_{k-1}(\log x) + \Delta_k(x)$$

where $P_k(x)$ is some polynomial of degree k , and $\Delta_k(x)$ is the error term. It seems that the essence of this problem is to establish some relationship between the order of $\Delta_k(x)$ and the property of $\zeta(s)$ since

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1),$$

where $s = \sigma + it$ and $\zeta(s)$ is the Riemann zeta function.

It is known that

$$\Delta_k(x) \ll x^{(k-1)/(k+1)+\varepsilon} \quad (0)$$

for every positive ε , and that the statement

$$\Delta_k(x) \ll x^{1/2+\varepsilon}$$

for every k is equivalent to the Lindelöf hypothesis. Finally it is conjectured that

$$\Delta_k(x) \ll x^{(k-1)/2k+\epsilon},$$

but any corresponding properties of $\zeta(s)$ is not revealed yet.

On the other hand some mathematician tried to generalize the divisor problem. Let $d_z(n)$ be a multiplicative function defined by

$$d_z(p^m) \equiv \binom{z+m-1}{m}$$

where z is a complex number. Then we have

$$\zeta^z(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-z} = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \quad (\sigma > 1)$$

where $\zeta^z(s) \equiv \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$.

The generalized divisor problem is to find an asymptotic formula for $\sum_{n \leq x} d_z(n)$, which was observed for real $z > 0$ by A. Kienast [7] and K. Iseki [5] independently. A. Selberg [15] considered it for all complex z , his result being

$$D_z(x) \equiv \sum_{n \leq x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O(x(\log x)^{\Re z - 2}) \quad (1)$$

uniformly for $|z| \leq A$, where A is any fixed positive number. This was extended by Rieger [14] to arithmetic progressions such that

$$D_z(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} d_z(n) = \left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\Gamma(z)\varphi(q)} (\log x)^{z-1} \\ + O\left(\left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\varphi(q)} (\log x)^{\Re z - 2} \log \log 4q\right)$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$, where A and τ are any fixed positive numbers. We note that when z is a natural number, $d_z(n)$ coincides with the divisor function in the Dirichlet divisor problem, and $d_{-1}(n)$ with the Möbius function.

Next, let $\pi_k(x)$ be the number of integers $\leq x$ which are product of k distinct primes. For $k = 1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exceeding x .

C.F. Gauss stated empirically that $\pi_2(x) \sim x(\log \log x)/\log x$, and E. Landau proved that $\pi_k(x) \sim x(\log \log x)^{k-1}/(k-1)!\log x$ by using the prime number theorem. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x) = \frac{xQ(\log \log x)}{\log x} + O\left(\frac{x(\log \log x)^k}{k!(\log x)^2}\right) \tag{2}$$

uniformly for $1 \leq k \leq A \log \log x$, where $Q(x)$ is polynomial of degree $k - 1$. Now we define $\pi_k(x, q, l)$ as a generalization of $\pi_k(x)$ by

$$\pi_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ n = p_1 \cdots p_k \ (p_i \neq p_j)}} 1.$$

In this paper we shall consider the connections between the asymptotic formulas of $D_z(x, q, l)$, $\pi_k(x, q, l)$ and the location of zeros of the Dirichlet L -function. In particular we shall establish some necessary and sufficient conditions for the truth of the Riemann hypothesis, so that this paper gives a generalization of [1] to arithmetic progressions.

The main term of (1) and (2) is, however, inconvenient for our aim so that we introduce the following integrals as the main terms of $D_z(x, q, l)$ and $\pi_k(x, q, l)$ respectively :

$$\begin{aligned} \Phi_z(x, q) &= \frac{1}{2\pi i} \int_L (L(s, \chi_0))^z \frac{x^s}{s} ds, \\ F_{k, \delta}(x, q) &= \frac{1}{(2\pi i)^2} \int_{|z|=1} \int_{L_\delta} (L(s, \chi_0))^z \\ &\quad \times \left\{ \prod_p \left(1 + \frac{z\chi_0(p)}{p^s}\right) \left(1 - \frac{\chi_0(p)}{p^s}\right)^z \right\} \frac{1}{z^{k+1}} \frac{x^s}{s} ds dz \end{aligned}$$

where L is, for any r ($0 < r < 1/2$), the path which begins at $1/2$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2$ along the real axis, and L_δ is, for every δ and any r ($\delta > 0, r > 0, \delta + r < 1/2$), the path which begins at $1/2 + \delta$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2 + \delta$ along the real axis. Here we denote by χ_0 the principal character mod q .

The error terms are defined by

$$\begin{aligned} \Delta_z(x, q, l) &= D_z(x) - \frac{1}{\varphi(q)} \Phi_z(x, q), \\ R_{k, \delta}(x, q, l) &= \pi_k(x, q, l) - \frac{1}{\varphi(q)} F_{k, \delta}(x, q). \end{aligned}$$

Let

$$\Theta(\chi) = \sup\{\sigma : L(\sigma + it, \chi) = 0\}, \quad \Theta_q = \max_{\chi \pmod{q}} \Theta(\chi).$$

Then the following Theorem 1 and 2 follow from more general results proved in sections 1 and 2 below.

THEOREM 1. *There exists some constant c such that*

$$\Delta_z(x, q, l) \ll x e^{-c\sqrt{\log x}}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$ where A and τ are any fixed positive numbers.

Further we have

$$\Delta_z(x, q, l) \ll x^{\Theta_q + \varepsilon}$$

uniformly for $|z| \leq A$, $q \leq x$, $(q, l) = 1$.

Conversely if $\Delta_z(x, q, l) \ll x^{\Xi + \varepsilon}$ for any l ($(q, l) = 1$) and for some $z \in C - Q^+$, where Q^+ denotes the set of all non negative rational numbers, then any $L(s, \chi) \pmod{q}$ has no zeros for $\sigma > \Xi$.

The main term $\Phi_z(x, q)$ has an asymptotic expansion

$$\Phi_z(x, q) = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z-m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$. Here N is any fixed positive integer and $B_m(z, q)$ ($0 \leq m \leq N-1$) are regular functions of z , especially $B_0(z, q) = (\varphi(q)/q)^z$.

THEOREM 2. *There is some constant c such that*

$$R_{k, \delta}(x, q, l) \ll x e^{-c\sqrt{\log x}}$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q, l) = 1$.

Further we have

$$R_{k, \delta}(x, q, l) \ll x^{\Theta_q + \varepsilon}$$

uniformly for $k \geq 1$, $q \leq x$, $(q, l) = 1$.

Conversely if $R_{k, \delta}(x, q, l) \ll x^{\Xi + \varepsilon}$ for any l ($(q, l) = 1$) and for some $k \geq 1$, then any $L(s, \chi) \pmod{q}$ has no zeros for $\sigma > \Xi$.

The main term $F_{k,\delta}(x, q)$ has an asymptotic expansion

$$F_{k,\delta}(x, q) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_{m,q}(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every k and q . Here N is any fixed positive integer and $Q_{m,q}(x)$ are polynomials of degree not exceeding $k - 1$, especially the coefficient of x^{k-1} of $Q_{0,q}(x)$ is 1.

Remark.

1. If we define $r_{k,q,l}$ by

$$r_{k,q,l} = \inf_{\delta} \inf\{r : R_{k,\delta}(x, q, l) \ll x^r\}$$

Theorem 2 shows that $\max_l r_{k,q,l} = \Theta_q$. The statement $\Theta_q = 1/2$ for every q is equivalent to the truth of the Riemann hypothesis for Dirichlet L -function.

2. For $k = 1$, we can express the main term in terms of the logarithmic integral. Namely,

$$F_{1,\delta}(x, q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}),$$

so that

$$\pi_1(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{du}{\log u} + O(xe^{-c\sqrt{\log x}}).$$

3. Similar results hold for $\omega_k(x, q, l)$ and $\Omega_k(x, q, l)$. Here

$$\omega_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ \omega(n)=k}} 1, \quad \Omega_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q} \\ \Omega(n)=k}} 1$$

where $\omega(n)$ means the number of distinct prime factors of n , and $\Omega(n)$ means the number of total prime factors allowing multiplicity.

§1. The Generalized Divisor Problem

Actually we prove a more general statement than Theorem 1 and Theorem 2.

Suppose $b_z(n)$ is an arithmetical function with a complex number z and let

$$f(s, z) = \sum_{n=1}^{\infty} \frac{b_z(n)}{n^s} \quad (\sigma > 1/2)$$

be absolutely convergent, and that $f(s, 0) = 1$.

We define the multiplicative function $a_z(n)$ by

$$\zeta^z(s) f(s, z) = \sum_{n=1}^{\infty} \frac{a_z(n)}{n^s} \quad (\sigma > 1).$$

If we put

$$f(s, z, \chi) = \sum_{n=1}^{\infty} \frac{b_z(n)\chi(n)}{n^s} \quad (\sigma > 1/2)$$

where χ is a Dirichlet character *mod* q , then

$$(L(s, \chi))^z f(s, z, \chi) = \sum_{n=1}^{\infty} \frac{a_z(n)\chi(n)}{n^s} \quad (\sigma > 1).$$

Non negative number δ represent 0 or arbitrary small positive number according as

$$\lim_{s \rightarrow 1/2} f(s, z) < \infty$$

or not.

LEMMA 1.1. We have

$$A_z(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} a_z(n) = \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{L_\delta} (L(s, \chi_0))^z f(s, z, \chi_0) \frac{x^s}{s} ds \\ + O(xe^{-c\sqrt{\log x}})$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$, where χ_0 is the principal character *mod* q .

Further, if we put

$$\Phi_{z, \delta}(x, q) \equiv \frac{1}{2\pi i} \int_{L_\delta} (L(s, \chi_0))^z f(s, z, \chi_0) \frac{x^s}{s} ds,$$

then $\Phi_{z,\delta}(x, q)$ has the following asymptotic expansion

$$\Phi_{z,\delta}(x, q) = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z-m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$. Here N is any fixed positive integer, $B_m(z, q)$ are regular functions of z to be defined in the proof, especially $B_0(z, q) = (\varphi(q)/q)^z f(1, z, \chi_0)$.

Proof. We put $A_z(x, \chi) \equiv \sum_{n \leq x} a_z(n) \chi(n)$ and restrict that $q \leq \exp\{\sqrt{\log x}\}$. Absolute constants C, c , and so on, are not necessary the same at each occurrence.

First, it should be shown that

$$A_z(x, \chi) = \xi^{-1} \int_x^{x+\xi} A_z(u, \chi) du + O(\xi \log^A x) + O(x^{(A+2)/(A+1)}) \quad (3)$$

where $\xi = \xi(x)$ satisfies that $1 < \xi < x/2$.

If we denote $D_z(x, \chi) \equiv \sum_{n \leq x} d_z(n) \chi(n)$,

$$\begin{aligned} & \sum_{x < n \leq x+\rho} a_z(n) \chi(n) \\ &= \sum_{n \leq x+\rho} b_z(n) \chi(n) D_z((x+\rho)/n, \chi) - \sum_{n \leq x} b_z(n) \chi(n) D_z(x/n, \chi) \\ &= \left(\sum_{n \leq x+\rho} b_z(n) \chi(n) D_z((x+\rho)/n, \chi) - \sum_{n \leq x+\rho} b_z(n) \chi(n) D_z(x/n, \chi) \right) \\ & \quad + \left(\sum_{n \leq x+\rho} b_z(n) \chi(n) D_z(x/n, \chi) - \sum_{n \leq x} b_z(n) \chi(n) D_z(x/n, \chi) \right) \\ &= \sum_{n \leq x+\rho} b_z(n) \chi(n) \sum_{x/n < m \leq (x+\rho)/n} d_z(m) \chi(m) \\ & \quad + \sum_{x < n \leq x+\rho} b_z(n) \chi(n) D_z(x/n, \chi). \end{aligned}$$

for $1 < \rho \leq \xi$. Here the second term is 0 whereas the first term is

$$\ll \rho \log^A x + x^{(A+1)/(A+2)}.$$

Because that $|d_z(n)| \leq d_k(n)$ where $k = \lceil |z| \rceil + 1$, and the well known result (0) make $\sum_{x < n \leq x+\rho} d_z(n) \ll \rho \log^{k-1} x + x^{k/(k+1)}$. Hence we have

$$\sum_{x < n \leq x+\rho} a_z(n)\chi(n) \ll \rho \log^A x + x^{(A+1)/(A+2)}.$$

On the other hand

$$\begin{aligned} \xi^{-1} \int_x^{x+\xi} A_z(u, \chi) du &= A_z(x, \chi) + O(\xi^{-1} \int_x^{x+\xi} \sum_{x < n \leq u} a_z(n)\chi(n) du) \\ &= A_z(x, \chi) + O(\sup_{0 < \rho \leq \xi} | \sum_{x < n \leq x+\rho} a_z(n)\chi(n) |) \end{aligned}$$

Hence we obtain (3).

We start from the expression

$$\int_0^x A_z(u, \chi) du = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{2-iT}^{2+iT} (L(s, \chi))^z f(s, z, \chi) \frac{x^{s+1}}{s(s+1)} ds.$$

By the Cauchy integral theorem, the path of integration can be deformed within the domain where the integrand is single-valued.

For $\chi = \chi_0$, we replace the path of integration to $\sum_{i=1}^7 L_i$ which is defined as follows :

L_1 is the segment $[2 - iT, 1 - \eta(-T, q) - iT]$,

L_2 is the curve $s = 1 - \eta(t, q) + it \quad (-T \leq t \leq -t_0)$

and two segments

$[\eta - it_0, \eta - i(1 - \eta) \tan \theta] + [\eta - i(1 - \eta) \tan \theta, 1/2 + \delta - i(1/2 - \delta) \tan \theta]$,

L_3 is the segment $[1/2 + \delta - i(1/2 - \delta) \tan \theta, 1 + re^{-i(\pi - \theta)}]$,

L_4 is the arc $s = 1 + re^{i\varphi} \quad (-(\pi - \theta) \leq \varphi \leq \pi - \theta)$,

L_5 is the segment $[1 + re^{i(\pi - \theta)}, 1/2 + \delta + i(1/2 - \delta) \tan \theta]$,

L_6 is two segments

$[1/2 + \delta + i(1/2 - \delta) \tan \theta, \eta + i(1 - \eta) \tan \theta] + [\eta + i(1 - \eta) \tan \theta, \eta + it_0]$

and the curve $s = 1 - \eta(t, q) + it \quad (t_0 \leq t \leq T)$,

L_7 is the segment $[1 - \eta(T, q) + iT, 2 + iT]$.

Here $\eta(t, q) = c/\log q(|t| + 2)$, $\eta = 1 - \eta(t_0, q)$ and t_0 is sufficiently large number to make $1/2 < \eta < 1$. Non negative number δ and any positive numbers r and θ are satisfying $1/2 + \delta \leq \eta < 1 - r$, $0 < (1 - \eta) \tan \theta < t_0$.

The contributions from the integral along $L_1 + L_2 + L_6 + L_7$ are seen to give the error term, while the integral along $L_3 + L_4 + L_5$ gives the principal term since that path becomes L_δ allowing $\theta \downarrow 0$.

We can see

$$\log L(s, \chi_0) \ll \log \log q(|t| + 3)$$

for $s \in L_1 + L_2 + L_6 + L_7$ by Hilfssatz 4 and 7 of Rieger [13], so that

$$(L(s, \chi_0))^z f(s, z, \chi_0) \ll (\log q(|t| + 2))^A.$$

Then,

$$\int_{L_1} + \int_{L_7} \ll \int_{1-\eta(T,q)}^2 (\log qT)^A \frac{x^3}{T(T+1)} d\sigma$$

which tend to 0 by $T \rightarrow \infty$, and

$$\begin{aligned} \int_{L_2} + \int_{L_6} &\ll \int_0^T (\log q(|t| + 2))^A \frac{x^{2-\eta(t,q)}}{t^2} dt \\ &+ \int_0^{t_0} x^{1+\eta} dt + \int_{1/2+\delta}^\eta x^\eta d\sigma \\ &\ll x^2 e^{-c\sqrt{\log x}} \end{aligned}$$

uniformly for $T \geq 1, q \leq \exp\{\sqrt{\log x}\}$.

Hence we have

$$\begin{aligned} &\int_0^x A_z(u, \chi_0) du \\ &= \frac{1}{2\pi i} \int_{L_\delta} (L(s, \chi_0))^z f(s, z, \chi_0) \frac{x^{s+1}}{s(s+1)} ds + O(x^2 e^{-c\sqrt{\log x}}) \\ &= \int_0^x \Phi_{z,\delta}(u, q) du + O(x^2 e^{-c\sqrt{\log x}}). \end{aligned}$$

By the way, since

$$\log\{(s-1)L(s, \chi_0)\} \ll \sqrt{\log 2q}$$

for $s \in L_\delta$ because of $L(s, \chi_0) = \zeta(s) \prod_{p|q} (1 - p^{-s})$, we have

$$\begin{aligned} \Phi'_{z,\delta}(x, q) &= \frac{d}{dx} \Phi_{z,\delta}(x, q) \\ &= \frac{1}{2\pi i} \int_{L_\delta} \{(s-1)L(s, \chi_0)\}^z f(s, z, \chi_0) (s-1)^{-z} x^{s-1} ds \\ &\ll e^{CA\sqrt{\log 2q}} \left(\int_{1/2+\delta}^{1-\tau} + \int_{|s-1|=\tau} \right) (s-1)^{-z} x^{s-1} ds \\ &\ll e^{CA\sqrt{\log 2q}} \{(\log x)^{A-1} \int_{\tau \log x}^{(\log x)/2} u^{-A} e^{-u} du + r^{1-A} x^\tau\} \end{aligned}$$

where we put $(1-s)\log x = u$. Now we choose $r = 1/\log x$, this is

$$\ll e^{CA\sqrt{\log 2q}}(\log x)^{A-1} \ll \exp\{C'A(\log x)^{1/4}\}.$$

Hence

$$\begin{aligned} & \int_x^{x+\xi} A_z(u, \chi) du \\ &= \int_0^{x+\xi} \Phi_{z,\delta}(u, q) du - \int_0^x \Phi_{z,\delta}(u, q) du + O(x^2 e^{-c\sqrt{\log x}}) \\ &= \xi \Phi_{z,\delta}(x, q) + \xi^2 \Phi'_{z,\delta}(x + \xi\theta, q) + O(x^2 e^{-c\sqrt{\log x}}) \quad (0 < \theta < 1) \\ &= \xi \Phi_{z,\delta}(x, q) + O(\xi^2 \exp\{C'A(\log x)^{1/4}\}) + O(x^2 e^{-c\sqrt{\log x}}). \end{aligned}$$

Hence using (3) and choosing $\xi = x \exp\{-c\sqrt{\log x}/2\}/2$ make

$$\begin{aligned} A_z(x, \chi_0) &= \Phi_{z,\delta}(x, q) + O(\xi \exp\{C'A(\log x)^{1/4}\}) + O(\xi^{-1} x^2 e^{-c\sqrt{\log x}}) \\ &\quad + O(\xi \log^A x) + O(x^{(A+1)/(A+2)}) \\ &= \Phi_{z,\delta}(x, q) + O(x e^{-c\sqrt{\log x}/4}). \end{aligned}$$

Next we consider for $\chi = \chi_1$ which is the exceptional character with respect to the zero on the real axis.

If Siegel zero β_1 of $L(s, \chi_1)$ exists in the range $\beta_1 > 1 - c/(2 \log 2q)$, we replace the path of integration to $\sum_{i=1}^7 L_i$ which is defined as follows.:

- L_1 is the segment $[2 - iT, 1 - \eta(-T, q) - iT]$,
- L_2 is the curve $s = 1 - \eta(t, q) + it \quad (-T \leq t \leq -t_0)$
- and the segment $[\eta - it_0, \eta - i(\beta_1 - \eta) \tan \theta]$,
- L_3 is the segment $[\eta - i(\beta_1 - \eta) \tan \theta, \beta_1 + r_1 e^{-i(\pi - \theta)}]$,
- L_4 is the arc $s = \beta_1 + r_1 e^{i\varphi} \quad (-(\pi - \theta) \leq \varphi \leq \pi - \theta)$,
- L_5 is the segment $[\beta_1 + r_1 e^{i(\pi - \theta)}, \eta + i(\beta_1 - \eta) \tan \theta]$,
- L_6 is the segment $[\eta + i(\beta_1 - \eta) \tan \theta, \eta + it_0]$
- and the curve $s = 1 - \eta(t, q) + it \quad (t_0 \leq t \leq T)$,
- L_7 is the segment $[1 - \eta(T, q) + iT, 2 + iT]$.

Here $\eta(t, q) = c/\log q(|t|+2)$, $\eta = 1 - \eta(t_0, q)$ and t_0 is sufficiently large number to make $1 - c/(\log 2q) < \eta < 1 - c/(2 \log 2q)$. Any positive numbers r_1 and θ are satisfying $\eta < \beta_1 - r_1$, $0 < (\beta_1 - \eta) \tan \theta < t_0$. Let $\lim_{\theta \rightarrow 0} (L_3 + L_4 + L_5) = \ell$.

As the same as the case for $\chi = \chi_0$, we have

$$\int_{L_1+L_2+L_6+L_7} (L(s, \chi_1))^z f(s, z, \chi_1) \frac{x^{s+1}}{s(s+1)} ds \ll x^2 e^{-c\sqrt{\log x}}.$$

Hence

$$\begin{aligned} & \int_0^x A_z(u, \chi_1) du \\ &= \frac{1}{2\pi i} \int_l (L(s, \chi_1))^z f(s, z, \chi_1) \frac{x^{s+1}}{s(s+1)} ds + O(x^2 e^{-c\sqrt{\log x}}) \\ &= \int_0^x \Phi_z(u, \chi_1) du + O(x^2 e^{-c\sqrt{\log x}}), \quad \text{say.} \end{aligned}$$

We can see

$$\log \frac{L(s, \chi_1)}{s - \beta_1} \ll \log \log 4q$$

for $s \in l$ by Hilfssatz 7 of Rieger [13], so that

$$\left(\frac{L(s, \chi_1)}{s - \beta_1} \right)^z f(s, z, \chi_1) \ll (\log 4q)^{CA}.$$

Hence

$$\begin{aligned} \Phi_z(x, \chi_1) &= \frac{1}{2\pi i} \int_l (L(s, \chi_1))^z f(s, z, \chi_1) \frac{x^s}{s} ds \\ &= x^{\beta_1} \frac{1}{2\pi i} \int_l \left(\frac{L(s, \chi_1)}{s - \beta_1} \right)^z f(s, z, \chi_1) (s - \beta_1)^z x^{s - \beta_1} \frac{ds}{s} \\ &\ll x^{\beta_1} (\log 4q)^{CA} \left(\int_{\eta}^{\beta_1 - r_1} + \int_{|s - \beta_1| = r_1} \right) (s - \beta_1)^z x^{s - \beta_1} ds \\ &\ll x^{\beta_1} (\log 4q)^{CA} \{ (\log x)^{A-1} \int_{r_1 \log x}^{(\log x)/2} u^{-A} e^{-u} du + r_1^{1-A} x_1^r \} \end{aligned}$$

where we put $(\beta_1 - s) \log x = u$. Now we choose $r_1 = 1/\log x$, this is

$$\ll x^{\beta_1} (\log 4q)^{CA} (\log x)^{A-1} \ll x^{\beta_1} (\log x)^{C'A}.$$

Here we have

$$\begin{aligned} & \int_x^{x+\xi} A_z(u, \chi_1) du \\ &= \int_0^{x+\xi} \Phi_z(u, \chi_1) du - \int_0^x \Phi_z(u, \chi_1) du + O(x^2 e^{-c\sqrt{\log x}}) \\ &= \xi \Phi_z(x + \xi\theta, \chi_1) + O(x^2 e^{-c\sqrt{\log x}}) \quad (0 < \theta < 1) \\ &\ll \xi x^{\beta_1} (\log x)^{C'A} + x^2 e^{-c\sqrt{\log x}}. \end{aligned}$$

Hence using (3) and choosing $\xi = x \exp\{-c\sqrt{\log x}/2\}/2$ make

$$\begin{aligned} A_z(x, \chi_1) &\ll x^{\beta_1}(\log x)^{C'A} + \xi^{-1}x^2e^{-c\sqrt{\log x}} \\ &\quad + \xi \log^A x + x^{(A+1)/(A+2)} \\ &\ll x^{\beta_1}(\log x)^{C'A} + xe^{-c\sqrt{\log x}/4} \end{aligned}$$

On the other hand if $\beta_1 \leq 1 - c/(2 \log 2q)$, we replace the path of integration to $\sum_{i=1,2,6,7} L_i$ which is defined as follows.:

L_1 is the segment $[2 - iT, 1 - \eta(-T, q) - iT]$,
 L_2 is the curve $s = 1 - \eta(t, q) + it \quad (-T \leq t \leq 0)$
 L_6 is the curve $s = 1 - \eta(t, q) + it \quad (0 \leq t \leq T)$,
 L_7 is the segment $[1 - \eta(T, q) + iT, 2 + iT]$.
Here $\eta(t, q) = c/(4 \log q(|t| + 2))$.

As the same as the last case we have

$$\int_{L_1+L_2+L_6+L_7} (L(s, \chi_1))^z f(s, z, \chi_1) \frac{x^{s+1}}{s(s+1)} ds \ll x^2 e^{-c\sqrt{\log x}}.$$

Hence

$$\int_0^x A_z(u, \chi_1) du \ll x^2 e^{-c\sqrt{\log x}}.$$

By using (3), we have

$$\begin{aligned} A_z(x, \chi_1) &\ll \xi^{-1}x^2e^{-c\sqrt{\log x}} + \xi \log^A x + x^{(A+1)/(A+2)} \\ &\ll xe^{-c\sqrt{\log x}/4}. \end{aligned}$$

Hence in either case we have

$$A_z(x, \chi_1) \ll x^{\beta_1}(\log x)^{C'A} + xe^{-c\sqrt{\log x}/4},$$

where we note that

$$x^{\beta_1}(\log x)^{C'A} \ll xe^{-c\sqrt{\log x}/4}$$

for the case $\beta_1 \leq 1 - c/(2 \log 2q)$.

Finally we consider for the case $\chi \neq \chi_0, \chi_1$. The deformation of the path of integration is as the same as the case $\beta_1 \leq 1 - c/(2 \log 2q)$ for $\chi = \chi_1$, apart from $\eta(t, q) = c/\log q(|t| + 2)$.

Similar observation leads

$$A_z(x, \chi) \ll x e^{-c\sqrt{\log x}/4}.$$

As a consequence of these, we have, by replacing C' to C and $c/4$ to c , we have

$$A_z(x, \chi) = E_0 \Phi_{z, \delta}(x, q) + E_1 O(x^{\beta_1} (\log x)^{CA}) + O(x e^{-c\sqrt{\log x}}),$$

where E_0 takes 1 or 0 according as $\chi = \chi_0$ or not, and E_1 does 1 or 0 according as $\chi = \chi_1$ or not.

Hence we have

$$\begin{aligned} A_z(x, q, l) &= \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) A_z(x, \chi) \\ &= \frac{1}{\varphi(q)} \Phi_{z, \delta}(x, q) + O\left(\frac{x^{\beta_1}}{\varphi(q)} (\log x)^{CA}\right) + O(x e^{-c\sqrt{\log x}}). \end{aligned}$$

Siegel's theorem makes

$$\frac{x^{\beta_1}}{\varphi(q)} (\log x)^{CA} \ll x e^{-c\sqrt{\log x}}$$

uniformly for $q \leq (\log x)^\tau$.

Now it remains the asymptotic expansion of $\Phi_{z, \delta}(x, q)$. We define regular functions $B_m(z, q)$ as Taylor coefficients

$$\{(s-1)L(s, \chi_0)\}^z f(s, z, \chi_0) s^{-1} = \sum_{m=0}^{N-1} B_m(z, q) (s-1)^m + R_N(s, z, q)$$

for $s \in L_\delta$, $|z| \leq A$, where N is any fixed positive integer, especially

$$\begin{aligned} B_0(z, q) &= \left(\frac{\varphi(q)}{q}\right)^z f(1, z, \chi_0) \\ B_1(z, q) &= \left(\frac{\varphi(q)}{q}\right)^z \left\{ (z\gamma + z \sum_{p|q} \frac{\log p}{p-1} - 1) f(1, z, \chi_0) + f_s(1, z, \chi_0) \right\}, \end{aligned}$$

where γ is Euler constant.

Since we can see $R_N(s, z, q) \ll (s-1)^N$,

$$\begin{aligned} \Phi_{z,\delta}(x, q) &= \frac{1}{2\pi i} \int_{L_\delta} \{(s-1)L(s, \chi_0)\}^z f(s, z, \chi_0) s^{-1} (s-1)^{-z} x^s ds \\ &= \frac{x}{2\pi i} \left(\int_\Gamma - \int_{\Gamma-L_\delta} \right) \sum_{m=0}^{N-1} B_m(z, q) (s-1)^{m-z} x^{s-1} ds \\ &\quad + \frac{x}{2\pi i} \int_{L_\delta} R_N(s, z, q) (s-1)^{-z} x^{s-1} ds \end{aligned}$$

where Γ is the path which consist of the segment $(-\infty, i-r]$, the arc $s = i + re^{i\varphi}$ ($-\pi \leq \varphi \leq \pi$) and the segment $[1-r, -\infty)$. By substituting $(s-1)\log x = \omega$, we find

$$\begin{aligned} \frac{x}{2\pi i} \int_\Gamma (s-1)^{m-z} x^{s-1} ds &= (\log x)^{z-m-1} \frac{1}{2\pi i} \int_{\Gamma'} \omega^{m-z} e^\omega d\omega \\ &= \frac{(\log x)^{z-m-1}}{\Gamma(z-m)} \end{aligned}$$

The remaining integrals are to be the error term which is proved in Ivić[6].

LEMMA 1.2. We have

$$\log L(s, \chi) \ll (\log q(|t| + 2))^{1+2\Theta(\chi)-2\sigma+\varepsilon}$$

uniformly for $\Theta(\chi) < \sigma_0 \leq \sigma \leq 1$, $|t| \geq E_0$, $q \geq 1$.

This is proved in the same way as Theorem 14.2 in Titchmarsh [16], which is the case of $q = 1$ and $\sigma = 1/2$.

The Lemma is also a generalization of Lemma 1.2 in [11] which is the case of $q = 1$.

We define the error terms

$$\begin{aligned} \Delta_{z,\delta}(x, \chi) &= A_z(x, \chi) - E_0 \Phi_{z,\delta}(x, q), \\ \Delta_{z,\delta}(x, q, l) &= A_z(x, q, l) - \frac{1}{\varphi(q)} \Phi_{z,\delta}(x, q), \end{aligned}$$

and let

$$\begin{aligned} \alpha_z(\chi) &= \inf_\delta \inf \{ \alpha : \Delta_{z,\delta}(x, \chi) \ll x^\alpha \}, \\ \alpha_{z,q,l} &= \inf_\delta \inf \{ \alpha : \Delta_{z,\delta}(x, q, l) \ll x^\alpha \}. \end{aligned}$$

THEOREM 1.3. *We have*

$$\alpha_z(\chi) \leq \Theta(\chi)$$

for any $z \in C$ under the assumption that $a_z(n) \ll n^{1/2+\epsilon}$.

Remark. Theorem 1.3 leads us easily to that

$$\alpha_{z,q,l} \leq \Theta_q$$

because of the relation

$$\Delta_{z,\delta}(x,q,l) = \frac{1}{\varphi(q)} \sum_{\chi(\text{mod } q)} \bar{\chi}(l) \Delta_{z,\delta}(x,\chi).$$

Proof. By Lemma 3.12 in Titchmarsh [16], $A_z(x,\chi)$ has the expression

$$A_z(x,\chi) = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} (L(s,\chi))^z f(s,z,\chi) \frac{x^s}{s} ds + O\left(\frac{x^2}{T}\right) + O(x^{1/2+\epsilon})$$

uniformly for $T \geq 1$.

For $\chi = \chi_0$, the path of integration can be replaced by $\sum_{i=1}^7 L_i$ which is defined as follows :

L_1 is the segment $[2 - iT, \eta - iT]$,

L_2 is two segments $[\eta - iT, \eta - i(1 - \eta) \tan \theta] + [\eta - i(1 - \eta) \tan \theta, 1/2 + \delta - i(1/2 - \delta) \tan \theta]$,

L_3, L_4 and L_5 are the same as in Lemma 1.1,

L_6 is two segments $[1/2 + \delta + i(1/2 - \delta) \tan \theta, \eta + i(1 - \eta) \tan \theta] + [\eta + i(1 - \eta) \tan \theta, \eta + iT]$,

L_7 is the segment $[\eta + iT, 2 + iT]$.

Here η is a constant such that $\Theta(\chi_0) < \eta < 1$, and non negative number δ and any positive numbers r and θ are such that $1/2 + \delta \leq \eta < 1 - r$, $0 < (1 - \eta) \tan \theta < 1$.

As in Lemma 1.1, $L_3 + L_4 + L_5$ becomes L_δ by allowing $\theta \downarrow 0$.

From Lemma 1.2, we have

$$(L(s,\chi_0))^z f(s,z,\chi_0) \ll (q(|t| + 2))^\epsilon$$

for $s \in L_1, L_2, L_6, L_7$. Therefore,

$$\int_{L_1} + \int_{L_7} \ll \int_{\eta}^2 (qT)^\epsilon \frac{x^2}{T} d\sigma \ll q^\epsilon T^{\epsilon-1} x^2,$$

$$\int_{L_2} + \int_{L_6} \ll \int_0^T (qt)^\varepsilon \frac{x^\eta}{t+1} dt + \int_{1/2+\delta}^\eta q^\varepsilon x^\eta d\sigma \ll q^\varepsilon T^\varepsilon x^\eta.$$

Hence

$$A_z(x, \chi_0) = \Phi_{z, \delta}(x, q) + O(q^\varepsilon T^{\varepsilon-1} x^2) + O((qT)^\varepsilon x^\eta) + O\left(\frac{x^2}{T}\right) + O(x^{1/2+\varepsilon}).$$

By taking $T = x^2$, $\eta = \Theta(\chi_0) + \varepsilon$ we have

$$A_z(x, \chi_0) = \Phi_{z, \delta}(x, q) + O(x^{\Theta(\chi_0)+4\varepsilon})$$

uniformly for $q \leq x$.

For $\chi \neq \chi_0$, the path of integration is replaced by $\sum_{i=1,2,6,7} L_i$.

L_1 is the segment $[2 - iT, \eta - iT]$, L_2 is the segment $[\eta - iT, \eta]$, L_6 is the segment $[\eta, \eta + iT]$, L_7 is the segment $[\eta + iT, 2 + iT]$, where η is a constant such that $\Theta(\chi) < \eta < 1$.

By similar way, we find

$$A_z(x, \chi) \ll x^{\Theta(\chi)+4\varepsilon}.$$

Hence

$$A_z(x, \chi) = E_0 \Phi_{z, \delta}(x, q) + O(x^{\Theta(\chi)+4\varepsilon}),$$

this proves the theorem.

THEOREM 1.4. We have

$$\Theta(\chi) \leq \alpha_z(\chi)$$

for any $z \in C - Q^+$.

Remark. Theorem 1.4 leads us easily to that

$$\Theta(\chi) \leq \max_l \alpha_{z, q, l}$$

because of the relation

$$\Delta_{z, \delta}(x, \chi) = \sum_l \chi(l) \Delta_{z, \delta}(x, q, l).$$

Proof. First, we suppose that $\sigma > 2$. Then,

$$\begin{aligned} s \int_1^\infty \frac{\Phi_{z, \delta}(x, q)}{x^{s+1}} dx &= s \int_1^\infty \left(\frac{1}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{x^\omega}{\omega} d\omega \right) \frac{1}{x^{s+1}} dx \\ &= \frac{s}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \left(\int_1^\infty x^{\omega-s-1} dx \right) \frac{d\omega}{\omega} \\ &= \frac{s}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{d\omega}{\omega(s-\omega)} \\ &= \frac{1}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{d\omega}{\omega} \\ &\quad + \frac{1}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{d\omega}{s-\omega}. \end{aligned}$$

The interchange of the order of integration is justified because of the absolute convergence. Hence

$$\begin{aligned} (L(s, \chi))^z f(s, z, \chi) &= s \int_1^\infty \frac{A_z(x, \chi)}{x^{s+1}} dx \\ &= E_0 s \int_1^\infty \frac{\Phi_{z, \delta}(x, q)}{x^{s+1}} dx + s \int_1^\infty \frac{\Delta_{z, \delta}(x, \chi)}{x^{s+1}} dx \\ &= \frac{E_0}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{d\omega}{\omega} \\ &\quad + \frac{E_0}{2\pi i} \int_{L_\delta} (L(\omega, \chi_0))^z f(\omega, z, \chi_0) \frac{d\omega}{s-\omega} + s \int_1^\infty \frac{\Delta_{z, \delta}(x, \chi)}{x^{s+1}} dx. \quad (4) \end{aligned}$$

We put

$$L_{\sigma_0} = \{s : \sigma_0 < \sigma\} - \{s : |s-1| \leq r\} - \{s : t = 0, \sigma < 1\}$$

for $1/2 \leq \sigma_0 \leq 2$. Then on the right hand side of (4), the first and the second term can be continued analytically for $s \in L_{1/2}$ as a function of s , while the third term can be continued for $\sigma > \alpha_z(\chi)$ since the involved integral converge uniformly for $\sigma > \alpha_z(\chi)$. Hence the right hand side of (4) is regular for $s \in L_{\alpha_z(\chi_0)}$ in the case for $\chi = \chi_0$, and for $\sigma > \alpha_z(\chi)$ in the case for $\chi \neq \chi_0$.

On the other hand the left hand side of (4) has logarithmic singularities at the zeros of $L(s, \chi)$ when $s \in C - Q^+$.

We therefore conclude that $\Theta(\chi) \leq \alpha_z(\chi)$ for any $z \in C - Q^+$, since $L(s, \chi_0)$ vanishes neither on the real axis ($\sigma \geq 1/2$) nor near the point $s = 1$.

Remark.

If we suppose that all the zeros of $L(s, \chi)$ are simple, Theorem 1.4 holds for all $z \in C - N$.

Now Theorem 1 follows by taking $f(s, z) \equiv 1$.

§2. The Asymptotic Formula for $\pi_k(x, q, l)$

Throughout this section, we suppose that $a_z(n)$ is regular for $|z| \leq A$, and has Taylor expansion at $z = 0$ such that $a_z(n) = \sum_{k=0}^{\infty} c_k(n)z^k$ for $|z| \leq A$ with $A > 1$.

LEMMA 2.1. We have

$$C_k(x, q, l) \equiv \sum_{\substack{n \leq x \\ n \equiv l \pmod{q}}} c_k(n) = \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z, \delta}(x, q)}{z^{k+1}} dz + O(xe^{-c\sqrt{\log x}})$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q, l) = 1$.

Further, if we put

$$F_{k, \delta}(x, q) \equiv \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z, \delta}(x, q)}{z^{k+1}} dz,$$

then $F_{k, \delta}(x, q)$ has the following asymptotic expansion

$$F_{k, \delta}(x, q) = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{Q_{m, q}(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^{N+1}}\right)$$

for every k , where $Q_{m, q}(x)$ are polynomials of degree not exceeding $k - 1$, especially the coefficient of x^{k-1} of $Q_0(x)$ is 1.

Proof. Since $A_z(x, q, l)$ is regular for $|z| \leq A$ as a function of z , and $C_k(x, q, l)$ is Taylor coefficient of z^k , it follows by using Lemma 1.1 that

$$\begin{aligned} C_k(x, q, l) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{A_z(x, q, l)}{z^{k+1}} dz \\ &= \frac{1}{\varphi(q)} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Phi_{z, \delta}(x, q)}{z^{k+1}} dz + \frac{1}{2\pi i} \int_{|z|=1} \frac{\Delta_{z, \delta}(x, q, l)}{z^{k+1}} dz, \end{aligned}$$

where

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=1} \frac{\Delta_{z, \delta}(x, q, l)}{z^{k+1}} dz &\ll \max_{|z|=1} |\Delta_{z, \delta}(x, q, l)| \\ &\ll xe^{-c\sqrt{\log x}} \end{aligned}$$

which proves the first half.

Now, we expand the principal term asymptotically. By using the asymptotic expansion of $\Phi_{z,\delta}(x, q)$ proved in Lemma 1.1, $F_{k,\delta}(x, q)$ has the following expression

$$F_{k,\delta}(x, q) = \frac{x}{\log x} \frac{1}{2\pi i} \int_{|z|=1} \frac{(\log x)^z}{z^{k-1}} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z-m)} dz \\ + O\left(\frac{x}{(\log x)^{N+1}} \frac{1}{2\pi i} \int_{|z|=1} \frac{(\log x)^{\Re z}}{z^{k+1}} dz\right)$$

Then, if we denote

$$\frac{B_m(z, q)}{\Gamma(z-m)} = \sum_{i=1}^{\infty} e_{m,i,q} z^i, \quad (\log x)^z = \sum_{l=0}^{\infty} \frac{(\log \log x)^l}{l!} z^l,$$

the leading term can be deformed

$$\frac{x}{\log x} \frac{1}{2\pi i} \int_{|z|=1} \frac{(\log x)^z}{z^{k+1}} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z-m)} dz \\ = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{1}{(\log x)^m} \\ \times \frac{1}{2\pi i} \int_{|z|=1} \sum_{l=0}^{\infty} \frac{(\log \log x)^l}{l!} \sum_{i=1}^{\infty} e_{m,i,q} z^{l+i-k-1} dz \\ = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{1}{(\log x)^m} \sum_{\substack{l+i=k \\ l \geq 0, i \geq 1}} \frac{e_{m,i,q}}{l!} (\log \log x)^l \\ = \frac{x}{\log x} \sum_{m=0}^{N-1} \frac{1}{(\log x)^m} \sum_{l=0}^{k-1} \frac{e_{m,k-l,q}}{l!} (\log \log x)^l \\ = \frac{x}{\log x} \sum_{m=0}^{N-2} \frac{Q_{m,q}(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^{N+1}}\right),$$

where $Q_{m,q}(x) = \sum_{l=0}^{k-1} e_{m,k-l,q} (l!)^{-1} x^l$ are polynomials of degree not exceeding $k-1$.

On the other hand

$$\frac{x}{(\log x)^{N+1}} \frac{1}{2\pi i} \int_{|z|=1} \frac{(\log x)^{\Re z}}{z^{k+1}} dz \ll \frac{x}{(\log x)^{N+1}} \int_{|z|=1} \frac{\log x}{|z|^{k+1}} |dz| \\ \ll \frac{x}{(\log x)^N}.$$

Hence

$$F_{k, \delta}(x, q) = \frac{x}{\log x} \sum_{m=0}^{N-2} \frac{Q_{m,q}(\log \log x)}{(\log x)^m} + O\left(\frac{x(\log \log x)^{k-1}}{(\log x)^N}\right) + O\left(\frac{x}{(\log x)^N}\right).$$

This proves the Lemma by replacing N to $N + 1$.

Remark.

For $k = 1$, we can express the main term in terms of the logarithmic integral. Namely, we start from the expression

$$\Phi_{z, \delta}(x, q) = \int_2^x \left(\frac{1}{2\pi i} \int_{L_\delta} (L(s, \chi_0))^z f(s, z, \chi_0) x^{s-1} ds \right) dx + O(1)$$

and define $\tilde{B}_m(z, q)$ by Taylor coefficients of

$$\begin{aligned} & \{(s-1)L(s, \chi_0)\}^z f(s, z, \chi_0) \\ &= \left(\frac{\varphi(q)}{q}\right)^z f(1, z, \chi_0) + \sum_{m=0}^{N-1} \tilde{B}_m(z, q)(s-1)^m + \tilde{R}_N(s, z, q) \end{aligned}$$

instead of $B_m(z, q)$. Then similar consideration to the asymptotic expansion in Lemma 1.1 and Lemma 1.3 make

$$F_{1, \delta}(x, q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}),$$

so that

$$C_1(x, q, l) = \frac{1}{\varphi(q)} \int_2^x \frac{du}{\log u} + O(xe^{-c\sqrt{\log x}}).$$

This satisfies the assertion.

We define the error terms

$$\begin{aligned} R_{k, \delta}(x, \chi) &= C_k(x, \chi) - E_0 F_{k, \delta}(x, q) \\ R_{k, \delta}(x, q, l) &= C_k(x, q, l) - \frac{1}{\varphi(q)} F_{k, \delta}(x, q) \end{aligned}$$

and let

$$\begin{aligned} r_k(\chi) &= \inf_{\delta} \inf\{r : R_{k, \delta}(x, \chi) \ll x^r\}, \\ r_{k, q, l} &= \inf_{\delta} \inf\{r : R_{k, \delta}(x, q, l) \ll x^r\}. \end{aligned}$$

THEOREM 2.2. *We have*

$$r_k(\chi) = \Theta(\chi)$$

for any $k \geq 1$.

Remark. Theorem 2.2 leads us easily to that

$$\max_l r_{k,q,l} = \Theta_q$$

by the relations that

$$R_{k,\delta}(x, q, l) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \bar{\chi}(l) R_{k,\delta}(x, \chi),$$

$$R_{k,\delta}(x, \chi) = \sum_l \chi(l) R_{k,\delta}(x, q, l).$$

Proof. From Theorem 1.3, we have

$$R_{k,\delta}(x, \chi) = \frac{1}{2\pi i} \int_{|z|=1} \frac{\Delta_{z,\delta}(x, \chi)}{z^{k+1}} dz \ll \max_{|z|=1} |\Delta_{z,\delta}(x, \chi)| \ll x^{\Theta(\chi)+4\epsilon}.$$

Hence $r_k(\chi) \leq \Theta(\chi)$.

Conversely,

$$\begin{aligned} C_k(x, \chi) &= \frac{1}{2\pi i} \int_{|z|=1} \frac{A_z(x, \chi)}{z^{k+1}} dz \\ &= \frac{1}{2\pi i} \int_{|z|=1} \left(\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} (L(s, \chi))^z f(s, z, \chi) \frac{x^s}{s} ds \right) \frac{1}{z^{k+1}} dz \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \left(\frac{1}{2\pi i} \int_{|z|=1} \frac{(L(s, \chi))^z f(s, z, \chi)}{z^{k+1}} dz \right) \frac{x^s}{s} ds \\ &= \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} G_k(s, \chi) \frac{x^s}{s} ds, \text{ say.} \end{aligned}$$

Here we have

$$G_k(s, \chi) = \sum_{\ell=0}^k \frac{1}{\ell!(k-\ell)!} (\log L(s, \chi))^\ell f^{(k-\ell)}(s, 0, \chi)$$

where $f^{(n)}(s, z, \chi)$ means the n -th derivative of $f(s, z, \chi)$ with respect to z . It follows that $G_k(s, \chi)$ is regular for $s \in L_{\Theta(\chi)}$, and has the expression

$$G_k(s, \chi) = \sum_{n=1}^{\infty} \frac{c_k(n)\chi(n)}{n^s} \quad (\sigma > 1).$$

Thus

$$F_{k, \delta}(x, q) = \frac{1}{2\pi i} \int_{L_\delta} G_k(s, \chi) \frac{x^s}{s} ds.$$

If we suppose $\sigma > 2$, then

$$\begin{aligned} s \int_1^\infty \frac{F_{k, \delta}(x, q)}{x^{s+1}} dx &= s \int_1^\infty \left(\frac{1}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \frac{x^\omega}{\omega} d\omega \right) \frac{1}{x^{s+1}} dx \\ &= \frac{s}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \left(\int_1^\infty x^{\omega-s-1} dx \right) \frac{d\omega}{\omega} \\ &= \frac{1}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \frac{d\omega}{\omega} + \frac{1}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \frac{d\omega}{s-\omega}. \end{aligned}$$

Hence

$$\begin{aligned} G_k(s, \chi) &= E_0 s \int_1^\infty \frac{C_k(x, \chi)}{x^{s+1}} dx \\ &= E_0 s \int_1^\infty \frac{F_{k, \delta}(x, q)}{x^{s+1}} dx + s \int_1^\infty \frac{R_{k, \delta}(x, \chi)}{x^{s+1}} dx \\ &= \frac{E_0}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \frac{d\omega}{\omega} \\ &\quad + \frac{E_0}{2\pi i} \int_{L_\delta} G_k(\omega, \chi) \frac{d\omega}{s-\omega} + s \int_1^\infty \frac{R_{k, \delta}(x, \chi)}{x^{s+1}} dx. \end{aligned} \tag{5}$$

Now on the right hand side of (5), the first and the second term can be continued analytically for $s \in L_{1/2}$ as a function of s , while the third term can be continued for $\sigma > r_k(\chi)$, since the involved integral converges uniformly for $\sigma > r_k(\chi)$. Hence the right hand side of (5) is regular for $s \in L_{r_k(\chi_0)}$ in the case for $\chi = \chi_0$, and for $\sigma > r_k(\chi)$ in the case for $\chi \neq \chi_0$.

But $G_k(s, \chi)$ has singularities at zeros $\rho = \beta + i\gamma$, say, of $L(s, \chi)$. In fact, if we consider the limit $G_k(\sigma + i\gamma, \chi)$ as $\sigma \downarrow \beta$, under the

assumption that ρ is a zero of order M ,

$$\begin{aligned}
 G_k(\sigma + i\gamma, \chi) & \\
 & \sim \sum_{\ell=0}^k \frac{1}{\ell!(k-\ell)!} (\log L(\sigma + i\gamma, \chi))^\ell f^{(k-\ell)}(\sigma + i\gamma, 0, \chi) \\
 & \sim \sum_{\ell=0}^k \frac{1}{\ell!(k-\ell)!} (M \log(\sigma - \beta))^\ell f^{(k-\ell)}(\rho, 0, \chi) \\
 & \sim \sum_{\ell=0}^k \frac{1}{\ell!(k-\ell)!} M^\ell (-t)^\ell f^{(k-\ell)}(\rho, 0, \chi) \quad (\sigma - \beta = e^{-t}) \\
 & \sim M^k t^k \quad (t \rightarrow \infty),
 \end{aligned}$$

for $f^{(k-\ell)}(\rho, 0, \chi)$ is bounded and $f(s, 0, \chi) = 1$. Hence we conclude that $\Theta(\chi) \leq r_k(\chi)$ whether $\chi = \chi_0$, or not, for any $k \geq 1$ since $L(s, \chi_0)$ vanishes neither on the real axis ($\sigma \geq 1/2$) nor near the point $s = 1$.

This completes the theorem.

Now Theorem 2 follows by taking

$$f(s, z) = \prod_p \left(1 + \frac{z}{p^s}\right) \left(1 - \frac{1}{p^s}\right)^z \quad \text{for } \pi_k(x, q, l)$$

$$f(s, z) = \prod_p \left(1 + \frac{z}{p^s - 1}\right) \left(1 - \frac{1}{p^s}\right)^z \quad \text{for } \omega_k(x, q, l)$$

$$f(s, z) = \prod_p \left(1 - \frac{z}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^z \quad \text{for } \Omega_k(x, q, l)$$

which satisfy the assumptions on $f(s, z)$ at the top of sections 1 and 2.

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