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Sci. Rep. Kanazawa Univ.
Vol. 33 No. 1 pp. 1～13
June 1988

Note on Ramification of Central Extensions and Leopoldt's Conjecture

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(Received April 26, 1988)

Abstract. Some equivalent conditions to Leopoldt's conjecture related with the structure of Galois groups of central extensions are studied.

Introduction

Let k be an algebraic number field of finite degree, and ℓ be a prime number. Throughout this paper, we always assume

(*) $\quad \sqrt{-1} \in k$ *when* $\ell=2$.

Denote by $G(K/k)$ the Galois group of a Galois extension K/k, and by $k^{(\ell)}$ the maximal ℓ-extension of k unramified outside ℓ. Then it is well-known that Leopoldt's conjecture for k and ℓ is equivalent to $H^{-3}(G(k^{(\ell)}/k), \mathbf{Z})=0$. This is connected with a certain problem of central extensions through the relationship between the structure of the Galois group of a central extension and (the dual of) Schur's multiplicator $H^{-3}(G,\mathbf{Z})$. The purpose of the present paper is to study some equivalent conditions to Leopoldt's conjecture in this point of view, especially the conditions related with the structure of the connected component of unity of idele class groups are obtained (Theorem 14 and 15).

1. Leopoldt's conjecture and abundant central extensions.

For any pro-finite group G and a natural number n, the cohomology group $H^{-n}(G,\mathbf{Z})$ of minus dimension is defined by

$$H^{-n}(G,\mathbf{Z})=\varprojlim H^{-n}(G/U_{\lambda},\mathbf{Z}),$$

where U_{λ} runs over open subgroups of G of finite index, and lim is of the deflation map.

Then we have $H^{-n}(G,\mathbf{Z})\simeq H^{n}(G,\mathbf{Z})^{\wedge}$. Hence $H^{-3}(G,\mathbf{Z})\simeq H^{3}(G,\mathbf{Z})^{\wedge}\simeq H^{2}(G,\mathbf{Q}/\mathbf{Z})^{\wedge}$, which is called (the dual of) Schur's multiplicator of G.

For a tower of Galois extensions $M\supset K\supset k$, denote by $K^{*}_{M/K}$ the genus field of K/k in M, which is by definition, the composite of K and the maximal abelian extension of K/k in M. Denote by $\hat{K}_{M/k}$ the maximal central extension of K/k in M, namely the maximal extension in M whose Galois group over K is contained in the center of the Galois group over k. Then we have the following theorem (Cf. Heider [3, §2], Furuta [1, Theorem 5]).

THEOREM 1.

$$G(\hat{K}_{M/k}/K^{*}_{M/k})=\frac{H^{-3}(G(K/k),\mathbf{Z})}{Def_{G(M/k)\to G(K/k)}H^{-3}(G(M/k,\mathbf{Z})}.$$

We call M *abundant for* K/k when $G(\hat{K}_{K/k}/K^{*}_{M/k})\simeq H^{-3}(G(K/k),\mathbf{Z})$, namely $Def_{G(M/k)\to G(K/k)}H^{-3}(G(M/k),\mathbf{Z})=0$.

For a Galois extension M/k, it follows from Theorem 1 and the definition of cohomology groups of pro-finite groups that $H^{-3}(G(M/k),\mathbf{Z})=0$ if and only if M is abundant for any finite Galois extension K over k contained in M.

Now denote by $\bar{k}$ the algebraic closure of k. Then it is well-known that

$$H^{-3}(G(\bar{k}/k),\mathbf{Z})=0 \tag{1.1}$$

(Cf. Serre [10, Theorem 4], Heider [3, §5], Yamashita [11, Theorem 3], Miyake [9]. See also Theorem 5 and its remark below). Hence we have

THEOREM 2. $G(\hat{K}_{\bar{k}/k}/K^{*}_{\bar{k}/k})\simeq H^{-3}(G(K/k),\mathbf{Z})$.

For an algebraic number field K, denote by $k^{\times}$ the multiplicative group of non-zero elements of K, J_K the group of ideles of K, in which $K^{\times}$ is embeded as the group of principal ideles. Let $C_K=J_K/K^{\times}$ the idele class group of K, and U_K the group of unit ideles of K. Denote further by $K^{\times}(\mathfrak{m})$ the ray group mod. $\mathfrak{m}$ for an integral divisor $\mathfrak{m}$ of K, i.e., $K^{\times}(\mathfrak{m})=\{\alpha\in K^{\times}\ ;\ \alpha\equiv 1 \bmod.\mathfrak{m}\}$ and $J_K(\mathfrak{m})$ the group of elements $\mathfrak{a}$ of J_K with a $\mathfrak{p}$-component $\mathfrak{a}_{\mathfrak{p}}$ such that $\mathfrak{a}_{\mathfrak{p}}\equiv 1 \bmod \mathfrak{m}_{\mathfrak{p}}$, where $\mathfrak{p}$ is any prime of K and $\mathfrak{m}_{\mathfrak{P}}$ is the $\mathfrak{P}$-component of $\mathfrak{m}$.

For a Galois extension K/k, an integral divisor $\mathfrak{m}$ of K is called *Scholz conductor* if $\mathfrak{m}$ is invariant by $G(K/k)$ and

$$k^{\times}\cap N_{K/k}J_K(\mathfrak{m})/N_{K/k}K^{\times}(\mathfrak{m})\simeq H^{-1}(G(K/k),\ C_K),$$

which is isomorphic to $H^{-3}(G(K/k),\mathbf{Z})$. The following theorem is well-known.

THEOREM 3 (Heider [3, §4, Satz]). *Let K/k be a galois extension of finite degree and E_k be the group of units of k. Then*

(i) *There always exists Scholz conductor, and it can be taken so that it contains only primes ramified in K/k.*

(ii) *The ray class field over K* mod. $\mathfrak{m}$ *is abundant for K/k if and only df* $\mathfrak{m}$ *is a Scholz conductor and*

$$E_k\cap N_{K/k}K^{\times}(\mathfrak{m})=E_k\cap N_{K/k}U_K(\mathfrak{m}).$$

Now we are interested in the following problem :

PROBLEM. *For any Galois exension K/k, does there exist an abundant extension M for K/k such that if a prime divisor $\mathfrak{P}$ of K is ramified in M/K, then $\mathfrak{P}$ is a ramified prime of K/k or a prime divisor of the degree* (K : k) ?

The above problem is closely related to Leopoldt's conjecture e. g. as follows. We assume always (*), and denote by $k^{(\ell)}$ the maximal ℓ-extension unramified outside ℓ as in Introduction. Then the following theorem is well-known (Cf. Heider [4, Satz 6, Bemerkung] , Heider [5, Satz 11] , Iwasawa [6] , Kuz'min [8, Theorem 7. 2]).

THEOREM 4. *Under the assumption* (*), *Leopoldt's conjecture for ℓ is true for k if and only if $k^{(\ell)}$ is abundant for any Galois extension of finite degree over k contained in $k^{(\ell)}$, i.e., $H^{-3}(G(k^{(\ell)}/k),\mathbf{Z})=0$.*

REMARK. In a previous report "Note on central extensions and Leopoldt's conjecture, RIMS Kōkyūroku 603 (1987), 137-151", we state that the above abundantness holds if it holds for any extensions in $k^{(\ell)}$ of type $(\ell,\ \ell)$. However the statement is not guaranteed, because the assertion (2.2') in the report is unexact. But the present paper does not use this result.

2. Some conditions of Leopoldt's conjecture.

For a group G of finite order and a G-module A, denote by I_G the augmentation ideal of the group ring $\mathbf{Z}[G]$, and set $N_G=\sum_{\sigma\in G}\sigma$ in $\mathbf{Z}[G]$. Denote further by A^G the submodule of A consists of G-invariant elements of A, and $O_G(A)$ the submodule of A consists of elements a of A such that $N_G a=0$. For a Galois extension M of K, we denote by $B_K(M)$ the idele group of K corresponding to the maximal abelian extension of K contained in M by class field theory, namely $B_K(M)=K^\times \cdot N_{M/K}J_M$. Put $B_K=B_K(\bar{K})$ and $D_K=B_K/K^\times$. Then D_K is the connected component of unity of the idele class group C_K, and the following is well-known (Cf. Iyanaga [7, Chap. 5, Theorem 6. 5]):

(2.1) $$H^s(G(K/k), D_K) \simeq \begin{cases} 0 & \text{when } s \text{ is odd} \\ (\mathbf{Z}/2\mathbf{Z})^{r_0} & \text{when } s \text{ is even,} \end{cases}$$

where r_0 is the number of real places of k ramified in K. Thus we have

(2.1′) $$H^s(G(K/k), D_K)=0$$

for any integer s under the assumption (*).

THEOREM 5. *Let $M/K/k$ be a tower of Galois extensions with Galois group $G=G(K/k)$ of finite order. Then M is abundant for K/k if $H^{-1}(G,B_K(M)/K^\times)=0$.*

Proof. An exact sequence of natural homomorphisms

$$0\to B_K(M)/K^\times \xrightarrow{\lambda} C_K=J_K/K^\times \xrightarrow{\rho} J_K/B_K(M)\to 0$$

implies a long exact sequence

(2.2) $$\cdots\to H^{-1}(G,B_K(M)/K^\times)\xrightarrow{\lambda^\#} H^{-1}(G,C_K)\xrightarrow{\rho^\#} H^{-1}(G,J_K/B_K(M))\xrightarrow{\bar{\delta}^\#} H^0(G,B_K(M)/K^\times)\to\cdots.$$

Now $H^{-1}(G,J_K(M))\simeq O_G(J_K/B_K(M))/I_G(J_K/B_K(M))$ and $H^0(G,B_K(M))/K^\times)\simeq (B_K(M)/K^\times)^G/N_{K/k}(B_K(M)/K^\times)$. By means of these isomorphisms, the connecting homomorphism $\bar{\delta}^\#$ is induced from the norm map. More precisely, let $\mathfrak{a}\, B_K(M)\in O_G(J_K/B_k(M))$, namely $N_{K/k}\mathfrak{a} \in B_k(M)$. Then $\bar{\delta}^\#(\mathfrak{a}\, B_K(M))=(\hat{\lambda}^{-1}\cdot \hat{N}_G \cdot \hat{\rho}^{-1})(\mathfrak{a}\, B_K(M))$, where $\bar{\lambda}$, $\hat{N}_G$ and $\hat{\rho}$ are the induced homomorphisms to the each corresponding residue classes. Thus we have Ker $\bar{\delta}^\# \simeq (X\cdot B_K(M)/B_K(M))/((I_G J_K\cdot B_K(M))$, where $X=\{\mathfrak{a}\in J_K\,;\, N_{K/k}\mathfrak{a}\in k^\times N_{K/k}B_K(M))=B_k(K^*_{M/k})$. Therefore $G(\hat{K}_{M/k}/K^*_{M/k})\simeq B_K(K^*_{M/k})/B_K(\hat{K}_{M/k})\simeq (B_K(K^*_{M/k})B_K(M)/B_K(M))/((I_G J_K\cdot B_K(M)/B_K(M))\simeq$ Ker $\bar{\delta}^\# =$ Im $\rho^\# \simeq H^{-1}(G,C_K)/\lambda^\# H^{-1}(G, B_K(M)/K^\times)$. Hence if $H^{-1}(G,B_K(M)/K^\times)=0$, then $G(\hat{K}_{M/k}/K^*_{M/k})\simeq H^{-1}(G,C_k)\simeq H^{-3}(G,\mathbf{Z})$, which is to be proved.

REMARK. Since $H^{-1}(G(K/k), D_K)=0$ by (2.2), Theorem 5 implies Theorem 2.

For a prime $\mathfrak{P}$ of K, let $K_{\mathfrak{P}}$ be the $\mathfrak{P}$-completion of K which is embeded in J_K as usual, and $U_{K,\mathfrak{P}}$ or $U_{\mathfrak{P}}$ the group of units of $K_{\mathfrak{P}}$. Let further $U'_K = \prod_{\mathfrak{P} \not| \ell} U_{\mathfrak{P}}$, the group of unit ideles whose $\mathfrak{L}$-components are equal to 1 for all prime divisors $\mathfrak{L}$ of ℓ. Denote by $C_K^{(\ell)}$ the group of elements $\mathfrak{c}$ of C_K such that $\mathfrak{c}^t \in D_K = B_K/K^\times$ by some natural number t prime to ℓ. Then $H^{-1}(G(K/k), C_K^{(\ell)}) = 0$.

THEOREM 6. *Leopoldt's conjecture for ℓ is true for k, if*

(B) $\qquad H^{-1}(G(K/k), U'_K B_K/K^\times) = 0$

for any Galois extension K over k of finite degree contained in $k^{(\ell)}$.

Proof. By (2.1′) and an exact sequence $0 \to B_K/K^\times \to U'_K B_K/K^\times \to (U'_K B_K/K^\times)/(B_K/K^\times) \to 0$, we have $H^{-1}(G(K/k), U'_K B_K/K^\times) \simeq^{-1}(G(K/k), (U'_K B_K/K^\times)/D_K)$, and in a same way $H^{-1}(G(K/k), (U'_K B_K/K^\times)C_K^{(\ell)}/D_K) \simeq H^{-1}(G(K/k), (U'_K B_K/K^\times)C_K^{(\ell)})$, which is isomorphic to $H^{-1}(G(K/k), U'_K B_K/K^\times)$. On the other hand we have $B_K(k^{(\ell)})/K^\times = (U'_K B_K/K^\times)C_K^{(\ell)}$. Hence the proposition follows from Theorem 4 and Theorem 5.

3. Structure of B_K.

Let K be any Galois extension of k of finite degree ℓ^n contained in $k^{(\ell)}$, fixed once for all, and assume $(*)$ as in Introduction. Other notation being as in the preceding section, put further $G = G(K/k)$.

Denote by $\mu_\ell(K_{\mathfrak{P}})$ the group of roots of unity contained in $K_{\mathfrak{P}}$ whose orders are powers of ℓ. Let $T'_K = \prod_{\mathfrak{P} \not| \ell} \mu_\ell(K_{\mathfrak{P}})$, which is contained in U'_K. Put $B'_K = T'_K B_K$, where B_K is as before the satulation of D_K to J_K, namely $D_K = B_K/K^\times$. Then $U'_K B_K = U'_K T'_K B_K = U'_K B'_K$. Hence we have

(3.1) $$H^{-1}(G, U'_K B_K/K^\times) = H^{-1}(G, U'_K B'_K/K^\times).$$

Therefore it follows from Theorem 6 that Leopoldt's conjecture for ℓ is true for k, if the following condition (B') is satisfied :

(B') $$H^{-1}(G, U'_K B'_K/K^\times) = 0.$$

In order to deal the condition (B') in the following section, we prepare some results.

LEMMA 7. $B_K^G = B_K$

Proof. The connected components D_K and D_k of unity of C_K and C_k are correspond to the maximal abelian extensions of K and of k respectively. Hence $N_{K/k}D_K = D_k$. On the other hand we have $H^0(G, D_K) = 0$ by (2.1'). Thus $D_K^G = N_{K/k}D_K = D_k$. It follows from Hilbert's theorem 90 that $D_K^G = (B_K/K^\times)^G = B_K^G/k^\times$. Since $D_k = B_k/k^\times$, we have the lemma.

LEMMA 8. *$N_{K/k}\mu_\ell(K_\mathfrak{P}) = \mu_\ell(k_\mathfrak{p})$ when $\mathfrak{P} \mid \mathfrak{p}$ and $\mathfrak{p} \not\mid \ell$.*

Proof. Put $q = N_{k/Q}\mathfrak{p}$ and $q^f = N_{K/Q}\mathfrak{P}$. Let ω and ξ be primitive roots of unity of orders $q-1$ and q^f-1, which are elements of $k_\mathfrak{p}$ and of $K_\mathfrak{P}$ respectively. Then we can chose ω and ξ so that $\xi = N_{K_\mathfrak{P}/k_\mathfrak{p}}\omega$, because $q^f-1 = (q-1)(q^{f-1}+q^{f-2}+\cdots+1)$ and $\omega^\sigma = \omega^q$ by a generator σ of the cyclic group $G(K_\mathfrak{P}/k_\mathfrak{p})$ of order f. Now let $q-1 = s\ell^t$, $(s, \ell) = 1$. Then ξ^s is a generator of $\mu_\ell(k_\mathfrak{p})$. Set $\omega_1 = \omega^s$. Then $\xi^s = N_{K_\mathfrak{P}/k_\mathfrak{p}}\omega_1$. Let u be the order of ω_1, that is, $\omega_1^u = 1$, and let $u = u'\ell^v$, $(u', \ell) = 1$. Then $\omega_1^{u'} \in \mu_\ell(K_\mathfrak{P})$ and $N_{K_\mathfrak{p}/k_\mathfrak{p}}\omega_1^{u'} = \xi^{su'}$, which is also a generator of $\mu_\ell(k_\mathfrak{p})$. This proves the lemma.

PROPOSITION 9. *$H^s(G, T'_K) = 0$ for all integers* s.

Proof. It follows from the definition of T'_K and semi-local theory of cohomology groups that $H^r(G, T'_K) \simeq \prod_{\mathfrak{p} \not\mid \ell} H^r(G_\mathfrak{P}, \mu_\ell(K_\mathfrak{P}))$, where $G_\mathfrak{P} = G(K_\mathfrak{P}/k_\mathfrak{p})$ and $\mathfrak{P}$ is any one of prime divisors of $\mathfrak{p}$. Let $\mathfrak{p}$ be a prime of k not dividing ℓ. Then Lemma 8 implies $H^0(G_\mathfrak{P}, \mu_\ell(K_\mathfrak{P})) = 0$. Then since $G_\mathfrak{P}$ is cyclic and $\mu_\ell(K_\mathfrak{P})$ is finite, we have further $H^1(G_\mathfrak{P}, \mu_\ell(K_\mathfrak{P})) = 0$. Hence $H^0(G, T'_K) = H^1(G, T'_K) = 0$, which implies the proposition by the well-known property of cohomology groups.

We recall the structure of B_k. For $\mathfrak{a} \in J_k$, set $|\mathfrak{a}| = \prod_\mathfrak{p} |\mathfrak{a}_\mathfrak{p}|_\mathfrak{p}$, the volume of $\mathfrak{a}$, where $|\ |_\mathfrak{p}$ stands for the canonical valuation and $\mathfrak{a}_\mathfrak{p}$ the $\mathfrak{p}$-component of $\mathfrak{a}$.

Let $\bar{\mathbf{Z}}$ be the completion of $\mathbf{Z}$ by means of subgroups of finite index, thus $\bar{\mathbf{Z}} \simeq \prod_p \mathbf{Z}_p$, where p runs over all rational primes and $\mathbf{Z}_p$ is the p-completion of $\mathbf{Z}$.

Let $B_k^o = \{\mathfrak{a} \in B_k \ ; \ |\mathfrak{a}| = 1\}$. Then any element $\mathfrak{a}$ of B_k^o is written as follows (Cf. Iyanaga [7, Chap. 5, Theorem 6. 3]) :

(3. 2) $\quad \mathfrak{a} = \alpha\eta_1^{\lambda_1} \cdots\cdots \eta_r^{\lambda_r}\phi_1(t_1) \cdots\cdots \phi_{r_2}(t_{r_2}),$

where $\alpha\in k^{\times}$, $\lambda_i=(s_i, \mathfrak{x}_i)\in\mathbf{R}\times\bar{\mathbf{Z}}$, $\mathfrak{x}_i=(x_{ip})$, $x_{ip}\in\mathbf{Z}_p$, r_2 the number of complex places of k, and $\phi_i(t)=e^{2\pi\sqrt{-1}\cdot t}$ for $t\in\mathbf{R}$.

Let $\mathfrak{p}$ be a finite prime of k, and set $N\mathfrak{p}=N_{k/Q}\mathfrak{p}$. For any global unit ε of k, set

$$\varepsilon=\zeta_{\mathfrak{p}}(\varepsilon)\cdot\varepsilon_{\mathfrak{p}}(1)$$

in $k_{\mathfrak{p}}$, where $\varepsilon_{\mathfrak{p}}(1)$ is a unit of $k_{\mathfrak{p}}$ such that $\varepsilon_{\mathfrak{p}}(1)\equiv1$ mod. $\mathfrak{p}$, and $\zeta_{\mathfrak{p}}(\varepsilon)$ is a root of unity of $k_{\mathfrak{p}}$ of order a divisor of $N\mathfrak{p}-1$. For $\mathfrak{x}=(x_{\mathfrak{p}})\in\bar{\mathbf{Z}}=\prod_p\mathbf{Z}_p$ with $x_p\in\mathbf{Z}_p$, let $(\varepsilon^{\mathfrak{x}})_{\mathfrak{p}}$ be the $\mathfrak{p}$-component of $\varepsilon^{\mathfrak{x}}$. Then we have

(3. 3) $\quad (\varepsilon^{\mathfrak{x}})_{\mathfrak{p}}=\zeta_{\mathfrak{p}}(\varepsilon)^{g(\mathfrak{x},\mathfrak{p})}\cdot\varepsilon_{\mathfrak{p}}(1)^{x_p}$,

where $g(\mathfrak{x}, \mathfrak{p})$ is a rational integer such that

(3. 4) $\quad g(\mathfrak{x}, \mathfrak{p})\equiv x_q \mod.(N\mathfrak{p}-1, q^s)$

for all rational primes q such that $q\mid(N\mathfrak{p}-1)$ and a sufficiently large s.

PROPOSITION 10. *Notation being as above, we have*

(Δ) $\quad U'_k\cap B'_k\subset N_{K/k}(U'_K\cap B'_K)$.

Proof. Let E_k and E_K be the group of global units of k and K respectively, and let $(K:k)=\ell^n$. Let $\varepsilon_1, \cdots\cdots, \varepsilon_r$ be a system of fundamental units of k. Put $\eta_i=\varepsilon_i^{\ell^n}=N_{K/k}\varepsilon_i$ for $i=1, \cdots\cdots, r$. Then they are independent over $\mathbf{Z}$ as $\mathbf{Z}$-module by means of power.

Now let $\mathfrak{a}\in U'_k\cap B'_k$. Then $\mathfrak{a}\in B^o_kT'_k$. Hence $\mathfrak{a}$ has the form as (3. 2) :

(3. 2') $\quad \mathfrak{a}=\alpha\eta_1^{\lambda_1}\cdots\cdots\eta_r^{\lambda_r}\phi_1(t_1)\cdots\cdots\phi_{r_2}(t_{r_2})\mathfrak{b}$,

where $\mathfrak{b}\in T'_k$. Moreover since $\mathfrak{a}\in U'_k$, we have $\alpha\in E_k$ and

(3. 5) $\quad \alpha=\zeta\,\varepsilon_1^{a_1}\cdots\cdots\varepsilon_r^{a_r}$,

where ζ is a root of unity in $k^{\times}$ and $a_i\in\mathbf{Z}$. Let $\eta_i=\zeta_{\mathfrak{p}}(\eta_i)\cdot\eta_{i\mathfrak{p}}(1)$ and $\varepsilon_i=\eta_{\mathfrak{p}}(\varepsilon_i)\cdot\varepsilon_{i\mathfrak{p}}(1)$ be decompositions by means of (3. 3) in $k_{\mathfrak{p}}$. Observing $\mathfrak{l}$-components of $\mathfrak{a}$ for $\mathfrak{l}\mid\ell$ and by means of (3. 3), it follows from (3. 2') that

(3. 6) $\quad 1=\alpha\prod_i\zeta_{\mathfrak{l}}(\eta_i)^{g(\mathfrak{x}_i,\mathfrak{l})}\eta_{i\mathfrak{l}}(1)^{x_{i\ell}}$.

where $x_{i\ell}\in\mathbf{Z}_\ell$ and $g(\mathfrak{x}_i, \mathfrak{l})$ be as (3. 4) with $\mathfrak{p}=\mathfrak{l}$. Let d be the order of ζ in (3. 5), and set $d=\ell^e d'$, $(d', \ell)=1$. Then by (3. 5) and (3. 6) we have

(3. 7) $\quad \prod_i\eta_{i\mathfrak{l}}(1)^{-x_{i\ell}d'}=\zeta^{d'}\prod_i\varepsilon_{i\mathfrak{l}}(1)^{a_id'}$, and

(3. 8) $\quad \prod_i\zeta_{\mathfrak{l}}(\eta_i)^{-g(\mathfrak{x}_i,\mathfrak{l})d'}=\prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{a_id'}$.

If p is any rational prime other than ℓ, then $1/\ell^n$ is an integer in $\mathbf{Z}_p$. Denote by

$U_{\mathfrak{p}}$ the group of $\mathfrak{p}$-adic units of k, $\mathfrak{p}$ being a prime divisor of p. Let $\mathfrak{y}_i=(y_{i\mathfrak{p}})$ be an element of $\bar{\mathbf{Z}}$ for $i=1, \cdots\cdots r$, whose p-component y_{ip} is defined by

$$(3.\ 9) \qquad y_{ip}=\begin{cases} a_i d'/\ell^n & \text{when } p\neq\ell \\ -x_{i\ell}d' & \text{when } p=\ell, \end{cases}$$

where a_i and $x_{i\ell}$ be as in (3. 5) and (3. 6). Let further $s_i=a_i d'/\ell^n$, the element of $\mathbf{R}$, for $i=1, \cdots\cdots, r$. Set

$$(3.\ 10) \qquad \mathfrak{b}=\eta_1^{\beta_1}\cdots\cdots\eta_r^{\beta r},$$

where $\beta_i=(s_i, \mathfrak{y}_i)\in\mathbf{R}\times\bar{\mathbf{Z}}$.

Let $\eta_i=\zeta_{\mathfrak{p}}(\eta_i)\cdot\eta_{i\mathfrak{p}}(1)$ be a decomposition in $k_{\mathfrak{p}}$ as above. Then for a finite prime $\mathfrak{p}$, we have

$$\mathfrak{b}_{\mathfrak{p}}=\prod_i(\eta_i^{\beta_i})_{\mathfrak{p}}=\prod_i\zeta_{\mathfrak{p}}(\eta_i)^{g(\mathfrak{y}_i,\ \mathfrak{p})}\cdot\prod_i\eta_{i\mathfrak{p}}(1)^{y_{ip}}$$

$$=\begin{cases}\prod_i\zeta_{\mathfrak{p}}(\varepsilon_i)^{g(\mathfrak{y}_i,\ \mathfrak{p})\ell^n}\cdot\prod_i\varepsilon_{i\mathfrak{p}}(1)^{y_{ip}\ell^n} & \text{when } \mathfrak{p}\nmid\ell \\ \prod_i\zeta_{\mathfrak{l}}(\eta_i)^{g(\mathfrak{y}_i,\ \mathfrak{l})}\cdot\prod_i\eta_{i\mathfrak{l}}(1)^{-x_{i\ell}d'} & \text{when } \mathfrak{l}\mid\ell\end{cases}$$

$$=\begin{cases}\prod_i\zeta_{\mathfrak{p}}(\varepsilon_i)^{g(\mathfrak{y}_i,\ \mathfrak{p})\ell^n}\cdot\prod_i\varepsilon_{i\mathfrak{p}}(1)^{a_id'} & \text{when } \mathfrak{p}\nmid\ell \\ \prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{g(\mathfrak{y}_i,\ \mathfrak{l})\ell^n}\cdot\zeta^{d'}\prod_i\varepsilon_{i\mathfrak{l}}(1)^{a_id'} & \text{when } \mathfrak{l}\mid\ell\end{cases}$$

by (3. 7) and (3. 9), where $g(\mathfrak{y}_i,\ \mathfrak{p})\ell^n$ mod.$(N\mathfrak{p}-1,\ q^s)$, $s>>0$ for rational primes q such that $q\mid(N\mathfrak{p}-1)$. Moreover (3. 9) implies for $\mathfrak{p}\nmid\ell$

$$g(\mathfrak{y}_i,\ \mathfrak{p})\ell^n\equiv\begin{cases} a_id' \quad \text{mod.}(N\mathfrak{p}-1,\ q^s) & \text{when } q\neq\ell \text{ and } q\mid(N\mathfrak{p}-1) \\ -x_{i\ell}d'\ell^n \ \text{mod.}(N\mathfrak{p}-1,\ \ell^s) & \text{when } q=\ell \text{ and } q\mid(N\mathfrak{p}-1)\end{cases}$$

by sufficiently large s. On the other hand (3. 5) implies

$$(\alpha^{d'})_{\mathfrak{p}}=\zeta^{d'}\prod_i\varepsilon^{a_id'}=\zeta^{d'}\prod_i\zeta_{\mathfrak{p}}(\varepsilon_i)^{a_id'}\cdot\prod_i\varepsilon_{i\mathfrak{p}}(1)^{a_id'}.$$

Hence $(\alpha^{d'})_{\mathfrak{p}}=\mathfrak{b}_{\mathfrak{p}}\mathfrak{c}_{\mathfrak{p}}$ by some $\mathfrak{c}_{\mathfrak{p}}\in\mu_\ell(k_{\mathfrak{p}})$ when $\mathfrak{p}\nmid\ell$.

For $\mathfrak{l}\mid\ell$, we have also

$$g(\mathfrak{y}_i,\ \mathfrak{l})\ell^n\equiv a_id' \quad \text{mod.}(N\mathfrak{l}-1,\ q^s),\ s\gg0,$$

for any prime $q\mid(N\mathfrak{l}-1)$. Hence we have

$$(3.\ 11) \qquad \prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{g(\mathfrak{y}_i,\ \mathfrak{l})\ell^n}=\prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{a_id'},$$

and (3. 5) implies

$$(\alpha^{d'})_{\mathfrak{l}}=\zeta^{d'}(\prod_i\varepsilon^{a_id'})_{\mathfrak{l}}=\zeta^{d'}\prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{a_id'}\prod_i\varepsilon_{i\mathfrak{l}}(1)^{a_id'}.$$

Hence $(\alpha^{d'})_{\mathfrak{l}}=b_{\mathfrak{l}}$.

Now we have $\alpha^{d'}=\mathfrak{b}\mathfrak{c}$ with $\mathfrak{c}\in T'_k$, and (3. 2') implies

$$\mathfrak{a}^{d'}=\alpha^{d'}\eta^{\lambda_1 d'}\cdots\cdots\eta_r^{\lambda_r d'}\phi_1(t_1 d')\cdots\cdots\phi_{r_2}(t_{r_2}d')\mathfrak{b}^{d'}$$
$$=\mathfrak{c}\,\eta_1^{\lambda_1 d'+\beta_1}\cdots\cdots\eta_r^{\lambda_r d'+\beta_r}\phi_1(t_1 d')\cdots\cdots\phi_{r_2}(t_{r_2}d')\mathfrak{b}^{d'}$$

Let $\hat{\mathfrak{c}}$ and $\hat{\mathfrak{b}}$ be elements of T'_K such that $\mathfrak{c}=N_{K/k}\hat{\mathfrak{c}}$ and $\mathfrak{b}=N_{K/k}\hat{\mathfrak{c}}$, whose existence are followed from Lemma 9. Let further $\hat{\phi}_i(t)=(\phi_i(t), 1, \cdots\cdots, 1)$, the element of C^{ℓ^n} such that $N_{K/k}(\hat{\phi}_i(t))=\phi_i(t)$, where $\ell^n=(K:k)$. Set

$$\hat{\mathfrak{a}}=\hat{\mathfrak{c}}\,\varepsilon_1^{\lambda_1 d'+\beta_1}\cdots\cdots\varepsilon_r^{\lambda_r d'+\beta_r}\hat{\phi}_1(t_1 d')\cdots\cdots\hat{\phi}_{r_2}(t_{r_2}d')\hat{\mathfrak{b}}^{d'}.$$

Then $\hat{\mathfrak{a}}\in B'_K$. Since the order of $\zeta_{\mathfrak{l}}(\varepsilon_i)$ is prime to ℓ, we have $\zeta_{\mathfrak{l}}(\varepsilon_i)^{g(\mathfrak{y}_i,\mathfrak{l})}=$

$\zeta_{\mathfrak{l}}(\varepsilon_i)^{a_i d'/\ell^n}$. Hence the $\mathfrak{L}$-component of $\hat{\mathfrak{a}}$ is equal to

$$\prod_i\zeta_{\mathfrak{l}}(\varepsilon_i)^{g(\mathfrak{x}_i,\mathfrak{l})d'+g(\mathfrak{y}_i,\mathfrak{l})}\cdot\prod_i\varepsilon_i(1)^{x_i\ell d'+y_i\ell}=1$$

by (3. 8) and (3. 9), when $\mathfrak{L}$ is a prime divisor of ℓ. Hence we have

$\hat{\mathfrak{a}}=U'_K\cap B'_K$ and $\mathfrak{a}^{d'}=N_{K/k}\hat{\mathfrak{a}}$.

Thus the condition (Δ) holds when $d'=1$.

Suppose that $d'>1$. Let $(K:k)=\ell^n$ as above. Since $(d', \ell)=1$, we can take integers u and v so that $d'u+\ell^n v=1$. Then we have $\mathfrak{a}=\mathfrak{a}^{d'u}\mathfrak{a}^{\ell^n v}=N_{K/k}(\hat{\mathfrak{a}}^u\mathfrak{a}^v)$ and $\hat{\mathfrak{a}}^u\mathfrak{a}^v\in(U'_K\cap B'_K)(U'_k\cap B'_k)\subset(U'_K\cap B'_K)$, which satisfies ($\Delta$).

4. Equivalent condition to (B).

Notation being as in the preceding section, let further ζ_{ℓ^m} be a primitive ℓ^m-th root of unity.

It follows from Theorem 6 and (3. 1) that Leopoldt's conjecture for ℓ is true for k, if

(B') $\qquad H^{-1}(G(K/k), U'_K B'_K/K^\times)=0$

for any finite Galois extension K over k contained in $k^{(\ell)}$.

The purpose of this section is to show (B') under the following assumption.

(#) $\qquad H^s(G(K/k), T'_K\cap B_K)=0 \qquad$ for all integers s.

PROPOSITION 11.

$$H^s(G, B'_K/K^\times)=0 \quad \textit{if and only if} \quad H^{s+1}(G, T'_K\cap B_K)=0.$$

Proof. Since $B'_K/B_K=T'_KB_K/B_K\simeq T'_K/(T'_K\cap B_K)$, we have an exact sequence $0\to B_K/K^\times\to T'_KB_K/K^\times\to T'_K/(T'_K\cap B_K)\to 0$. This implies $H^s(G, B'_K/K^\times)\simeq H^s(G, T'_K/(T'_K\cap B_K))$ for any integer s. Because $H^s(G, B_K/K^\times)=H^s(G, D_K)=0$ by (2. 1'). Moreover an exact sequence

$$0\to(T'_K\cap B_K)\to T'_K\to T'_K/(T'_K\cap B_K)\to 0$$

implies a long exact sequence

$$\cdots\to H^s(G, T'_K)\to H^s(G, T'_K/(T'_K\cap B_K))\to H^{s+1}(G, T'_K\cap B_K)\to H^{s+1}(G, T'_K)\to \cdots.$$

Hence the proposition follows from Proposition 9.

LEMMA 12. $B'^{\,G}_K=B'_K$, *if* $H^1(G, T'_K\cap B_K)=0$.

Proof. We have $(B'_K/K^\times)^G=N_{K/k}(B'_K/K^\times)=N_{K/k}(B_K/K^\times)\cdot N_{K/k}(T'_KK^\times/K^\times)$ by Proposition 11. Moreover $N_{K/k}(B_K/K^\times)=B_k/K^\times$ by Lemma 7, and $N_{K/k}(T'_K)=T'_k$ by Proposition 9. Thus the lemma follows.

PROPOSITION 13. *If the condition* (#) *is satisfied, then the condition* (B') *holds* :

$$H^{-1}(G, U'_KB'_K/K^\times)=0$$

Proof. Set $H^s(A)=H^s(G, A)$ for a G-module A and an integer s. An exact sequence $0\to U'_K\cap B'_K\to U'_K\to U'_K/(U'_K\cap B'_K)\simeq U'_KB'_K/B'_K\to 0$ implies the exact sequence

$$(4.\ 1)\qquad \cdots\to H^{-1}(U'_K)\to H^{-1}(U'_KB'_K/B'_K)\to H^0(U'_K\cap B'_K)\to\cdots.$$

A prime divisor $\mathfrak{P}$ of K is unramified in K/k if $\mathfrak{P}\nmid \ell$ by the assumption that $K\subset k^{(\ell)}$. Hence we have $H^{-1}(U'_K)=0$. We have further $H^0(U'_K\cap B'_K)=(U'_K\cap B'_K)^G/N_{K/k}(U'_K\cap B'_K)$ and $(U'_K\cap B'_K)^G=U'_k\cap B'_k$ by Lemma 12. Hence Proposition 10 implies $H^0(U'_K\cap B'_K)=0$ and further $H^{-1}(U'_KB'_K/B'_K)=0$ by (4. 1). On the other hand an exact sequence $0\to B'_K/K^\times\to U'_KB'_K/K^\times\to U'_KB'_K/B'_K\to 0$ implies

$$\cdots\to H^{-1}(B'_K/K^\times)\to H^{-1}(U'_KB'_K/K^\times)\to H^{-1}(U'_KB'_K/B'_K)\to\cdots$$

Then the Proposition follows from the above result and Proposition 11.

THEOREM 14. *Leopoldt's conjecture for k and ℓ is true, if the condition* (#) *is satisfied for any finite Galois extension* K over k contained in $k^{(\ell)}$.

Proof. It follows from Theorem 6 that Leopoldt's conjecture for ℓ is true for k if

$H^{-1}(G(K/k), U'_K B_K/K^\times)=0$, which is equivalent to $H^{-1}(G(K/k), U'_K B'_K/K^\times)=0$ by (3. 1). This holds by Proposition 13, if (#) is satisfied.

Denote by $\mu_\ell(k_\infty)$ the group of ideles of k whose finite components are all equal to *1* and infinite components are roots of unity of order ℓ-th power.

THEOREM 15. *Leopoldt's conjecture for ℓ is true for k, if and only if*

$$T'_k \cap B_k = \mu_\ell(k_\infty).$$

Proof. By the assumption (*), it is easy to see that $H^s(G(K/k), T'_K \cap B_K)=0$ for $s=1, 2$, if $T'_k \cap B_k = \mu_\ell(k_\infty)$. Hence the "if part" of the theorem is implied from Theorem 14 immediately. In order to prove "only if part", let $\varepsilon_1, \cdots, \varepsilon_r$ be a system of fundamental units of k, and assume $\mathfrak{a} \in T'_k \cap B_k$. Then $\mathfrak{a} = \alpha\varepsilon_1^{\lambda_1}\cdots\varepsilon_r^{\lambda_r}\phi_1(t_1)\cdots\phi_{r_2}(t_{r_2})$, where $\alpha \in k^\times$, $\lambda_i=(s_i, \mathfrak{x}_i)\in \mathbf{R}\times\bar{\mathbf{Z}}$, and other notation be as (3. 2). Moreover since $\mathfrak{a}\in T'_k$, we see that α is a global unit of k. Hence we can set

(4. 2) $\qquad \mathfrak{a}' = \varepsilon_1^{\lambda_1}\cdots\varepsilon_r^{\lambda_r}\phi_1(t_1)\cdots\phi_{r_2}(t_{r_2}).$

Let n be any natural number and p be a rational prime other than ℓ. Then there are rational integers y_i such that $y_i \equiv \mathfrak{x}_i$ mod. p^n $(i=1, \cdots, r)$. Put $\eta=\varepsilon_1^{y_1}\cdots\varepsilon_r^{y_r}$. Now observing finite parts of ideles, we have

(4. 3) $\qquad \eta = \bar{\mathfrak{a}}\,\varepsilon_1^{y_1-\mathfrak{x}_1}\cdots\varepsilon_r^{y_r-\mathfrak{x}_r} = \bar{\mathfrak{a}}\,(\varepsilon_1^{\mathfrak{y}_1}\cdots\varepsilon_r^{\mathfrak{y}_r})^{p^n} = (\bar{\mathfrak{b}}\,\varepsilon_1^{\mathfrak{y}_1}\cdots\varepsilon_r^{\mathfrak{y}_r})^{p^n},$

where $\bar{\mathfrak{a}}$ stands for the finite part of $\mathfrak{a}$, and $\bar{\mathfrak{b}}$ is an idele of k such that $\bar{\mathfrak{a}}=(\bar{\mathfrak{b}})^{p^n}$, whose existence is followed from $\mathfrak{a}\in T'_k$, and further $\mathfrak{y}_i$ is an element of $\bar{\mathbf{Z}}$ such that $p^n\mathfrak{y}_i=y_i-\mathfrak{x}_i$ for $i=1, \cdots, r$. Then there is further a unit η_1 of k such that $\eta=\eta_1^{p^n}$ (Cf. Iyanaga [7, Chap. 5, Theorem 6. 2]). Let $\eta_1=\varepsilon_1^{\nu_1}\cdots\varepsilon_r^{\nu_r}$, where $\nu_i \in \mathbf{Z}$. Then $\eta=\eta_1^{\nu_1 p^n}\cdots\eta_r^{\nu_r p^n}=\varepsilon_1^{y_1}\cdots\varepsilon_r^{y_r}$. Hence we have $y_i=\nu_i p^n$, and further $p^n \mid \mathfrak{x}_i$ for $i=1, \cdots, r$. This holds for any rational prime p other than ℓ and any natural number n. Therefore the p-part of $\mathfrak{x}_i$ is equal to 0 for any rational prime p other than ℓ, and hence $\mathfrak{x}_i$ can be considered as an element of $\mathbf{Z}_\ell$. By the assumption $\mathfrak{a}\in T'_k$, we have an ℓ-component of $\mathfrak{a}$ is equal to 1. Hence we have $1=\varepsilon_1^{\mathfrak{x}_1}\cdots\varepsilon_r^{\mathfrak{x}_r}$ by $\mathfrak{x}_i \in \mathbf{Z}_\ell$. Now if Leopoldt's conjecture is true for ℓ, then we have $\mathfrak{x}_i=0$ for $i=1, \cdots, r$. Then $\mathfrak{a}\in\mu_\ell(k_\infty)$, which is to be proved.

Now combining Theorem 4, 6, 14 and 15, we have

THEOREM 16. *Under the assumption* (*), *the following statements are equivalent.*

(a) *Leopoldt's conjecture for ℓ is true for k.*

(b) *$k^{(\ell)}$ is abundant for any Galois extension of finite degree over k contained in $k^{(\ell)}$.*

(c) $H^{-3}(G(k^{(\ell)}/k), \mathbf{Z})=0$.

(d) $H^{-1}(G(K/k), U'_K B_K/K^{\times})=0$ *for any Galois extension* K *over* k *of finite degree contained in* $k^{(\ell)}$.

(e) $H^s(G(K/k), T'_K \cap B_K)=0$ *for all integers s and for any Galois extension* K *over* k *of finite degree contained in* $k^{(\ell)}$.

(f) $T'_k \cap B_k = \mu_\ell(k_\infty)$.

As a case of the condition (f) to be satisfied, the following result is obtained in the same way as the proof of Theorem 15.

PROPOSITION 17. *The condition* (f) *is satisfied, if there is a natural number* N *such that* $k_n(\sqrt[\ell^n]{\varepsilon})$ *is ramified over* k_n *for any integer* $n>N$ *and for any unit* ε *of* k, *where* $k_n = k(\zeta_{\ell^n})$.

Proof. In order to prove the proposition, it is enough to show $T'_k \cap B_k = \mu_\ell(k_\infty)$. Let $\mathfrak{a} \in T'_k \cap B_k$. Then in the same way as to (4. 2) in the proof of Theorem 15, we have

$$\mathfrak{a} = \varepsilon_1^{\lambda_1} \cdots \varepsilon_r^{\lambda_r} \phi_1(t_1) \cdots \phi_{r_2}(t_{r_2}).$$

Let n be any natural number. Now we take a prime number ℓ instead of p in the proof of Theorem 15. Then there are rational integers y_i such that $y_i \equiv \mathfrak{x}_i$ mod. $\ell^n (i=1, \cdots, r)$. Put $\varepsilon = \varepsilon_1^{y_1} \cdots \varepsilon_r^{y_r}$. Now observing finite parts of ideles, we have

(4. 4) $$\varepsilon = \bar{\mathfrak{a}}\, \varepsilon_1^{y_1 - \mathfrak{x}_1} \cdots \varepsilon_r^{y_r - \mathfrak{x}_r} = \bar{\mathfrak{a}}\, (\varepsilon_1^{\mathfrak{y}_1} \cdots \varepsilon_r^{\mathfrak{y}_r})^{\ell^n},$$

where $\bar{\mathfrak{a}}$ stands for the finite part of $\mathfrak{a}$ and $\mathfrak{y}_i$ is an element of $\bar{\mathbf{Z}}$ such that $\ell^n \mathfrak{y}_i = y_i - \mathfrak{x}_i$ for $i=1, \cdots, r$. Let $L = k_n(\sqrt[\ell^n]{\varepsilon})$. Now by definition of T'_k, the $\mathfrak{l}$-components of $\bar{\mathfrak{a}}$ are equal to 1 for all $\mathfrak{l} \mid \ell$, and the other $\mathfrak{p}$-components are roots of unity of order ℓ-th power contained in $k_\mathfrak{p}$. Hence L is unramified over k_n. Therefore if $n>N$, then $\varepsilon \in k_n^{\ell^n}$ by assumption of the proposition, which implies further $\varepsilon \in k^{\ell^n}$ (Cf. Iyanaga [7, Chap. 5, the proof of Theorem 6. 2]). Then we have $\bar{\mathfrak{a}} \in U_k^{\ell^n} \cap T'_k$ by (4. 2) for all $n>N$. Thus we have $\bar{\mathfrak{a}} = 1$.

COROLLARY 18. *Leopoldt's conjecture for* ℓ *is true for* k, *when the class number of* $k(\zeta_{\ell^n})$ *is not divisible by* ℓ *for any sufficiently large* n.

REMARK. By using Iwasawa's version of Leopoldt's conjecture, the above corollary is proved simply as follows, which due to S. Shirai.

For any natural number n, take a natural number m which is sufficiently large and depending to n so that if $\varepsilon \in E_k$ and $\varepsilon \equiv 1$ mod. ℓ^m, then $k_n(\sqrt[\ell^n]{\varepsilon})$ is unramified over k_n. Then by the assumption of the corollary, we have $\varepsilon \in (k_n^\times)^{\ell^n}$, which implies further $\varepsilon \in$

$(k^{\times})^{\ell^n}$ (Cf. Iyanaga [7, Chap. 5, the proof of Theorem 6. 2]), hence $\varepsilon \in E_k^{\ell^n}$. Thus we have $\{\varepsilon \in E_k\ ;\ \varepsilon \equiv 1 \bmod. \ell^m\} \subset E_k^{\ell^n}$, which is Iwasawa's version of Leopoldt's conjecture.

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