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On Nilpotent Factors of Maximam Abelian Extensions of Algebraic Number Fields

Hiroshi YAMASHITA*

Department of Mathematics, Faculty of Science, Kanazawa University

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Abstract. For a finite Galois extension K of an algebraic number field k and an abelian extension L of K which is a Galois extension of k , let L_0 be the genus field of L with respect to K/k and L_n be the n -th central field of L with respect to K/k . Then the structure of $\text{Gal}(L_n/L_{n-1})$ is studied for $n \geq 1$.

1. Let k be an algebraic number field and K be a finite Galois extension of k with the Galois group G . Let L be an abelian extension of K which is also a Galois extension of k . Let H be the Galois group of L/K . Let $\sigma \in G$ and S_σ be any extension of σ to L . Then $S_\sigma^{-1} x S_\sigma$ does not depend on the choice of S_σ for any $x \in H$. Hence G acts on H by $\sigma x = S_\sigma^{-1} x S_\sigma$. Let I_G be the augmentation ideal of the group ring $\mathbf{Z}[G]$ over the ring of integers \mathbf{Z} . Let I_G^n be the product of n -copies of I_G . Then we have a sequence $H \supset I_G H \supset I_G^2 H \supset \dots$, and a sequence of the corresponding intermediate fields of L ; $K \subset L_1 \subset L_2 \subset \dots$. We call L_n the n -th central field of L with respect to K/k . Let L_0 be the genus field of L with respect to K/k . Then it is known that $\text{Gal}(L_1/L_0)$ is a homomorphic image of the Schur multiplier $M(G)$ of G , and that there exists a finite extension L of K such that $\text{Gal}(L_1/L_0) \cong M(G)$. This field L_1 is called an abundant central extension of K/k in [3]. Let $[G, G]$ be the commutator group of G and $G_0 = G/[G, G]$. We give an alternative proof of the existence of the abundant central extension, using the cohomology group of the maximam connected component of the idèle class group of K (Theorem 3). Secondly we prove that $\text{Gal}(L_{n+1}/L_n)$ is an infinite torsion group with the exponent dividing the order of G_0 when L is the maximam abelian extension of K and $G_0 \neq 1$ (Theorem 4).

2. Let C_K (resp. C_k) be the idèle class group of K (resp. k). Let $N_{K/k} : C_K \rightarrow C_k$ be the norm map and $H_{K/k}$ be its kernel. Let H_L be the closed subgroup of C_K corresponding to L .

*Present address: Kanazawa Women's Junior College.

PROPOSITION 1. $\text{Gal}(L/L_0) \cong H_{K/k}/H_L \cap H_{K/k}$.

PROOF. By class field theory, $\text{Gal}(L/L_0) \cong N_{K/k}^{-1}(N_{K/k}(H_L))/H_L \cong H_L \cdot H_{K/k}/H_L \cong H_{K/k}/H_L \cap H_{K/k}$.

COROLLARY 2. $\text{Gal}(L_1/L_0) \cong H_{K/k}/C_K^{lc} \cdot (H_{K/k} \cap H_L)$, where we denote by C_K^{lc} the group $\{xa \mid a \in C_K, x \in I_G\}$.

Let $M(G)$ be the Schur multiplier of G . Then $M(G) \cong H^{-3}(G, \mathbf{Z}) \cong H^{-1}(G, C_K)$. We prove the following theorem in other way of [3] or [4].

THEOREM 3. *There exists an abundant central extension of K/k .*

PROOF. Let \mathfrak{m} be a G -invariant divisor of K and $K_{\mathfrak{m}}$ be the ray class field mod \mathfrak{m} over K . We put $L=K_{\mathfrak{m}}$ and $H_{\mathfrak{m}}=H_{K_{\mathfrak{m}}}$. By Corollary 2 we have the sequence

$$1 \rightarrow C_K^{lc} \cdot (H_{\mathfrak{m}} \cap H_{K/k})/C_K^{lc} \rightarrow H_{K/k}/C_K^{lc} \rightarrow \text{Gal}(L_1/L_0) \rightarrow 1.$$

Let D_K be the maximum connected component of C_K . Then we have $D_K = \bigcap_{\mathfrak{m}} H_{\mathfrak{m}}$, where \mathfrak{m} runs through all of the G -invariant divisors of K . We note that C_K^{lc} is a compact subgroup of C_K and $H^{-1}(G, D_K) \cong H_{K/k} \cap D_K/D_K^{lc} = 1$. $H_{K/k}/C_K^{lc} \cong H^{-1}(G, C_K)$ is a discrete group, and $\bigcap (H_{K/k} \cap H_{\mathfrak{m}}) = H_{K/k} \cap D_K \subset C_K^{lc}$. Thus we have a G -invariant divisor \mathfrak{m} of K such that $H_{K/k} \cap H_{\mathfrak{m}} \subset C_K^{lc}$. Since $H_{K/k}/C_K^{lc} \cong M(G)$, we have Theorem 3.

3. Let $G_0^{(n)}$ be the tensor product $G_0 \otimes \cdots \otimes G_0$ of n -copies of G_0 . We have by [2] that $\text{Gal}(L_{n+1}/L_n)$ is a torsion group with the exponent dividing $\#G_0$. Moreover $\text{Gal}(L_{n+1}/L_n) = 1$ for all $n \geq 1$ if $G_0 = 1$.

THEOREM 4. *Let L be the maximum abelian extension of K .*

- (1) *If $G_0 = 1$, we have $\text{Gal}(L_{n+1}/L_n) = 1$ for all $n \geq 1$.*
- (2) *If $G_0 \neq 1$, we have $\text{Gal}(L_{n+1}/L_n)$ is an infinitely generated torsion groups with the exponent dividing $\#G_0$.*

To prove Theorem 4, we need some lemmas. Let K_1 be an intermediate Galois extension of K/k . Let M be an abelian extension of K_1 which is also a Galois extension of k . Let $L=MK$. We denote by M_n (resp. L_n) the n -th central field of M (resp. L) with respect to K_1/k (resp. K/k).

LEMMA 5. *$\text{Gal}(L_{n+1}/L_n)$ is finite iff $\text{Gal}(M_{n+1}/M_n)$ is so.*

PROOF. Let $K^* = K \cap M$. Let $G = \text{Gal}(K/k)$, $G_1 = \text{Gal}(K_1/k)$, $H = \text{Gal}(M/K_1)$, and $H' = \text{Gal}(M/K^*) \cong \text{Gal}(L/K)$. It is easily verified that $I_G^n \text{Gal}(L/K) \cong I_{G_2}^n H'$, where $G_2 = \text{Gal}(K^*/k)$. Since H is abelian and $\text{Gal}(K^*/K_1)$ is its factor group, $\text{Gal}(K^*/K_1)$ acts trivially on H' . We have $I_{G_2}^n H' = I_{G_1}^n H'$. We define the surjection $\varphi : I_{G_1}^n \otimes H/H' \rightarrow I_{G_1}^n H/I_{G_1}^n H'$ by $\varphi(x \pmod{H'}) = xa \pmod{I_{G_1}^n H'}$. Since H/H' is finite, so is $I_{G_1}^n H/I_{G_1}^n H'$. Hence $[I_{G_1}^n H : I_{G_1}^{n+1} H] = [I_{G_1}^n H : I_{G_1}^n H'] [I_{G_1}^n H' : I_{G_1}^{n+1} H'] / [I_{G_1}^{n+1} H : I_{G_1}^{n+1} H']$. We have Lemma 5.

By Lemma 5, it is sufficient to prove (2) of Theorem 4 when K is a cyclic extension over k of order p which is a prime factor of $\#G_0$. Let G be a cyclic group of order p . In the following, we denote by I_G the augmentation ideal of the group ring $\mathbf{Z}_p[G]$, where \mathbf{Z}_p is the ring of p -adic integers.

Let A be a finite p -abelian group such that $\mathbf{Z}_p[G]$ acts on it. We denote by I_G^n the product of n -copies of I_G . Let $I_G^n A = \{xa \mid x \in I_G^n, a \in A\}$. We define $\text{nil}(A)$ to be I_G^n for the smallest natural number n such that $I_G^n A = 0$. Let X_1, \dots, X_p be indeterminants. Let $S_n = \sum_{1 \leq i_1 < \dots < i_n \leq p} X_{i_1} \dots X_{i_n}$ for $1 \leq n \leq p$ and $S_0 = 1$. Let $S_n^{(k)} = \sum_{1 \leq i_1 < \dots < i_n \leq p} X_{i_1} \dots X_{i_n}$, where we chose i_1, \dots, i_n so that any of them are not equal to k . Let $S_0^{(k)} = 1$. Let $T_m = X_1^m + \dots + X_p^m$. Then $S_n = X_k S_{n+1}^{(k)} + S_n^{(k)}$. Hence we have $S_1 S_n = \sum_{k=1}^p X_k S_n = \sum X_k^2 S_{n-1}^{(k)} + \sum X_k S_n^{(k)} = \sum X_k^2 (S_{n-1} - X_k S_{n-2}^{(k)}) + \sum X_k S_n^{(k)} = T_2 S_{n-1} - \sum X_k^3 S_{n-2}^{(k)} + \sum X_k S_n^{(k)}$ when $p-1 \geq n \geq 2$. Moreover $\sum X_k S_n^{(k)} = (n+1) S_{n+1}$. Hence we have $S_1 S_n = \sum_{i=2}^{n+1} (-1)^i T_i S_{n+1-i} + (n+1) S_{n+1}$. We substitute σ^i for X_i in the following, where σ is a generator of G .

LEMMA 6. $I_G^p = pI_G$.

PROOF. Let X be an indeterminant. Let σ be a generator of G . Put $\prod_{i=1}^p (X - \sigma^i) = X^p - S_1 X^{p-1} + \dots + (-1)^{p-1} S_{p-1} X + (-1)^p S_p$ for $S_i \in \mathbf{Z}_p[G]$. Obviously $(\sigma - 1)S_1 = 0$. Let $T_i = \sigma^i + \sigma^{2i} + \dots + \sigma^{pi}$. Then we have $0 = (\sigma - 1) S_i S_n = (\sigma - 1) \sum_{i=2}^{n+1} (-1)^i T_i S_{n+1-i} + (\sigma - 1) (n+1) S_{n+1}$ for $1 \leq n \leq p-2$. Since $(\sigma - 1) T_i = 0$ for $2 \leq i \leq p-1$ and $(n+1) \in \mathbf{Z}_p$, we have $(\sigma - 1) S_{n+1} = 0$. Put $\prod_{i=1}^{p-1} (x - \sigma^i) = \underline{X}^{p-1} - S'_1 X^{p-2} + \dots + (-1)^{p-2} S'_{p-2} X + (-1)^{p-1} S'_{p-1}$. Since $S'_n = S_n - S'_{n-1}$ for $2 \leq n \leq p-1$, we have $(\sigma - 1) S'_n = -(\sigma - 1) S'_{n-1}$. Hence $(\sigma - 1) S'_n = (-1)^{n-1} (\sigma - 1) S'_1 = (-1)^n (\sigma - 1)$. We substitute 1 for X . Then $(\sigma - 1) \prod_{i=1}^{p-1} (1 - \sigma^i) = p(\sigma - 1)$. Since $(\sigma - 1) \prod (1 - \sigma^i)$ generates I_G^p , we have $I_G^p = pI_G$.

The following Lemma 7 is obvious.

LEMMA 7. Let A be a finite p -abian group such that $\mathbf{Z}_p[G]$ acts on it. Then we have

- (1) $\text{nil}(A) \supset I_G^n \Leftrightarrow I_G^n A = I_G^{n+1} A$
 (2) $A \neq 0 \Leftrightarrow A$ has a G -invariant element x which is not 0.

LEMMA 8. Put $A = (\mathbf{Z}/p^n\mathbf{Z})[G]$. Let m be an integer such that $p^n I_G = I_G^m$. Then we have $I_G^{m-1} A \neq 0$.

PROOF. Since $I_G A = I_G/p^n I_G$, we have $I_G^{m-1} A \cong I_G^{m-1}/I_G^m$. If $I_G^{m-1} A = 0$, then $I_G^{m-1} = I_G^m$ and moreover $I_G^m = I_G^{m+1} = \dots$. Then $I_G^m = p^k I_G$ for any $k \geq n$ by Lemma 6. Hence $I_G^m \subset \bigcap_{k=n}^{\infty} p^k I_G = 0$. This contradicts to the fact $I_G^m \neq 0$. Thus we have Lemma 8.

The following Lemma 9 is known in [5].

LEMMA 9. Let k be an algebraic number field. Let c be an integral divisor of k and n be a natural number. Then there exists an integral divisor \mathfrak{d} of k which is prime to c , such that

$$E_k(\mathfrak{d}) \supset E_k^n$$

where $E_k(\mathfrak{d})$ is the group of units x of k such that $x \equiv 1 \pmod{\mathfrak{d}}$.

4. Let ξ_n be a primitive p^n -th root of 1. Let n_1 be the natural number such that $\mathbf{Q}(\xi_{n_1}) \cap k \subsetneq \mathbf{Q}(\xi_n)$. Let $k_n = k(\xi_n)$. Let $n \geq n_1$ and σ be a generator of $\text{Gal}(k_{n+1}/k_n)$. Let \mathfrak{q} be a prime ideal of k which is completely decomposed in K/k , which is prime to p , and whose Artin symbol $(\frac{k_n/k}{\mathfrak{q}})$ equals σ . Let $\mathfrak{q}' = N_{k_n/k}(\mathfrak{q})$. Since the restriction of σ onto $\mathbf{Q}(\xi_{n+1})$ is a generator of $\text{Gal}(\mathbf{Q}(\xi_{n+1})/\mathbf{Q}(\xi_n))$, we have $\mathfrak{q}' \equiv 1 \pmod{p^n}$ but $\mathfrak{q}' \not\equiv 1 \pmod{p^{n+1}}$. Let \mathfrak{B} be the prime ideal of K such that $\mathfrak{B} | \mathfrak{q}$. Then the residue field of \mathfrak{B} contains the primitive p^n -th root of 1, but does not contain the primitive p^{n+1} -th root of 1. Let \mathfrak{L} be the set of all of such prime ideals \mathfrak{B} of K . This set is a infinite set. Let $\mathfrak{B}_1 \in \mathfrak{L}$ and $\mathfrak{p}_1 = N_{K/k} \mathfrak{B}_1$. We apply Lemma 9 for p^n and \mathfrak{p}_1 . Then we have an ideal \mathfrak{m}_1 such that $(\mathfrak{m}_1, \mathfrak{p}_1) = 1$ and $E_K(\mathfrak{m}_1) \subset E_K^{p^n}$. We can take \mathfrak{m}_1 to be G -invariant. Let K_1 be the ray class field mod $\mathfrak{m}_1 \mathfrak{p}_1$ of K , which is a Galois extension over k . Then we have $\text{Gal}(K_1/K) \cong J_K/K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1)$, where J_K is the idèle group of K and $U_K(\mathfrak{m}_1 \mathfrak{p}_1)$ is the group of unit idèles u such that $u \equiv 1 \pmod{\mathfrak{m}_1 \mathfrak{p}_1}$. Let ξ be the primitive p^n -th root of 1 contained in the completion of K at \mathfrak{B}_1 . Let $j_{\mathfrak{B}_1} : \mathfrak{B}_1 \rightarrow J_K$ be the canonical injection. We put $\mathfrak{a} = j_{\mathfrak{B}_1}(\xi)$. We denote by \mathfrak{B}_1^σ the conjugate ideal of \mathfrak{B}_1 with respect to $\sigma \in G$ and by \mathfrak{a}^σ the conjugate of \mathfrak{a} with respect to $\sigma \in G$. Let T be the subgroup of J_K generated by \mathfrak{a}^σ for $\sigma \in G$. Let $au \in K^\times \cdot U_K(\mathfrak{m}_1 \mathfrak{p}_1) \cap T$, where $a \in K^\times$ and $u \in U_K(\mathfrak{m}_1 \mathfrak{p}_1)$.

Then $au = \prod_{\sigma \in G} \mathfrak{a}^{\sigma \mathfrak{b} \sigma}$ for certain $\mathfrak{b}_\sigma \in \mathbf{Z}$. Since $\mathfrak{a} = u^{-1} \prod \mathfrak{a}^{\sigma \mathfrak{b} \sigma} \in U_K(\mathfrak{m}_1) \cap K^\times = E_K(\mathfrak{m}_1)$, we have

an element $a_1 \in E_K$ such that $a_1^{p^n} = a$. Since $a = a_1^{p^n} = \mathbf{a}^{\sigma b \sigma} \pmod{\mathfrak{P}_1^\sigma}$, we have $b_\sigma = 0$ for any $\sigma \in G$. Hence $K^\times \cdot U_K(\mathfrak{m}_1, \mathfrak{p}_1) \cap T = \{1\}$ and $T \cong T \cdot K^\times \cdot U_K(\mathfrak{m}_1, \mathfrak{p}_1) / K^\times \cdot U_K(\mathfrak{m}_1, \mathfrak{p}_1)$. Let M_1 be the subfield of K_1 corresponding to $T \cdot K^\times \cdot U_K(\mathfrak{m}_1, \mathfrak{p}_1) / K^\times \cdot U_K(\mathfrak{m}_1, \mathfrak{p}_1)$. Then $\text{Gal}(K_1/M_1) \cong T \cong (\mathbb{Z}/p^n \mathbb{Z})[G]$. Let \mathfrak{P}_2 be an element of L which is prime to $\mathfrak{m}_1, \mathfrak{p}_1$. We apply Lemma 9 for p^n and $\mathfrak{m}_1, \mathfrak{p}_1, \mathfrak{p}_2$, where $\mathfrak{p}_2 = N_{K/K} \mathfrak{P}_2$. Then we obtain a G -invariant divisor \mathfrak{m}_2 such that $(\mathfrak{m}_2, \mathfrak{m}_1, \mathfrak{p}_1, \mathfrak{p}_2) = 1$ and $E_K(\mathfrak{m}_2) \in E_K^{p^n}$. Let K_2 be the ray class field mod $\mathfrak{m}_2, \mathfrak{p}_2$ over K and M_2 be its subfield such that $\text{Gal}(K_2/M_2) = (\mathbb{Z}/p^n \mathbb{Z})[G]$. In this way, we have ray class fields K_i over K and its subfield M_i for $i=1, 2, 3, \dots$. Put $L_t = K_1 \cdots K_t$. Then L_t contains the Hilbert class field H over K . Let A be the p -Sylow subgroup of $\text{Gal}(L_t/K)$, A_1 be that of $\text{Gal}(L_t/H)$, and A_2 be that of $\text{Gal}(H/K)$. Let $\text{nil}(A_2) = I_G^1$. Then we have $I_G^1 A \subset A_1$, and further $I_G^l A \supset I_G^l A_1 \supset I_G^{l+j+1} A_1 \supset I_G^{l+2j+1} A$. Hence we have $[I_G^l A : I_G^{l+2j+1} A] \geq [I_G^l A_1 : I_G^{l+j+1} A_1]$.

Let B_i be the p -Sylow subgroup of $\text{Gal}(K_i/H)$. Then $A_1 \cong \prod_{i=1}^t B_i$. Let $p^n I_G = I_G^m$ by Lemma 6. Since $I_G^{m-1} \text{Gal}(K_i/M_i) \neq 0$ by Lemma 8, we have $I_G^{m-1} B_i \neq 0$. Since $I_G^{m-1} A_1 / I_G^m A_1 \cong$

$\prod_{i=1}^t I_G^{m-1} B_i / I_G^m B_i$ and $I_G^{m-1} B_i / I_G^m B_i \neq 0$ by Lemma 7, we have $[I_G^{m-1} A_1 : I_G^m A_1] \geq p^t$. Let $m > j+1$.

Let $l = m-1$. Then $[I_G^{m-1} A : I_G^{m+2j} A] \geq [I_G^{m-1} A_1 : I_G^m A_1] \geq p^t$ because $I_G^{m-1} A_1 \supset I_G^m A_1 \supset I_G^{m+2j} A_1$.

Let σ be a generator of G . Then we have the surjection $\varphi : I_G^k A / I_G^{k+1} A \rightarrow I_G^{k+1} A / I_G^{k+2} A$ by $\varphi(a \pmod{I_G^{k+1} A}) = (\sigma-1)a \pmod{I_G^{k+2} A}$ for $a \in A$. Hence $[I_G^k A : I_G^{k+1} A] \geq [I_G^{k+1} A : I_G^{k+2} A]$.

Thus $[I_G^{m-1} A : I_G^{m+2j} A] = \prod_{i=0}^{2j} [I_G^{m-1+i} A : I_G^{m+i} A] \leq [I_G^{m-1} A : I_G^m A]^{2j+1}$. We have $p^t \leq [I_G^{m-1} A : I_G^m A]^{2j+1}$. Moreover we have $[I_G^k A : I_G^{k+1} A] \geq [I_G^{m-1} A : I_G^m A] \geq p^{2j+1} \sqrt{p^t}$ for $1 \leq k \leq m-1$.

Let L be the maximal abelian extension of K and $H = \text{Gal}(L/K)$. We denote by J_G the augmentation ideal of $\mathbb{Z}[G]$. Let L_t be the maximam p -extension over K which is contained in L . Then $\text{Gal}(L_t/K) = A$. Let π be the restriction map of the Galois group from L onto L_t . Then we have $\pi(J_G^n H) = J_G^n A = I_G^n A$. Hence $[I_G^n A : I_G^{n+1} A] = [J_G^n H \cdot \text{Ker } \pi : J_G^{n+1} H \cdot \text{Ker } \pi] = [J_G^n H : J_G^{n+1} H (J_G^n H \cdot \text{Ker } \pi)] \leq [J_G^n H : J_G^{n+1} H]$. We conclude the proof of Theorem 4.

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