On a limit set of geometrically finite Kleinian group

Sci. Rep. Kanazawa Univ.
Vol. 27, No 2 pp. 85—116
December 1982

On the limit set of a geometrically finite Kleinian group

Tohru AKAZA and Katsumi INOUE

Department of Mathematics, Faculty of Science, Kanazawa University and Tôhoku University

(Received May 20, 1982)

Abstract. Let G^* be a finitely generated Kleinian group with a fundamental domain bounded by a finite number of circles which are mutually disjoint or tangent externally to each other. Then it holds $d(\Lambda(G^*))=P(G^*)/2$ between the Hausdorff dimension $d(\Lambda(G^*))$ of the limit set $\Lambda(G^*)$ and the Poincaré dimension $P(G^*)$ for G^*.

Introduction. Let G be a finitely generated and geometrically finite Kleinian group. Let

$$S_i(z)=(a_iz+b_i)/(c_iz+d_i),\ a_id_i-b_ic_i=1,\ (i=0,\ 1,\ 2,\ \ldots)$$

be any element of G, where $S_0(z)$ is the identity transformation. The Poincaré dimension $P(G)$ and the Hausdorff dimension $d(\Lambda(G))$ of the limit set $\Lambda(G)$ for G are defined as follows, respectively:

$$P(G) = \inf \{t\ ;\ \sum_{S_i \in G} |\ c_i\ |^{-t} < +\infty\}$$

and

$$d(\Lambda(G))=\inf \{d\ ;\ M_d\,(\Lambda(G))=0\}\ ,$$

where $M_d(\Lambda(G))$ denotes the d-dimensional Hausdorff measure of $\Lambda(G)$.

Suppose that G is a Schottky group. Then the former author proved the following relation ([2]):

$$(*) \qquad d(\Lambda(G)) = \frac{P(G)}{2}.$$

If G is a Fuchsian group of the first kind, the above (*) is trivial. It is proved by Patterson ([7]) that (*) holds for a Fuchsian group of the second kind without parabolic elements and for one with parabolic elements in the case $d(\Lambda(G))\geqq\frac{2}{3}$ and the problem whether or not (*) holds for $\frac{1}{2}<d(\Lambda(G))<\frac{2}{3}$ is presented by him. D. Sullivan ([9]) solved this problem positively by using the method of space group and further showed that (*) holds for the group with $G(\delta)$ finite volume.

Let us denote by G^* a finitely generated Kleinian group with a fundamental domain bounded by a finite number of circles which are mutually disjoint or tangent externally to each other. The purpose of this paper is to show that (*) is valid for G^*

We are interested in the relation between G^* and the group with $G(\delta)$ finite volume in the sense of Sullivan and have nothing yet.

It is easily seen that G^* is a free group. It seems still open that (*) holds for more general finitely generated and geometrically finite free groups.

We shall state preliminaries and notations about our Kleinian groups in §1. We shall give the relation between the Hausdorff measure and the measure defined by the special covering formed by the isometric circles for the limit set of our group in §2. We shall prove the main theorem giving the relation between the computing function and the Hausdorff measure of the limit set of our group in §3. At last in §4 we shall give the relation (*) between the Poincaré dimension and the Hausdorff dimension of the limit set for our Kleinian group G^* by using the main theorem.

We shall state our hearty thanks to Prof. T. Kuroda for his valuable and suitable advices given to this paper.

§1. Geometrically finite groups.

1. Let Γ be a group of automorphisms of the extended complex plane $\hat{C}$. A point $z_0 \in \hat{C}$ is called a limit point of Γ, if there are a sequence of distinct elements $\{\gamma_n\} \subset \Gamma$ and a point $z \in \hat{C}$ such that $\gamma_n(z) \to z_0$ as $n \to +\infty$. A set of all limit points of Γ is called the limit set of Γ and is denoted by $\Lambda(\Gamma)$. If z is not a limit point, we say it is an ordinary point of Γ, and we denote the set of all ordinary points by $\Omega(\Gamma)$ and call it the region of discontinuity of Γ. It is obvious $\Omega(\Gamma) = \hat{C} - \Lambda(\Gamma)$.

2. Let us denote by $c_1, \cdots, c_p$ $(p \geqq 2)$ the circles in the complex plane C such that Int $(c_i) \subset$ Ext (c_j) for $i, j = 1, \cdots, p$ $(i \neq j)$.

We shall consider the group generated by the inversions γ_i $(i = 1, \cdots, p)$ with respect to the circles c_i $(i = 1, \cdots, p)$ and denote it by $G = <\gamma_1, \cdots, \gamma_p>$.

We put $B_i = \mathrm{Cl}(\mathrm{Int}(c_i))$ $(i = 1, \cdots, p)$, where Cl (A) denotes the closure of A and

$$G_n = \{\gamma_{\nu_n} \circ \cdots \circ \gamma_{\nu_1} \mid \gamma_i = 1, \cdots, p \ (i = 1, \cdots, n), \ \nu_j \neq \nu_{j+1} \ (j = 1, \cdots, n-1)\}$$

for any positive integer n.

If $n \geqq 2$, we shall define $B_n(\gamma) = B_n(\gamma_{\nu_n} \circ \cdots \circ \gamma_{\nu_1}) = (\gamma_{\nu_n} \circ \cdots \circ \gamma_{\nu_2})(B_{\nu_1})$ for any $\gamma = \gamma_{\nu_n} \circ \cdots \circ \gamma_{\nu_1} \in G_n$. In particular, if $n = 1$, we put $B_1(\gamma) = B_{\gamma_1}$ for any $\gamma = \gamma_{\nu_1} \in G_1$.

Now put $L_n = \bigcup_{\gamma \in G_n} B_n(\gamma)$ for $n \in N$. Then it is obvious $L_n \supset L_{n+1}$ and $\bigcap_{n=1}^{\infty} L_n \neq \phi$. We shall prove the following proposition due to [6], [8].

PROPOSITION 1. *The group G defined in the above has the following properties:*
(i) *G is a discontinuous inversion group with $\Lambda(G)=\cap_{n=1}^{\infty} L_n$.*
(ii) *G has the defining relation $\gamma_1^2=\cdots=\gamma_p^2=id$.*

Proof. Take any point $z\in\cap_{n=1}^{\infty}L_n$. Since $z\in L_n$ for any n, there exist a sequence of distinct elements $\{S_n\}\subset G$ and a set of closed discs $\{B_{k(n)} \mid 1\leqq k(n)\leqq p,\ n=1, 2, \cdots\}$ such that $S_n(B_{k(n)})\ni z$ for any n. Hence there exist a subsequence $\{S_{n_i}\}$ ($\subset \{S_n\}$) and B_k ($1\leqq k\leqq p$) such that $S_{(n_i)}(B_k)\ni z$ for all i. This means that $z\in\Lambda(G)$ and so $\Lambda(G)\supset\cap_{n=1}^{\infty}L_n$.

To obtain the converse inclusion, it suffices to show $\Lambda(G)\subset L_n$ for any n. If we take any point $z\in\Lambda(G)$, then there exist a sequence of distinct elements $\{S_n\}$ of G and some point $x\in\hat{C}$ such that $S_n(x)\to z$ as $n\to\infty$.

Assume that $z\notin L_1$. Then it is obvious that $z\in\cap_{i=1}^{p}\mathrm{Ext}\,(c_i)$. Since $\cap_{i=1}^{p}\mathrm{Ext}\,(c_i)$ is open, there exists a positive integer n_0 such that $S_n(x)\in\cap_{i=1}^{p}\mathrm{Ext}\,(c_i)$ for all $n\geqq n_0$. This is a contradicicition. Hence we get $\Lambda(G)\subset L_1$. We shall use the induction.

Assume that $\Lambda(G)\subset L_n$. By the invariance of $\Lambda(G)$ under the action of G, we have $\Lambda(G)=\cap_{i=1}^{p}\gamma_i(\Lambda(G))\subset\cap_{i=1}^{p}\gamma_i(L_n)=L_{n+1}$. Hence we obtain $\Lambda(G)\subset\cap_{n=1}^{\infty}L_n$.

Noting $\cap_{n=1}^{\infty}L_n \subsetneqq \hat{C}$, we can easily see $\Omega(G)=\hat{C}-\Lambda(G)\neq\phi$. This means that G is discontinuous.

(ii) For any fixed element $S\in G$, we put $S=\gamma_{\nu_n}\circ\cdots\circ\gamma_{\nu_1}$, where $\nu_i=1, \cdots, p$ ($i=1, \cdots, n$), $\nu_j\neq\nu_{j+1}$ ($j=1, \cdots, n-1$) and $n\geqq 1$. Taking a fixed point $z\in\cap_{i=1}^{p}\mathrm{Ext}\,(B_i)$, we find that $\gamma_{\nu_1}(z)\in\mathrm{Int}(B_{\nu_1})$ and so $\gamma_{\nu_1}(z)\notin B_{\nu_2}$. From the form of S, we have $(\gamma_{\nu_2}\circ\gamma_{\nu_1})(z)\in\mathrm{Int}\,(B_{\nu_2})$ and so $(\gamma_{\nu_2}\circ\gamma_{\nu_1})(z)\notin B_{\nu_3}$. Repeating these procedures, we obtain $S(z)=(\gamma_{\nu_n}\circ\cdots\circ\gamma_{\nu_1})(z)\in B_{\nu_n}$. Hence we have $\gamma(z)\neq z$ and $\gamma\neq id$. This completes the proof of (ii).

q. e. d.

3. Now let us put $T_i=\gamma_p\circ\gamma_i$ for $i=1, \cdots, p-1$. Denote by G^* the group generated by $\{T_1, \cdots, T_{p-1}\}$. Then we can easily see that G^* consists of linear transformations in G. We shall put $\mathscr{G}=\{T_1, T_1^{-1}, \cdots, T_{p-1}, T_{p-1}^{-1}\}$ and call it the generator system of G^*.

At first we shall give the following theorem.

THEOREM 1. *G^* is a geometrically finite free Kleinian group with $\Lambda(G^*)=\Lambda(G)$.*

To prove this theorem, we shall need two lemmas.

LEMMA 1. *G^* is the maximal subgroup of G of linear transformations in G.*

Proof. To prove this lemma, we shall introduce a new set $G'=\{\gamma_{\nu_{2n}}\circ\cdots\circ\gamma_{\nu_1} \mid \nu_i=1, \cdots, p\ (i=1, \cdots, 2n),\ \nu_j\neq\nu_{j+1}\ (j=1, \cdots, 2n-1) \text{ and } n=0, 1, 2, \cdots\}$. Since G^* is the maximal subgroup of G of linear transformations, it suffices to show $G^*=G'$.

Since any element $S\in G^*$ can be written in the form $S=T_{\nu_k}\circ\cdots\circ T_{\nu_1}$, we have $G^*\subset G'$, where $T_{\nu_i}=\gamma_p\circ\gamma_{\nu_i}\in G'$ ($\nu_i=1, \cdots, p-1$; $i=1, \cdots, k$). To obtain the converse inclusion, take any element $S=\gamma_{\nu_{2n}}\circ\cdots\circ\gamma_{\nu_1}\in G'$. If $n=1$, then $S=\gamma_{\nu_2}\circ\gamma_{\nu_1}=(\gamma_p\circ\gamma_{\nu_2})^{-1}\circ(\gamma_p\circ\gamma_{\nu_1})=T_{\nu_2}^{-1}\circ T_{\nu_1}\in G^*$. Assume $\gamma_{\nu_{2n-2}}\circ\gamma_{\nu_{2n-3}}\circ\cdots\circ\gamma_{\nu_1}\in G^*$. Then we can see $S=\gamma_{\nu_{2n}}\circ\gamma_{\nu_{2n-1}}\circ(\gamma_{\nu_{2n-2}}\circ\cdots\circ\gamma_{\nu_1})\in(\gamma_{\nu_{2n}}\circ\gamma_{\nu_{2n-1}})G^*=(\gamma_p\circ\gamma_{\nu_{2n}})^{-1}\circ(\gamma_p\circ\gamma_{2n-1})G^*=T_{\nu_{2n}}^{-1}\circ T_{\nu_{2n-1}}G^*=G^*$. Hence $G^*\supset G'$ and our lemma is established.

q.e.d.

LEMMA 2. *It holds* $G^*\lhd G$ *and* $[G:G^*]=2$.

Proof. First we shall prove the former assertion. Take any elements S and T from G and $G^*=G'$, respectively. Then $S\circ T\circ S^{-1}$ is a word consisting of an even number of elements, that is, $2n$ elements of $\mathscr{G}$. From (ii) of Proposition 1, $S\circ T\circ S^{-1}$ can be written as a reduced word of an even number of elements, that is, $2m$ elements of $\mathscr{G}$ $(m\leqq n)$. Hence we have $S\circ T\circ S^{-1}\in G'=G^*$ and so $SG^*S^{-1}\subset G^*$ for any $S\in G$.

Let us show the remainder. Take any element $S=\gamma_{\nu_n}\circ\cdots\circ\gamma_{\nu_1}\in G$, where $\nu_i=1,\cdots,p$ $(i=1,\cdots,n)$ and $\nu_j\neq\nu_{j+1}$ $(j=1,\cdots,n-1)$. If n is even, then $S\in G'=G^*$. Assume that n is odd. If we put $n=2k+1$, then we have $S=\gamma_{2k+1}\circ(\gamma_{\nu_{2k}}\circ\cdots\circ\gamma_{\nu_1})=\gamma_p\circ(\gamma_p\circ\gamma_{\nu_{2k+1}})\circ(\gamma_{\nu_{2k}}\circ\cdots\circ\gamma_{\nu_1})\in\gamma_pG'=\gamma_pG^*$. Hence we have $G=G^*\cup\gamma_pG^*$ and so $[G:G^*]=2$.

q.e.d.

Now we can give the proof of Theorem 1.

Proof. From Proposition 1 we can easily see that G^* is a Kleinian group. First we shall show that G^* is geometrically finite. For this purpose let us consider the action of G^* on the upper half-space, $H^3=\{(z,t)\mid z\in C,\ t>0\}$. Let H_i be a sphere in E^3, the Eucliden 3-space, which is orthogonal to C with $H_i\cap C=c_i$ $(i=1,\cdots,p)$. We put $P=H^3\cap\{\cap_{i=1}^{p}\mathrm{Ext}(H_i)\}$. Then P is a convex finite-sided fundamental polyhedron for G in H^3. Hence $P^*=\mathrm{Int}\ \{P\cup\mathrm{Cl}(\gamma_p(P))\}$ is a convex fundamental polyhedron for G^* in H^3. Since P is finite sided, so is P^*. Therefore G^* is geometrically finite.

Next we shall prove that G^* is free.

Let $S\in G^*$ be of the form $S=T_{\nu_n}^{\varepsilon_n}\circ\cdots\circ T_{\nu_1}^{\varepsilon_1}$, where $\nu_i=1,\cdots,p-1$, $\varepsilon_i=1$ or -1 $(i=1,\cdots,n)$ and $T_{\nu_j}^{\varepsilon_j}\neq T_{\nu_{j+1}}^{-\varepsilon_{j+1}}$ $(j=1,\cdots,n-1)$. First we consider the case of $n=2$, that is, $S=T_{\nu_2}^{\varepsilon_2}\circ T_{\nu_1}^{\varepsilon_1}$. It can be represented by $\gamma_1,\cdots,\gamma_p$ as the below table.

ε_2 \ ε_1	1	−1
1	$\gamma_p\circ\gamma_{\nu_2}\circ\gamma_p\circ\gamma_{\nu_1}$	$\gamma_p\circ\gamma_{\nu_2}\circ\gamma_{\nu_1}\circ\gamma_p$
−1	$\gamma_{\nu_2}\circ\gamma_{\nu_1}$	$\gamma_{\nu_2}\circ\gamma_p\circ\gamma_{\nu_1}\circ\gamma_p$

In any case $T_{\nu_2}^{\varepsilon_2}\circ T_{\nu_1}^{\varepsilon_1}$ can be represented as a reduced word of $\gamma_1,\cdots,\gamma_p$ whose length is at least two. By the similar argument to the above, we can form the similar table and find that $T_{\nu_3}^{\varepsilon_3}\circ T_{\nu_2}^{\varepsilon_2}\circ T_{\nu_1}^{\varepsilon_1}$ is represented as a reduced word whose length is at least four. By repeating these procedure, we see that S is not an identical transformation. Hence G^* is free.

Finally we shall show that $\Lambda(G^*)=\Lambda(G)$. Since $\Lambda(G^*)\subset\Lambda(G)$, it suffices to show the converse inclusion. Take any point $z\in\Lambda(G)$. Then there exist a sequence of distinct elements $\{S_n\}\subset G$ and $x\in\hat{C}$ such that $S_n(x)\to z$ as $n\to+\infty$. We can easily see from Lemma 2 that there exist an element $U_n\in G^*$ and $\delta_n=0$ or 1 for any element S_n such that $S_n=\gamma_p^{\delta_n}\circ U_n$. If $\delta_n=0$ for infinitely many δ_n, then $z\in\Lambda(G^*)$. Assume that there exists a positive integer n_0 such that $\delta_n=1$ for all $n\geqq n_0$. Noting $S_n(x)=(\gamma_p\circ U_n)(x)=(\gamma_p\circ U_n\circ\gamma_p)(\gamma_p(x))$ and $\gamma_p\circ U_n\circ\gamma_p\in G^*$ from Lemma 2, we can easily see that $z\in\Lambda(G^*)$. Hence we complete the proof of Theorem 1.

§2. Isometric circles and a covering of $\Lambda(G^*)$.

1. Since G^* is free, there exist $T_{\nu_1},\cdots,T_{\nu_n}\in\mathcal{G}$ for any $S\in G^*$ such that S can be represented uniquely in the form $S=T_{\nu_n}\circ\cdots\circ T_{\nu_1}$. So we shall call the number n the grade of S and use the notation $S_{(n)}$ to clarify the grade n of S.

Let $S, T\in G^*-\{id\}$ be two arbitrary elements with $S\circ T\neq id$. Denote by I_S, I_T and $I_{S\circ T}$ the isometric circles of S, T and $S\circ T$, respectively. Let R_S, R_T and $R_{S\circ T}$ be the radii of I_S, I_T and $I_{S\circ T}$, respectively. The following equalities are well known (see [5]):

$$(2.1)\qquad R_{S\circ T}=\frac{R_S\cdot R_T}{|T(\infty)-S^{-1}(\infty)|}$$

$$(2.2)\qquad |(S\circ T)^{-1}(\infty)-T^{-1}(\infty)|=\frac{R_{S\circ T}\cdot R_T}{R_S}=\frac{R_T^2}{|T(\infty)-S^{-1}(\infty)|}.$$

2. The purpose of §2 is to give the relation between the Hausdorff measure and the measure defined by the special covering formed by the isometric circles of elements of G^* for $\Lambda(G^*)$. We shall arrange the results as Theorem 2 on the end of §2. For this purpose we shall give some lemmas.

First of all we shall prove the following lemma.

LEMMA 3. *Let $\{S_{(n)}\}$ be a sequence of G^* satisfying $S_{(n)}=T_{\nu_n}\circ\cdots\circ T_{\nu_1}$ $(T_{\nu_1},\cdots,T_{\nu_n}\in\mathcal{G})$ and $S_{(n+1)}=T_{\nu_{n+1}}\circ S_{(n)}$ for all $n\in N$. Then there exist two positive constants $k_0=k_0(G^*)<1$ and $k_1=k_1(G^*)$ depending only on G^* such that*

$$(2.3)\qquad k_0\leqq\frac{R^2_{S_{(n+1)}}}{R^2_{S_{(n)}}}\leqq k_1$$

for all n.

Proof. Since

$$R_{S_{(n+1)}}=R_{T_{\nu_{n+1}}\circ S_{(n)}}=\frac{R_{T_{\nu_{n+1}}}\cdot R_{S_{(n)}}}{|S_{(n)}(\infty)-T^{-1}_{\nu_{n+1}}(\infty)|}$$

from (2.1), we have

$$(2.4)\qquad \frac{R^2_{S_{(n+1)}}}{R^2_{S_{(n)}}}=\frac{R^2_{T_{\nu_{n+1}}}}{|S_{(n)}(\infty)-T^{-1}_{\nu_{n+1}}(\infty)|^2}.$$

Put $\min_{T\in\mathcal{G}} R_T=\sigma$, $\sup_{z,\zeta\in \bigcup_{S\in G^*-\{id\}} I_S} |z-\zeta|=\rho$ and

$\inf_{T\in\mathcal{G}, S\in G^*-\{id,T\}} |T^{-1}(\infty)-S^{-1}(\infty)|=\delta$. Clearly σ, ρ and δ are positive constants depending only on G^* and $\rho>\sigma$. From the definitions of these constants, we have immediately

$|S_{(n)}(\infty)-T^{-1}_{\nu_{n+1}}(\infty)|\geqq\delta$ and $R^2_{T_{\nu_{n+1}}}\leqq\rho^2$ for all n.

The right hand side of (2.3) is easily gotten by putting $\rho^2\delta^{-2}=k_1$. Similarily we have $R^2_{T_{\nu_{n+1}}}\geqq\sigma^2$ and $|S_{(n)}(\infty)-T^{-1}_{\nu_{n+1}}(\infty)|^2\leqq\rho^2$ for all n. Putting $\sigma^2\rho^{-2}=k_0$, we have the left hand side of (2.3).

q.e.d.

3. A.F. Beardon and B. Maskit discussed the necessary and sufficient condition for a Kleinian group to be geometrically finite ([4]). Here we shall state a part of their results, which we need later, briefly.

Let Γ be a Kleinian group. We assume $\infty\in\Omega(\Gamma)$. A point $z\in C$ is called a point of approximation of Γ if there exist $\{g_n\}\subset\Gamma$, $k=k(z)>0$ and $k'=k'(\Gamma)>0$ such that

$$(2.5)\qquad k'R^2_{g_n}<|z-g^{-1}_{g_n}(\infty)|<kR^2_{g_n},$$

where k (resp. k') is a positive constant depending on z (resp. Γ). Another condition, which is equivalent to (2.5), for $z\in C$ to be a point of approximation is the following: there exist $\{g\ \}\subset\Gamma$, $k''=k''(z)>0$ and $x\in\Omega(\Gamma)$ such that

$$(2.6)\qquad |g_n(z)-g_n(x)|>k'',$$

here k'' is a constant depending only on z.
We denote the set of all points of approximation of Γ by $\Lambda_a(\Gamma)$.

Let z be a parabolic fixed point of Γ. Denote the stabilizer of z by Γ_z. A set U with the following properties is called a cuped region for z: (i) U is a union of two disjoint, non-empty half-planes (or discs). (ii) $g(U)=U$ for all $g\in\Gamma_z$ and $g(U)\cap U=\phi$ for all $g\in\Gamma-\Gamma_z$. We say that z is a cusped parabolic fixed point of Γ if either z has a cusped region U or Γ_z has a free abelian subgroup of rank two.

The following two propositions are due to Beardon and Maskit ([4]).

PROPOSITION 2. *A Kleinian group is geometrically finite if and only if every limit point of Γ is a point of approximation or is a cusped parabolic fixed point.*

PROPOSITION 3. *Let* Γ *be a geometrically finite Kleinian group. Then, for every* $z\in\Lambda_a(\Gamma)$, there exist $\{g_n\}\subset\Gamma$ *and* $\tilde{k}=\tilde{k}(\Gamma)$ (>0) *depending on* Γ *such that*

$$|z-g_n^{-1}(\infty)|<\tilde{k}R_{g_n}^2.$$

We find from Theorem 1 that G^* is a geometrically finite Kleinian group. Noting (2.5) and Propsition 3, we see easily that, for any point $z\in\Lambda_a(G^*)$, there exist $\{S_n\}\subset G^*$, $k_2=k_2(G^*)>0$ and $k_3(G^*)>0$ depending on G^* such that

$$\text{(2.7)} \qquad k_2R_{S_n}^2<|z-S_n^{-1}(\infty)|<k_3R_{S_n}^2.$$

4. Now let us give an important property of points of approximation for G^* by using Lemma 3.

LEMMA 4. *For any* $z\in\Lambda_a(G^*)$, *there exist* $\{S_{(n)}\}\subset G^*$ *and* $K=K(G^*)$ (>0) *depending on* G^* such that

$$\text{(2.8)} \qquad |z-S_{(n)}^{-1}(\infty)|<KR_{S_{(n)}}^2.$$

Proof. From (2.7) of Proposition 3, for any $z\in\Lambda_a(G^*)$, there exists a sequence of distinct elements $\{S_{(n_i)}\}\subset G^*$ such that $|z-S_{(n_i)}^{-1}(\infty)|<k_3R_{S_{(n_i)}}^2$ and $S_{(n_i)}=U_{(m_i)}\circ S_{(n_i)}$ $(m_i\geqq 1)$.

Assume that (2.8) is not true for some $z\in\Lambda_a(G^*)$. Then, for any fixed $m>0$ which is sufficiently large, there exist a sequence of distinct elements $\{S_{(n)}\}\subset G^*$ which contains $\{S_{(n_i)}\}$ and a subsequence $\{S_{(n_i')}\}\subset\{S_{(n)}\}$ such that

$$\text{(2.9)} \qquad |z-S_{(n_i')}^{-1}(\infty)|<mk_1R_{S_{(n_i')}}^2$$

$$\text{(2.10)} \qquad |z-S_{(n_i'+1)}^{-1}(\infty)|>mR_{S_{(n_i'+1)}}^2$$

Put

$$\text{(2.11)} \qquad S_{(n_i'+1)}=T_{\mu_{i+1}'}\circ S_{(n_i')} \quad (T_{\mu_{i+1}'}\in\mathcal{G}).$$

For any $S\in G^*-\{id\}$ and any fixed $x_0\in\Omega(G^*)-\{T^{-1}(\infty)\mid T\in G^*\}$, the following equality holds (see [3], p. 3):

$$\text{(2.12)} \qquad |S(z)-S(x_0)|=\frac{R_S^2|z-x_0|}{|z-S^{-1}(\infty)|\,|x_0-S^{-1}(\infty)|}.$$

We put

$$d_1=\min_{\xi\in\Lambda(G^*)}|\xi-x_0|,\qquad d_2=\max_{\xi\in\Lambda(G^*)}|\xi-x_0|,$$

$$\rho_1=\inf_{S\in G^*-\{id\}}|x_0-S^{-1}(\infty)|,\qquad \rho_2=\sup_{S\in G^*-\{id\}}|x_0-S^{-1}(\infty)|,$$

$$\rho_3=\inf_{T\in\mathscr{G},U_{(n)}\in G^*,n\geqq 2}|U_{(n)}^{-1}(\infty)-T^{-1}(\infty)|.$$

Obviously these constants are positive and depend on x_0 and G^*. We have from (2.2), (2.9), (2.11) and (2.12)

$$(2.13)\qquad |S_{(n_i'+1)}^{-1}(\infty)-S_{(n_i')}^{-1}(\infty)|=\frac{R_{S_{(n_i')}}^2}{|S_{(n_i')}(\infty)-T_{\mu'_{i+1}}(\infty)|}\leqq\frac{1}{\rho_3}R_{S_{(n_i')}}^2$$

$$(2.14)\qquad |S_{(n_i')}(z)-S_{(n_i')}(x_0)|=\frac{R_{S_{(n_i)}}^2|z-x_0|}{|z-S_{(n_i')}^{-1}(\infty)|\;|x_0-S_{(n_i')}^{-1}(\infty)|}\geqq\frac{d_1}{mk_1\rho_2}.$$

Combining (2.3) of Lemma 3, (2.7) of Proposition 3, (2.12) and (2.13), we have

$$\frac{|S_{(n_i')}(z)-S_{(n_i')}(x_0)|}{|S_{(n_i'+1)}(z)-S_{(n_i'+1)}(x_0)|}=\frac{R_{S_{(n_i')}}^2}{R_{S_{(n_i'+1)}}^2}\cdot\frac{|z-S_{(n_i'+1)}^{-1}(\infty)|}{|z-S_{(n_i')}^{-1}(\infty)|}\cdot\frac{|x_0-S_{(n_i'+1)}^{-1}(\infty)|}{|x_0-S_{(n_i')}^{-1}(\infty)|}$$

$$\leqq\frac{1}{k_0}\cdot\frac{\rho_2}{\rho_1}\cdot\frac{|z-S_{(n_i'+1)}^{-1}(\infty)|}{|z-S_{(n_i')}^{-1}(\infty)|}\leqq\frac{1}{k_0}\cdot\frac{\rho_2}{\rho_1}\cdot\frac{|z-S_{(n_i')}^{-1}(\infty)|+|S_{(n_i')}^{-1}(\infty)-S_{(n_i'+1)}^{-1}(\infty)|}{|z-S_{(n_i')}^{-1}(\infty)|}$$

$$\leqq\frac{1}{k_0}\cdot\frac{\rho_2}{\rho_1}\cdot\Bigl(1+\frac{\frac{1}{\rho_3}R_{S_{(n_i')}}^2}{k_2R_{S_{(n_i')}}^2}\Bigr)=\frac{1}{k_0}\cdot\frac{\rho_2}{\rho_1}\Bigl(1+\frac{1}{k_2\rho_3}\Bigr).$$

This inequality and (2.14) yield

$$(2.15)\qquad |S_{(n_i'+1)}^{-1}(z)-S_{(n_i'+1)}^{-1}(x_0)|\geqq\frac{d_1k_0k_2\rho_1\rho_3}{mk_1\rho_2^2(1+k_2\rho_3)}.$$

Hence we have from (2.12) and (2.15)

$$|z-S_{(n_i'+1)}^{-1}(\infty)|=\frac{R_{S_{(n_i'+1)}}^2|z-x_0|}{|S_{(n_i'+1)}(z)-S_{(n_i'+1)}(x_0)|\;|x_0-S_{(n_i'+1)}^{-1}(\infty)|}$$

(2.16)

$$\leqq\frac{mk_1\rho_2^2(1+k_2\rho_3)}{d_1k_0k_2\rho_1\rho_3}\cdot\frac{d_2}{\rho_1}\cdot R_{S_{(n_i'+1)}}^2.$$

Putting $m'=\dfrac{d_2k_1\rho_2^2(1+k_2\rho_3)}{d_1k_0k_2\rho_1^2\rho_3}>0$, we have from (2.16)

$$|z-S_{(n_i'+1)}^{-1}(\infty)|\leqq mm'R_{S_{(n_i'+1)}}^2.$$

This contradicts the assumption that (2.8) is not true. Thus we could prove Lemma 4.

q.e.d.

5. For any $\delta>0$, we denote by $I(\delta)$ a family of a countable number of closed discs $\{D_\lambda\}$ of radii $l_\lambda\leqq\delta$ such that every point of $\Lambda(G^*)$ is contained in some $\mathrm{Int}(D_\lambda)$. For any $S_{(n)}\epsilon G^*-\{id\}$, we put

$$B_{S_{(n)}}^{(2)}=\{z\,|\,|z-S_{(n)}^{-1}(\infty)|\leqq k^*R_{S_{(n)}}^2\},$$

where $k^*=k^*(G^*)$ is a constant K in Lemma 4 depending on G^*. Since $\Lambda(G^*)$ is a compact set, then there exists $D_1,\cdots,D_k\epsilon I(\delta)$ for any small $\delta>0$ such that

$$\cup_{j=1}^k D_j\supset\Lambda(G^*).$$

Let δ be a sufficiently small number. Then we obtain from Lemma 4 that there exist $S_{(n)}\epsilon G^*$ and $T_{\nu_{n+1}}\epsilon\mathscr{G}$ for a fixed disc D_i $(1\leqq i\leqq k)$ and for any point $z\epsilon\Lambda_a(G^*)\cap D_i$ such that $B_{S_{(n)}}^{(2)}\ni z$, $k^*R_{S_{(n)}}^2>l_i$, $B_{T_{\nu_{n+1}}\circ S_{(n)}}^{(2)}\ni z$ and $k^*R_{T_{\nu_{n+1}}\circ S_{(n)}}^2\leqq l_i$. Note that the number of $\{B_{S_{(n)}}^{(2)}\}$ with $k^*R_{S_{(n)}}^2>l_i$ is finite for all i. So there exist a positive integer $N(i)$, $z_j\epsilon\Lambda_a(G^*)\cap D_i$, $S_{(n(j))}\epsilon G^*$ and $T_{\mu(j)}\epsilon\mathscr{G}$ $(j=1,\cdots,N(i))$ satisfying the following:

$$(2.17)\qquad B_{S_{(n(j))}}^{(2)}\ni z,\quad k^*R_{S_{(n(j))}}^2>l_i,$$

$$(2.18)\qquad B_{T_{\mu_{j+1}}\circ S_{(n(j))}}^{(2)}\ni z,\quad k^*R_{T_{\mu_{j+1}}\circ S_{(n(j))}}^2\leqq l_i$$

for $j=1,2,\cdots,N(i)$ $(i=1,\cdots,k)$.

Furthermore we can prove that the set of all such $N(i)$'s is a bounded set.

LEMMA 5. *For any small $\delta>0$ and any $D_i\epsilon I(\delta)$, there exists a positive integer $N_0=N_0(G^*)$ such that $N(i)\leqq N_0$ for all $i=1,\cdots,k$.*

Proof. Take two arbitrary distinct transformations $S_{(n(l))}$, $S_{(n(m))}$ satisfying (2.17) and (2.18). Then they can be written in the forms:

$$S_{(n(l))}=U_{\nu_q}\circ\cdots\circ U_{\nu_1}\circ S_{(n_0)}$$

and

$$S_{(n(m))}=U_{\mu_r}\circ\cdots\circ U_{\mu_1}\circ S_{(n_0)},$$

where $S_{(n_0)}\in G^*$ ($n_0\geqq 1$), $U_{\nu_j}, U_{\mu_k}\in \mathcal{G}$ ($j=1,\cdots,q$; $k=1,\cdots,r$) and $n(l)=q+n_0$, $n(m)=r+n_0$. Putting $T^{-1}=S_{(n(l))}$ and $S=S_{(n(m))}$ in (2.2), we have

$$|S^{-1}_{(n(l))}(\infty)-S^{-1}_{(n(m))}(\infty)| \tag{2.19}$$
$$=\frac{R^2_{S_{(n(l))}}}{|(U_{\nu_q}\circ\cdots\circ U_{\nu_1}\circ U^{-1}_{\mu_1}\circ\cdots\circ U^{-1}_{\mu_r})(\infty)-(U_{\nu_q}\circ\cdots\circ U_{\nu_1}\circ S_{(n_0)})(\infty)|}.$$

Since the set of points $\bigcup_{S\in G^*-\{id\}} B^{(2)}_S$ is bounded, then there exists $\rho>0$ such that $\{z\,|\,|z|\leqq\rho\}\supset B^{(2)}_S$ for all $S\in G^*-\{id\}$. Thus it follows from (2.19)

$$|S^{-1}_{(n(l))}(\infty)-S^{-1}_{(n(m))}(\infty)|\geqq\frac{1}{\rho}R^2_{S_{(n(l))}}>\frac{1}{k^*\rho}l_i. \tag{2.20}$$

And also it can be seen from (2.3) and (2.18)

$$k^*R^2_{S_{(n(j))}}\leqq\frac{1}{k_0}l_i$$

for all j. So the distance from the center of D_i to the center of $B^{(2)}_{S_{(n(j))}}$ is not greater than $(1+1/k_0)\,l_i$. Hence we obtain from (2.18) and (2.20)

$$\pi\left(\frac{1}{2k^*\rho}l_i\right)^2N(i)<\pi\left\{\left(1+\frac{1}{k_0}+\frac{1}{2k^*\rho}\right)l_i\right\}^2$$

and so $\quad N(i)<4(k^*\rho)^2\left(1+\frac{1}{k_0}+\frac{1}{2k^*\rho}\right)^2.$

Obviously $4(k^*\rho)^2(1+1/k_0+1/2k^*\rho)^2$ is a positive number which depends only on G^* and is denoted by $N_0=N_0(G^*)$. Thus our lemma is established.

q.e.d.

We have from Lemma 3

$$R^2_{T_{\mu_j}\circ S_{(n(j))}}\geqq k_0R^2_{S_{(n(j))}}$$

for any $j=1,2,\cdots,N(i)$ ($\leqq N_0$). So (2.17) and (2.18) yield

$$k^*R^2_{S_{(n(j))}}>l_i\geqq k^*k_0R^2_{S_{(n(j))}}. \tag{2.21}$$

From now on we shall assume $0<\mu\leqq 4$. Then we have the following theorem ([1] , [2]).

THEOREM 2. *Let $\delta>0$ be a sufficiently small number and let $\{D_1,\cdots,D_k\}\subset I(\delta)$ be any covering of $\Lambda(G^*)$. Then there exist a positive interger N_0 and $S_{(n(1))},\cdots,S_{(n(N(i)))}\in G^*$ satisfying the following properties*: (i) *$N(i)\leqq N_0$ for $i=1,\cdots,k$.* (ii) $\bigcup_{i=1}^{k}(\bigcup_{j=1}^{N(i)}B^{(2)}_{S_{(n(j))}})\supset$

$\Lambda_a(G^*)$.

(iii) $\sum_{i=1}^{k}\sum_{j=1}^{N(i)}(k^*k_0R^2_{S(n(j))})^{\frac{\mu}{2}}\leqq N_0\sum_{i=1}^{k}l_i^{\frac{\mu}{2}}$.

Proof. Properties (i) and (ii) are immediate corollaries of Lemma 5 and (iii) is a consequence of (ii) and (2.21).

q.e.d.

Now we shall call the quantity

$$M_{\frac{\mu}{2}}(\Lambda(G^*))=\lim_{\delta\to 0}\left[\inf_{\{I(\delta)\}}\left\{\sum_{D_\lambda\in I(\delta)}(2l_\lambda)^{\frac{\mu}{2}}\right\}\right]$$

the $\frac{\mu}{2}$-dimensional Hausdorff measure of $\Lambda(G^*)$.

Since G^* is geometrically finite, the set $\Lambda(G^*)-\Lambda_a(G^*)$ is a countable set. So it holds $M_{\frac{\mu}{2}}(\Lambda(G^*))=M_{\frac{\mu}{2}}(\Lambda_a(G^*))$.

Putting

$$F^{\delta/k_0}_{n_0}=\{B^{(2)}_{S(n)}\mid S_{(n)}\in G^*,\ n\geqq n_0 \text{ and } k^*R^2_{S(n)}\leqq\delta/k_0\}$$

for any number $\delta>0$ and any positive integer n_0, we obtain the following from Theorem 2.

Corollary 1. *For any number* μ $(0<\mu\leqq 4)$, *it holds*

$$\lim_{\delta\to 0}\left[\inf_{\{F^{\delta/k_0}_{n_0}\}}\left\{\sum_{B^{(2)}_{S(n)}\in F^{\delta/k_0}_{n_0}}(2R^2_{S(n)})^{\frac{\mu}{2}}\right\}\right]$$

$$\leqq N_0(k^*k_0)^{-\frac{\mu}{2}}\lim_{\delta\to 0}\left[\inf_{\{I(\delta)\}}\left\{\sum_{D_\lambda\in I(\delta)}(2l_\lambda)^{\frac{\mu}{2}}\right\}\right] \tag{2.22}$$

$$=N_0(k^*k_0)^{-\frac{\mu}{2}}M_{\frac{\mu}{2}}(\Lambda(G^*)).$$

§3. Computing functions and Hausdorff measure of $\Lambda(G^*)$.

1. For any $T_k=\gamma_p\circ\gamma_k\in\mathcal{G}$ $(k=1,\cdots,p-1)$, we put $D_{T_k}=\mathrm{Cl}\,(\mathrm{Int}(\gamma_p(c_k))$ and $D_{T_k^{-1}}=\mathrm{Cl}\,(\mathrm{Int}(c_k))$. Let $S_{(n)}=T_{\nu_n}\circ\cdots\circ T_{\nu_1}\in G^*$ be of the form

$$S_{(n)}(z)=(az+b)/(cz+d),\quad ad-bc=1.$$

Taking the derivative of $S_{(n)}(z)$, we obtain

$$(3.1)\qquad \left|\frac{dS_{(n)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\frac{1}{|cz+d|}\right)^{\mu}=\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-z|}\right)^{\mu}.$$

Take any fixed element $T\in\mathcal{G}$. Forming the sum of $(2p-3)^n$ terms with respect to all $S_{(n)}$ in (3.1) with $T_{\nu_1}\neq T^{-1}$, we have the following function

$$(3.2)\qquad \sum_{S_{(n)}}\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-z|}\right)^{\mu}=\sum_{S_{(n)}}\left|\frac{dS_{(n)}(z)}{dz}\right|^{\frac{\mu}{2}}$$

We denote it by $\chi_n^{(\mu\,;\,T)}(z)$ and call it the μ-dimensional computing function of order n on T. The domain of definition of $\chi_n^{(\mu\,;\,T)}(z)$ is D_T.

We assume that $S_{(l)}\in G^*$ is of the form $S_{(l)}=T\circ S_{(l-1)}$ $(T\in\mathcal{G})$. Then we can easily obtain from (2.1) and (3.2)

$$(3.3)\qquad \chi_n^{(\mu\,;\,T)}(S_{(l)}(\infty))=\sum_{S_{(n)}}\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-S_{(l)}(\infty)|}\right)^{\mu}=\sum_{S_{(n)}}\left(\frac{R_{S_{(n)}S_{(l)}}}{R_{S_{(l)}}}\right)^{\mu},$$

where $S_{(n)}S_{(l)}=S_{(n+l)}$.

2. The purpose of §3 is to seek for the relation between the computing function and the Hausdorff measure of $\Lambda(G^*)$ and we shall summerize the main result as Theorem 3 ([2]). For this purpose we need many lemmas. First let us consider the relation between two computing functions on the different elements of $\mathcal{G}$.

LEMMA 6. *For any two computing functions on the different elements of $\mathcal{G}$, it holds that there exists a positive constant $k(l,\mu)$ depending on l and μ such that*

$$\chi_{n+l}^{(\mu\,;\,T)}(z)\geqq k(l,\mu)\sum_{S_{(l)}}\chi_n^{(\mu\,;\,T_{\nu_l})}(S_{(l)}(z)),$$

where $\lim_{l\to\infty}k(l,\mu)=0$ *and* $S_{(l)}=T_{\nu_l}\circ\cdots\circ T_{\nu_1}$.

Proof. Let $S_{(n+l)}\in G^*$ be of the form $S_{(n+l)}=T_{\nu_{n+l}}\circ\cdots\circ T_{\nu_{l+1}}\circ T_{\nu_l}\circ\cdots\circ T_{\nu_1}$, where $T_{\nu_i}\in\mathcal{G}$ $(i=1,\cdots,n+l)$, $T_{\nu_j}\neq T_{\nu_{j+1}}^{-1}$ $(j=1,\cdots,n+l-1)$ and $T_{\nu_1}\neq T^{-1}$. Taking the derivative of $S_{(n+l)}$, we have

$$\frac{dS_{(n+l)}(z)}{dz}=\frac{dS_{(n+l)}(z)}{dS_{(l)}(z)}\cdot\frac{dS_{(l)}(z)}{dz}.$$

Then we obtain from (3.1)

$$(3.5)\qquad \left(\frac{R_{S_{(n+l)}}}{|S_{(n+l)}^{-1}(\infty)-z|}\right)^{\mu}=\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-S_{(l)}(z)|}\right)^{\mu}\left(\frac{R_{S_{(l)}}}{|S_{(l)}^{-1}(\infty)-z|}\right)^{\mu}.$$

Forming the sum of $(2p-3)^{n+l}$ terms in (3.5) with respect to $S_{(n+l)}$, we can easily see

$$\chi_{n+l}^{(\mu;T)}(z)=\sum_{S_{(l)}}\{\chi_{n}^{(\mu;T_{\nu_l})}(S_{(l)}(z))\left(\frac{R_{S_{(l)}}}{|S_{(l)}^{-1}(\infty)-z|}\right)^{\mu}\}, \tag{3.6}$$

where $S_{(l)}=T_{\nu_l}\circ\cdots\circ T_{\nu_1}$. Putting

$$\rho=\max_{\xi\in\partial D_U,z\in\partial D_V,U,V\in\mathcal{G}}|z-\xi|,$$

we have $|S_{(l)}^{-1}(\infty)-z|\leqq\rho$. Then, from (3.6), we have

$$\chi_{n+l}^{(\mu;T)}(z)\geqq(R_{S_{(l)}}/\rho)^{\mu}\sum_{S_{(l)}}\chi_{n}^{(\mu;T_{\nu_l})}(S_{(l)}(z)).$$

Putting $k(l,\mu)=(R_{S_{(l)}}/\rho)^{\mu}$, we obtain $\lim_{l\to\infty}k(l,\mu)=0$.

q.e.d.

Next we shall seek for the relation between two computing functions on the same T of different orders.

LEMMA 7. *Take any element $T\in\mathcal{G}$ and $z\in D_T\cap\Lambda(G^*)$. Then for any positive integer n there exist two constants $k_1(n,\mu)>0$ and $k_2(n,\mu)>0$ such that*

$$k_1(n,\mu)\chi_l^{(\mu;T)}(z)\leqq\chi_{n+l}^{(\mu;T)}(z)\leqq k_2(n,\mu)\chi_l^{(\mu;T)}(z). \tag{3.7}$$

Proof. For any fixed integer $n>0$, we have from (3.5) the following

$$\chi_{n+l}^{(\mu;T)}(z)=\sum_{S_{(n+l)}}\left\{\frac{R_{S_{(n)}}^{\mu}}{|S_{(n)}^{-1}(\infty)-S_{(l)}(z)|^{\mu}}\cdot\frac{R_{S_{(l)}}^{\mu}}{|S_{(l)}^{-1}(\infty)-z|^{\mu}}\right\}. \tag{3.8}$$

We put

$$\rho_1=\sup_{S_{(n)}\in G^*,\xi\in\Lambda(G^*)}|S_{(n)}^{-1}(\infty)-\xi|,\quad \rho_2=\inf_{S_{(n)}\in G^*,\xi\in\Lambda(G^*)}|S_{(n)}^{-1}(\infty)-\xi|,$$

$$\sigma_1=\max_{S_{(n)}\in G^*}R_{S_{(n)}},\qquad \sigma_2=\min_{S_{(n)}\in G^*}R_{S_{(n)}}.$$

Since n is fixed, these constants are positive and depend only on n. Noting $S_{(l)}(z)\in\Lambda(G^*)$, we have

$$\frac{\sigma_2^{\mu}}{\rho_1^{\mu}}\leqq\frac{R_{S_{(n)}}^{\mu}}{|S_{(n)}^{-1}(\infty)-S_{(l)}(z)|^{\mu}}\leqq\frac{\sigma_1^{\mu}}{\rho_2^{\mu}}. \tag{3.9}$$

By combining (3.8) with (3.9), we obtain

$$(2p-3)^n\left(\frac{\sigma_2}{\rho_1}\right)^{\mu}\chi_l^{(\mu;T)}(z)\leqq \sum_{S_{(n+l)}}\left\{\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-S_{(l)}(z)|}\right)^{\mu}\cdot\left(\frac{R_{S_{(l)}}}{|S_{(l)}^{-1}(\infty)-z|}\right)^{\mu}\right\}$$

$$\leqq (2p-3)^n\left(\frac{\sigma_1}{\rho_2}\right)^{\mu}\chi_l^{(\mu;T)}(z).$$

Putting $(2p-3)^n(\sigma_2/\rho_1)^{\mu}=k_1(n,\mu)$ and $(2p-3)^n(\sigma_1/\rho_2)^{\mu}=k_2(n,\mu)$, we have (3.7).

q.e.d.

3. Now let us give two lemmas on a sequence of computing functions $\{\chi_n^{(\mu;T)}(z)\}$.

LEMMA 8. *Let* $\{\chi_n^{(\mu;T)}(z)\}$ *be a sequence of computing functions. Suppose that* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=\infty$ *on some* $T^*\in\mathcal{G}$ *and for some* $z_0\in\mathrm{Int}\,(D_{T^*})$. *Then* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z)=\infty$ *uniformly on* $\mathrm{Int}\,(D_{T^*})$.

Proof. Let ρ_{T^*} be the distance from z_0 to ∂D_{T^*}. Put $D(z_0,d)=\{z\mid |z-z_0|\leqq d\}$ for any fixed d $(0<d<\rho_{T^*})$. We shall define $\eta_n^{(\mu;T)}(z)=\{\chi_n^{(\mu;T)}(z)\}^{-1}$. We divide the proof of this lemma into two cases: $z\in D(z_0,d)$ and $z\in\mathrm{Int}\,(D_{T^*})-D(z_0,d)$.

Case I. Let z be any point of $D(z_0,d)$. Then we have from the definition of $\chi_n^{(\mu;T)}(z)$ the following estimation:

$$|\eta_n^{(\mu;T^*)}(z)-\eta_n^{(\mu;T^*)}(z_0)|=\left|\frac{1}{\chi_n^{(\mu;T^*)}(z)}-\frac{1}{\chi_n^{(\mu;T^*)}(z_0)}\right|$$

(3.10)

$$\leqq \eta_n^{(\mu;T^*)}(z)\eta_n^{(\mu;T^*)}(z_0)\sum_{S_{(n)}}R_{S_{(n)}}^{\mu}\frac{\big||S_{(n)}^{-1}(\infty)-z_0|^{\mu}-|S_{(n)}^{-1}(\infty)-z|^{\mu}\big|}{|S_{(n)}^{-1}(\infty)-z|^{\mu}\,|S_{(n)}^{-1}(\infty)-z_0|^{\mu}}.$$

Since $S_{(n)}^{-1}(\infty)=(T_{\nu_1}^{-1}\circ\cdots\circ T_{\nu_n}^{-1})(\infty)$ and z are contained in $D_{T_{\nu_1}^{-1}}$ and $D(z_0,d)$, respectively, and $T\neq T_{\nu_1}^{-1}$, we obtain

$$(3.11)\qquad |S_{(n)}^{-1}(\infty)-z_0|>\rho_{T^*}$$

for all n.

Denote the rectangular coordinates of z_0, z and $S_{(n)}^{-1}(\infty)$ by (x_0,y_0), (x,y) and (a_n,b_n), respectively. Let us put $f(z)=f(x,y)=|S_{(n)}^{-1}(\infty)-z|^{\mu}$. Since $f(x,y)=\{(x-a_n)^2+(y-b_n)^2\}^{\mu/2}$, we have the partial derivatives:

$$\begin{cases} f_x(x,y)=\mu\,|S_{(n)}^{-1}(\infty)-z|^{\mu-2}(x-a_n)\\ f_y(x,y)=\mu\,|S_{(n)}^{-1}(\infty)-z|^{\mu-2}(y-b_n).\end{cases}$$

By use of the mean value theorem to $f(z)$ on $D(z_0, d)$, we obtain

$$|f(x, y)-f(x_0, y_0)| \leqq |f_x(x_0, y_0)||x-x_0| + |f_y(x_0, y_0)||y-y_0|$$
$$\leqq (|f_x(x_0, y_0)| + |f_y(x_0, y_0)|)|z-z_0|.$$

Hence we have the following inequality:

$$||S_{(n)}^{-1}(\infty)-z|^\mu - |S_{(n)}^{-1}(\infty)-z_0|^\mu| = |f(x, y)-f(x_0, y_0)|$$

(3.12)

$$\leqq 2\mu |S_{(n)}^{-1}(\infty)-z_0|^{\mu-1}|z-z_0|.$$

Applying (3.11) and (3.12) to (3.10), we have

$$(3.13) \qquad |\eta_n^{(\mu;T^*)}(z)-\eta_n^{(\mu;T^*)}(z_0)| \leqq \frac{2\mu d}{\rho_{T^*}}\eta_n^{(\mu;T^*)}(z_0).$$

Since (3.13) is symmetric with respect to z, $z_0 \in D(z_0, d)$, we obtain

$$(3.14) \qquad \frac{1}{K_1}\eta_n^{(\mu;T^*)}(z_0) \leqq \eta_n^{(\mu;T^*)}(z) \leqq K_1\eta_n^{(\mu;T^*)}(z_0),$$

where $K_1 = 2\mu d\rho_{T^*}^{-1}+1$.

Case II. Let z be any point of $\mathrm{Int}(D_{T^*})-D(z_0, d)$. Taking $r>0$ sufficiently large, we have for $i=1, \cdots, p$

$$D(z_0, r) = \{z \mid |z_0-z| \leqq r\} \supset c_i.$$

Obviously we can take a number K_2 (>0) such that it holds $r<K_2\rho_{T^*}$. Since $S_{(n)}^{-1}(\infty) \notin D_T$, we see

$$|S_{(n)}^{-1}(\infty)-z| \leqq 2r < 2K_2\rho_{T^*} \leqq 2K_2 |S_{(n)}^{-1}(\infty)-z_0|.$$

Hence we obtain from the above

$$(3.15) \qquad \chi_n^{(\mu;T^*)}(z) = \sum_{S_{(n)}}\left(\frac{R_{S_{(n)}}}{|S_{(n)}^{-1}(\infty)-z|}\right)^\mu \geqq \frac{1}{(2K_2)^\mu}\chi_n^{(\mu;T^*)}(z_0).$$

Putting $K^{-1} = \max(K_1, (2K_2)^\mu)$, we have from (3.14) and (3.15) the following inequality:

$$\chi_n^{(\mu;T^*)}(z) > K\chi_n^{(\mu;T^*)}(z_0)$$

for all $z \in \mathrm{Int}(D_{T^*})$.

This shows that $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z)=\infty$ uniformly on Int (D_{T^*}).

q.e.d.

LEMMA 9. *Let* $\{\chi_n^{(\mu;T)}(z)\}$ *be a sequence of computing functions. Suppose that* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=0$ *for some* $T^*\in\mathcal{G}$ *and some* $z_0\in\mathrm{Int}\,(D_{T^*})$. *Then* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z)=0$ *uniformly on* Int (D_{T^*}).

Proof. For any small $\varepsilon>0$, there exists a positive integer $l=l(\varepsilon)$ such that

$$(3.16)\qquad |(S_{(l)}\circ S_{(n)}^{-1})(\infty)-S_{(l)}(\infty)|<\varepsilon/2\mu k^*$$

for any n, where $S_{(l)}=S_{(l-1)}\circ T^*$, $S_{(n)}^{-1}=T_{\nu_1}^{-1}\circ\cdots\circ T_{\nu_n}^{-1}$ and $T_{\nu_1}^{-1}\neq T^*$. Let ε and l be fixed.

From Lemma 4, there exist $z_1,\cdots,z_k\in\Lambda_a(G^*)\cap D_{T^*}$ and $S_{1,(l)},\cdots,S_{k,(l)}\in G^*$ such that $B_{S_{j,(l)}}^{(2)}\ni z_j$, that is, $|z_j-S_{j,(l)}^{-1}(\infty)|<k^*R_{S_{j,(l)}}^2$ for $j=1,\cdots,k$ and $\cup_{j=1}^k B_{S_{j,(l)}}^2\supset\Lambda_a(G^*)\cap D_{T^*}$. Since $l\in N$ is sufficiently large, we may assume $(S_{j,(l)})^{-1}(\infty)\in\mathrm{Int}(D_{T^*})$ for $j=1,\cdots,k$. Then there exists a compact set K in $\mathrm{Int}(D_{T^*})$ so that Int $(K)\ni z_0$ and $K\ni S_{j,(l)}^{-1}(\infty)$ for $j=1,\cdots,k$. Let K be fixed. We set

$$d=\max_{z,\xi\in K}|z-\xi|>0,\quad \rho=\min_{z\in K,\xi\in\partial D_T}|z-\xi|>0,$$

$$r=\max_{z,\xi\in\cup_{T\in\mathcal{G}}D_T}|z-\xi|>0.$$

We divide the proof of this lemma into three cases, that is, z is contained in K, $(\cup_{j=1}^k B_{S_{j,(l)}}^{(2)})\cap\mathrm{Int}(D_{T^*})$ and Int $(D_{T^*})-\{K\cup(\cup_{j=1}^k B_{S_{j,(l)}}^{(2)})\}$

Case I. Take any point $z\in K$. In the analogous way to the proof of Lemma 8, we see

$$|\chi_n^{(\mu;T^*)}(z)-\chi_n^{(\mu;T^*)}(z_0)|\leqq\sum_{S_{(n)}}\frac{R_{S_{(n)}}^\mu}{|S_{(n)}^{-1}(\infty)-z_0|^\mu}\,\frac{||S_{(n)}^{-1}(\infty)-z_0|^\mu-|S_{(n)}^{-1}(\infty)-z|^\mu|}{|S_{(n)}^{-1}(\infty)-z|^\mu}$$

$$\leqq\frac{2\mu r^{\mu-1}d}{\rho^\mu}\,\chi_n^{(\mu;T^*)}(z_0).$$

Putting $M_1=2\mu r^{\mu-1}d/\rho^{\mu+1}$, we have from the above inequality

$$(3.17)\qquad \chi_n^{(\mu;T^*)}(z)\leqq M_1\chi_n^{(\mu;T^*)}(z_0).$$

Case II. Take any point $z\in(\cup_{j=1}^k B_{S_{j,(l)}}^{(2)})\cap\mathrm{Int}\,(D_{T^*})$. Then we have the following estimation:

$$| \chi_n^{(\mu\,;\,T^*)}((S_{j,(l)})^{-1}(\infty)) - \chi_n^{(\mu\,;\,T^*)}(z) |$$

$$\leqq \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{| S^{-1}_{(n)}(\infty) - z |^{\mu}} \cdot \frac{| | S^{-1}_{(n)}(\infty) - z |^{\mu} - | S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) |^{\mu} |}{| S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) |^{\mu}}$$

(3.18)

$$\leqq \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{| S^{-1}_{(n)}(\infty) - z |^{\mu}} \cdot \frac{2\mu \, | S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) |^{\mu-1} \, | z - (S_{j,(l)})^{-1}(\infty) |}{| S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) |^{\mu}}$$

$$= \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{| S^{-1}_{(n)}(\infty) - z |^{\mu}} \cdot \frac{2\mu \, | z - (S_{j,(l)})^{-1}(\infty) |}{| S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) |} .$$

Since $z \epsilon B^{(2)}_{S_{j,(l)}}$ for some j, we have

$$| z - (S_{j,(l)})^{-1}(\infty) | \leqq k^* R^2_{S_{j,(l)}} .$$

Here we note that $(S_{j,(l)})^{-1} = T^* \circ (S_{j,(l-1)})^{-1}$ and $T^{-1}_{\nu_1} \neq T^*$, where $S_{(n)} = T_{\nu_n} \circ \cdots \circ T_{\nu_1}$.

Putting $S_{j,(l)} = T^{-1}$ and $S_{(n)} = S$ in (2.2), we get from (2.2) and (3.16) the following inequality:

$$| S^{-1}_{(n)}(\infty) - (S_{j,(l)})^{-1}(\infty) | = \frac{R^2_{S_{j,(l)}}}{| (S_{j,(l)} \circ S^{-1}_{(n)})(\infty) - S_{j,(l)}(\infty) |}$$

(3.19)

$$> \frac{2\mu k^*}{\varepsilon} \cdot R^2_{S_{j,(l)}} .$$

Applying (3.19) to (3.18), we see easily

$$| \chi_n^{(\mu\,;\,T^*)}(S^{-1}_{j,(l)}(\infty)) - \chi_n^{(\mu\,;\,T^*)}(z) | < \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{| S^{-1}_{(n)}(\infty) - z |^{\mu}} \cdot \frac{2\mu\varepsilon k^* R^2_{S_{j,(l)}}}{2\mu k^* R^2_{S_{j,(l)}}}$$

$= \varepsilon \chi_n^{(\mu\,;\,T^*)}(z)$. Hence it holds

$$\chi_n^{(\mu\,;\,T^*)}(z) < \frac{1}{1-\varepsilon} \chi_n^{(\mu\,;\,T^*)}(S^{-1}_{j,(l)}(\infty)). \tag{3.20}$$

Noting $S^{-1}_{j,(l)}(\infty) \epsilon K$, we have $\chi_n^{(\mu\,;\,T^*)}(S^{-1}_{j,(l)}(\infty)) \leqq M_1 \chi_n^{(\mu\,;\,T^*)}(z_0)$. Hence we obtain from (3.20)

$$\chi_n^{(\mu\,;\,T^*)}(z) < \frac{M_1}{1-\varepsilon} \chi_n^{(\mu\,;\,T^*)}(z_0) \tag{3.21}$$

for all $z \in (\cup_{j=1}^{k} B^{(2)}_{S_{j,(l)}}) \cap \mathrm{Int}(D_{T^*})$.

Case III. Take any point $z \in \mathrm{Int}(D_{T^*}) - \{K \cup (\cup_{j=1}^{k} B^{(2)}_{S_{j,(l)}})\}$. Let us put $\mathfrak{D} = \mathrm{Int}\,(D_{T^*}) - \{K \cup (\cup_{j=1}^{k} B^{(2)}_{S_{j,(l)}})\}$. It can be easily seen that $\mathfrak{D} \subset \Omega(G^*)$ *and* $D_{T^*} - \mathfrak{D} \supset \Lambda(G^*) \cap D_{T^*}$. Hence we may assume

$$\delta = \inf_{\substack{\zeta \in \mathfrak{D},\, \xi \in \cup D_T \\ T \in G - \{T^*\}}} |z - \xi| > 0.$$

Put $\tau = \max_{\zeta \in \partial K} |\zeta - z_0| > 0$ and $\sigma = \max_{\zeta \in \partial D_{T^*}} |\zeta - z_0| > 0$.

In the analogous way to the proof of Lemma 8, we have

$$|\chi_n^{(\mu\,;\,T^*)}(z_0) - \chi_n^{(\mu\,;\,T^*)}(z)| \leqq \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{|S^{-1}_{(n)}(\infty) - z_0|^{\mu}} \frac{||S^{-1}_{(n)}(\infty) - z|^{\mu} - |S^{-1}_{(n)}(\infty) - z_0|^{\mu}|}{|S^{-1}_{(n)}(\infty) - z|^{\mu}}$$

$$\leqq \sum_{S_{(n)}} \frac{R^{\mu}_{S_{(n)}}}{|S^{-1}_{(n)}(\infty) - z_0|^{\mu}} \cdot \frac{2\mu\, |S^{-1}_{(n)}(\infty) - z_0|^{\mu-1}\, |z - z_0|}{|S^{-1}_{(n)}(\infty) - z|^{\mu}} \leqq \frac{2\mu\sigma^{\mu-1}\tau}{\delta^{\mu}} \chi_n^{(\mu\,;\,T^*)}(z_0).$$

Hence we obtain for any point $z \in \mathfrak{D}$

$$\chi_n^{(\mu\,;\,T^*)}(z) \leqq M_2 \chi_n^{(\mu\,;\,T^*)}(z_0),$$

where $M_2 = 2\mu\sigma^{\mu-1}\tau\delta^{-\mu} + 1$. Putting $M = \max(M_1/(1-\varepsilon),\, M_2)$, we have $\chi_n^{(\mu\,;\,T^*)}(z) \leqq M \chi_n^{(\mu;\,T^*)}(z_0)$ for any point $z \in \mathrm{Int}$ (D_{T^*}). This completes the proof of Lemma 9.

q.e.d.

4. Now let us give the main theorem.

THEOREM 3. *The following three propositions are equivalent to each other :*

(i) $\lim_{n\to\infty} \chi_n^{(\mu\,;\,T^*)}(z_0) = \infty$ (or 0) *on some* $T^* \in \mathcal{G}$ *and some* $z_0 \in \mathrm{Int}\,(D_{T^*})$.

(ii) $\lim_{n\to\infty} \chi_n^{(\mu\,;\,T)}(z) = \infty$ (or 0) *uniformly on* $\mathrm{Int}\,(D_T)$ *for any* $T \in \mathcal{G}$.

(iii) $M_{\frac{\mu}{2}}(\Lambda(G^*)) = \infty$ (or 0).

As the proof of this theorem is complicated, so we divide it into five lemmas. First we shall prove that (i) is equivalent to (ii). For this purpose, it suffices to show that (i) implies (ii) in the following.

LEMMA 10. *Suppose that* $\lim_{n\to\infty} \chi_n^{(\mu\,;\,T^*)}(z_0) = \infty$ (or 0) *on some* $T^* \in \mathcal{G}$ *and some* $z_0 \in \mathrm{Int}\,(D_{T^*})$.

Then it holds $\lim_{n\to\infty}\chi_n^{(\mu\,;\,T)}(z)=\infty$ (or 0) *uniformly on* Int (D_T) *for any* $T\in\mathcal{G}$.

Proof. (I) The case of the limit ∞. From the proof of Lemma 8, there exists a constant $K>0$ such that

$$\chi_n^{(\mu\,;\,T^*)}(z) \;>\; \frac{1}{K}\,\chi_n^{(\mu\,;\,T^*)}(z_0)$$

for any positive integer n and any point $z\in\text{Int}\,(D_{T^*})$. For any large number $M_0>0$, there exists an integer $n_0=n_0(M_0,\ T^*)>0$ depending on M_0 and T^* so that $\chi_n^{(\mu\,;\,T^*)}(z_0)>KM_0$ for any $n\geqq n_0(M_0,\ T^*)$. Hence we have

$$\text{(3.23)}\qquad \chi_n^{(\mu\,;\,T^*)}(z) \;>\; M_0$$

for any $z\in\text{Int}\,(D_{T^*})$ and for any $n\geqq n_0(M_0,\ T^*)>0$.

Now from (3.4) of Lemma 6, there exists a constant $k(1,\ \mu)>0$ depending only on μ for any $T\in\mathcal{G}-\{T^*\}$ such that

$$\text{(3.24)}\qquad \chi_{n+1}^{(\mu\,;\,T)}(z') > k(1,\ \mu)\ \chi_n^{(\mu\,;\,T^*)}(T^*(z'))$$

for any $z'\in\text{Int}\,(D_T)$ and any $n\geqq n_0(M_0,\ T^*)$. As $T^*(z')\in\text{Int}\,(D_{T^*})$, we have $\lim_{n\to\infty}\chi_n^{(\mu\,;\,T^*)}(T^*(z'))=\infty$ for any $z'\in\text{Int}\,(D_T)$. Hence from (3.23) and (3.24), there is an integer $n_0(M_0,T)\in N$ satisfying $\chi_{n+1}^{(\mu\,;\,T)}(z)>M_0$ for any $z\in\text{Int}\,(D_T)$ and any $n\geqq n_0(M_0,\ T)$. If we put $n^*(M_0)=\max_{T\in\mathcal{G}}\{n_0(M_0,\ T)\}+1$, it can be easily seen that it holds $\chi_n^{(\mu\,;\,T)}(z)>M_0$ for any $T\in\mathcal{G}$ and any $n\geqq n^*(M_0)$ and any $z\in\text{Int}\,(D_T)$. Hence we obtain for any $T\in\mathcal{G}$ $\lim_{n\to\infty}\chi_n^{(\mu\,;\,T)}(z)=\infty$ uniformly on Int (D_T).

(II) The case of the limit 0. From the proof of Lemma 9, there exists a constant $M>0$ such that $\chi_n^{(\mu\,;\,T^*)}(z)\leqq M\chi_n^{(\mu\,;\,T^*)}(z_0)$ for any $z\in\text{Int}\,(D_{T^*})$. From the assumption there exists an integer $n_0=n_0(\varepsilon,\ T^*)\in N$ for any small $\varepsilon>0$ such that it holds for any $n\geqq n_0(\varepsilon,\ T^*)$ $\chi_n^{(\mu\,;\,T^*)}(z_0)<\varepsilon/M$, where $n_0(\varepsilon,\ T^*)$ depends on ε and T^*. Then we have

$$\text{(3.25)}\qquad \chi_n^{(\mu\,;\,T^*)}(z) \;<\; \varepsilon$$

for any $z\in\text{Int}\,(D_{T^*})$ and any $n\geqq n_0(\varepsilon,\ T^*)$. Now from (3.4) of Lemma 6, there exists a constant $k(1,\ \mu)>0$ depending only on μ such that

$$\text{(3.26)}\qquad \chi_{n+1}^{(\mu\,;\,T^*)}(z') \;>\; k(1,\ \mu)\ \chi_n^{(\mu\,;\,T^*)}(T(z'))$$

for any $T\in\mathcal{G}-\{T^*\}$ and any $z'\in\text{Int}\,(D_{T^*})$. Hence, from (3.25), (3.26) and Lemma 9, there exists an integer $n_0(\varepsilon,\ T)\in N$ depending on any $\varepsilon>0$ and any $T\in\mathcal{G}$ such that $\chi_n^{(\mu\,;\,T)}(z)<\varepsilon$ for any $z\in\text{Int}\,(D_T)$ and any $n\geqq n_0(\varepsilon,\ T)$. Let us put $n^*(\varepsilon)=\max_{T\in\mathcal{G}}\{n_0(\varepsilon,\ T)\}$. Then we have

$\chi_n^{(\mu;T)}(z) < \varepsilon$ for any $n \geqq n^*(\varepsilon)$, any $z \in \mathrm{Int}(D_T)$ and any $T \in \mathscr{G}$.

q.e.d.

5. Next we shall show that (ii) implies (iii).

LEMMA 11. *Suppose that* $\lim_{n\to\infty} \chi_n^{(\mu;T)}(z) = \infty$ (or 0) *for any* $T \in \mathscr{G}$ *uniformly on* Int (D_T). *Then it holds* $M_{\frac{\mu}{2}}(\Lambda(G^*)) = \infty$ (or 0).

Proof. (I) The case of the limit ∞. From the assumption of this lemma, for any $T \in \mathscr{G}$ and any $M > 1$, there exists an integer $n_0 = n_0(M) \in N$ depending on M such that

$$(3.27) \qquad \chi_n^{(\mu;T)}(z) \quad > \quad M$$

for any $z \in \mathrm{Int}(D_T)$ and any $n \geqq n_0(M)$. Let an integer n_1 $(\geqq n_0)$ be fixed. Consider $(2p-2)(2p-3)^{n_1-1}$ elements of the grade n_1. Take an element $S_{(n_1)} = S_{(n_1-1)} \circ T^{-1}$ of the grade n_1 among them and let it be fixed. Let $F_{\tilde{n}_0}^{\delta/k_0}$ be a covering of $\Lambda_a(G^*)$ defined in §2. We take a covering consisting of a finite number of closed discs $B_{S_{(m_1)}}^{(2)}, \cdots, B_{S_{(m_Q)}}^{(2)} \in F_{\tilde{n}_0}^{\delta/k_0}$ of $\Lambda_a(G^*) \cap B_{S_{(n_1)}}^{(2)}$ i.e. $\cup_{j=1}^{Q} B_{S_{(m_j)}}^{(2)} \supset \Lambda_a(G^*) \cap B_{S_{(n_1)}}^{(2)}$. Here we assume that $\delta > 0$ is sufficiently small such that $\tilde{n}_0 - n_1 > n_0$.

We shall put $m^* = \min_{1 \leqq j \leqq Q} \{m_j\}$. We amend these closed discs $B_{S_{(m_1)}}^{(2)}, \cdots, B_{S_{(m_Q)}}^{(2)}$ in the following;

(i) if $m_j - m^* = n_0 r$ $(r \in Z,\ r \geqq 0)$, then we put $m_j = m'_j$, and

(ii) if $m_j - m^* = n_0 r + s$ $(r, s \in Z,\ r \geqq 0,\ 1 \leqq s \leqq n_0 - 1)$, then we replace the closed disc $B_{S_{(m_j)}}^{(2)}$ with $(2p-3)^{n_0-s}$ discs of the grade $m'_j = m^* + n_0(r+1) = m_j + (n_0 - r)$, $B_{S_{1,(m'_j)}}^{(2)}, \cdots, B_{S_{(2p-3)^{n_0-s},(m'_j)}}^{(2)}$.

By this procedure, we get a new covering of $\Lambda_a(G^*) \cap B_{S_{(n_1)}}^{(2)}$ consisting of $B_{S_{(m'_1)}}^{(2)}, \cdots, B_{S_{(m'_R)}}^{(2)}$ $(Q \leqq R)$. Then there exists a constant $K(n_0, \mu) > 0$ depending only on n_0 and μ from (2.3) such that

$$(3.28) \qquad \sum_{j=1}^{Q} (R_{S_{(m_j)}})^{\mu} \geqq K(n_0, \mu) \sum_{j=1}^{R} (R_{S_{(m'_j)}})^{\mu}.$$

We again amend these closed discs $B_{S_{(m'_1)}}^{(2)}, \cdots, B_{S_{(m'_R)}}^{(2)}$ in the following manner.

In the set of closed discs $B_{S_{(m'_1)}}^{(2)}, \cdots, B_{S_{(m'_R)}}^{(2)}$, there exist a finite number of systems $W_{m_k^*}$ $(1 \leqq k \leqq n)$ with the following properties: (i) each $W_{m_k^*}$ has $(2p-3)^{n_0}$ closed discs of the grade m_k^* and (ii) the grades of closed discs in different systems are not necessarily equal. Here we note that $W_{m_k^*} = \{B_{S_{j,(n_0)} \circ S_{(m_k^* - n_0)}}^{(2)} \mid j = 1, 2, \cdots, (2p-3)^{n_0}\}$. We replace these $(2p-3)^{n_0}$ closed discs in each system $W_{m_k^*}$ by closed discs whose grade numbers are $m_k^* - 1$. We repeat such procedure n_0 times for each $W_{m_k^*}$ $(1 \leqq k \leqq n)$. Then we see from (3.27)

$$(3.29) \qquad \sum_{S_{(n_0)}} (R_{S_{(m_k^*-n_0)} \circ S_{(n_0)}})^{\mu} > (R_{S_{(m_k^*-n_0)}})^{\mu}.$$

After such replacement we reach to a new covering of $\Lambda_a(G^*) \cap B^{(2)}_{S_{(n_1)}}$ consisting of closed discs $B^{(2)}_{S_{(m_1'')}}, \cdots, B^{(2)}_{S_{(m''_U)}}$ $(U < R)$.

Repeating such amendment of $B^{(2)}_{S_{(m_1')}}, \cdots, B^{(2)}_{S_{(m'_R)}}$, to $B^{(2)}_{S_{(m_1'')}}, \cdots, B^{(2)}_{S_{(m''_U)}}$, we obtain the following inequality:

$$(3.30) \qquad \sum_{j=1}^{R} (R_{S_{(m_j')}})^{\mu} \geqq \sum_{S_{(m^*-n_1)}} (R_{S_{(m^*)}})^{\mu},$$

where $S_{(m^*)} = S_{(n_1)} \circ S_{(m^*-n_1)}$ and the summation in the right hand side is taken over all transformations in G^* of the form $S_{(m^*)} = S_{(n_1)} \circ S_{(m^*-n_1)}$. Then we have from (3.3) and (3.27)

$$(3.31) \qquad \begin{aligned} \sum_{S_{(m^*-n_1)}} (R_{S_{(m^*)}})^{\mu} &= \sum_{S^{-1}_{(m^*-n_1)}} \{(R_{S^{-1}_{(m^*-n_1)} \circ S^{-1}_{(n_1)}})^{\mu} / (R_{S^{-1}_{(n_1)}})^{\mu}\} \times (R_{S^{-1}_{(n_1)}})^{\mu} \\ &= \chi^{(\mu;T)}_{m^*-n_1}(S^{-1}_{(n_1)}(\infty)) \times (R_{S^{-1}_{(n_1)}})^{\mu} \geqq M\ (R_{S^{-1}_{(n_1)}})^{\mu}, \end{aligned}$$

where $S_{(n_1)} = S_{(n_1-1)} \circ T^{-1}$ and the summation in (3.31) is taken over all transformations of the form $S_{(n_1)} = S_{(n_1-1)} \circ T^{-1}$. Hence we obtain from (3.29), (3.30) and (3.31)

$$(3.32) \qquad \begin{aligned} \sum_{j=1}^{Q} (R_{S_{(m_j)}})^{\mu} &\geqq K(n_0, \mu) \sum_{j=1}^{R} (R_{S_{(m_j')}})^{\mu} \\ &\geqq K(n_0, \mu) \sum_{S_{(m^*-n_1)}} (R_{S_{(m^*)}})^{\mu} \geqq K(n_0, \mu) \cdot M \cdot (R_{S_{(n_1)}})^{\mu}. \end{aligned}$$

Noting that (3.32) holds for any closed disc $B^{(2)}_{S_{(n_1)}}$, we obtain from (2.22) and (3.32) the following inequality:

$$(3.33) \qquad \begin{aligned} &N_0(2k^*k_0)^{-\frac{\mu}{2}} M_{\frac{\mu}{2}}(\Lambda(G^*) \cap D_T) \geqq \\ &K(n_0, \mu)\ (\sum_{S_{(n_1)}} (R_{S^{-1}_{(n_1)}})^{\mu} / (R_{T^{-1}})^{\mu}) \times (R_{T^{-1}})^{\mu} \times M. \end{aligned}$$

Since M is any positive number and n_1 is any fixed integer greater than n_0, we obtain from (3.33) that $M_{\frac{\mu}{2}}(\Lambda(G^*)) = \infty$ by taking n_1 to the infinity.

(II) The case of the limit 0. From the assumption, for any $T \in \mathcal{G}$ and any $\varepsilon > 0$, there exists a positive integer $n_0 = n_0(\varepsilon)$ depending only on ε such that

$$(3.34) \qquad \chi^{(\mu;T)}_{n_0}(z) < \varepsilon$$

for any $z \in \mathrm{Int}\,(D_T)$. Take any sufficiently large integer l ($\geqq n_0$) and let it be fixed. Then there exist $S_{j,(l)} \in G^*$ ($j=1, \cdots, (2p-2)(2p-3)^{l-1}$) such that

$$\bigcup_{j=1}^{(2p-2)(2p-3)^{l-1}} B^{(2)}_{S_{j,(l)}} \supset \Lambda_a(G^*).$$

Note that $S_{(k)}(\infty) \in \mathrm{Int}\,(D_T)$ if $S_{(k)} = T \circ S_{(k-1)}$. So we have from (3.3) and (3.34)

$$\chi_{n_0}^{(\mu\,;\,T)}(S_{(k)}(\infty)) = \sum_{S_{(n_0)}} \left(\frac{R_{S_{(n_0)}\circ S_{(k)}}}{R_{S_{(k)}}}\right)^{\mu} < \varepsilon.$$

Hence we have

$$\text{(3.35)} \qquad \sum_{S_{(n_0)}} (R_{S_{(n_0)}\circ S_{(k)}})^{\mu} < \varepsilon (R_{S_{(k)}})^{\mu}.$$

Let us put $l = rn_0 + s$ ($r, s \in N$, $s \leqq n_0 - 1$). Since $S_{(l)} = S_{(rn_0+s)} = S_{(n_0)} \circ S_{((r-1)n_0+s)}$, we can see from (3.35)

$$\text{(3.36)} \qquad \sum_{S_{(n_0)}} (R_{S_{(l)}})^{\mu} < \varepsilon (R_{S_{((r-1)n_0+s)}})^{\mu}.$$

Taking the summation in the both sides of (3.36) over all transformations of the grade $(r-1)n_0+s$, we obtain

$$\sum_{j=1}^{(2p-2)(2p-3)^{l-1}} (R_{S_{j,(l)}})^{\mu} < \varepsilon \sum_{j=1}^{(2p-2)(2p-3)^{(r-1)n_0+s-1}} (R_{S_{j,((r-1)n_0+s)}})^{\mu}.$$

If we repeat this procedure $(r-1)$ times, we obtain

$$\text{(3.37)} \qquad \begin{aligned} \sum_{j=1}^{(2p-2)(2p-3)^{l-1}} (R_{S_{j,(l)}})^{\mu} &< \varepsilon^{r} \sum_{j=1}^{(2p-2)(2p-3)^{l-1}} (R_{S_{j,(s)}})^{\mu} \\ &\leqq \varepsilon^{r} \max_{1 \leqq m \leqq s} \Big(\sum_{j=1}^{(2p-2)(2p-3)^{m-1}} (R_{S_{j,(m)}})^{\mu}\Big). \end{aligned}$$

Since the ritht hand side of (3.37) tends to zero as r tends to the infinity, we have

$$\lim_{l\to\infty} \sum_{j=1}^{(2p-2)(2p-3)^{l-1}} (R^{2}_{S_{j,(l)}})^{\frac{\mu}{2}} = 0.$$

Hence we can conclude $M_{\frac{\mu}{2}}(\Lambda_a(G^*)) = M_{\frac{\mu}{2}}(\Lambda(G^*)) = 0$.

q.e.d.

6. Now we shall prove that (iii) implies (i). At first we shall show this fact in the case of the limit ∞ as follows.

LEMMA 12. *If* $M_{\frac{\mu}{2}}(\Lambda(G^*))=\infty$, *then* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=\infty$ *for some* $T^*\in\mathcal{G}$ *and some* $z_0\in\mathrm{Int}(D_{T^*})$.

Proof. Let n_0 be a fixed natural number. From the assumption, we have $S_{(n_0)}\in G^*$ such that

$$(3.38)\qquad M_{\frac{\mu}{2}}(\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_0)}})=\infty.$$

Let us put $S_{(n_0)}=T^*\circ S_{(n_0-1)}$. Then for any integer n_1 ($\geqq n_0$) and any $z\in\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_0)}}$, there exists $S_{(n_1)}\in G^*$ such that $z\in B^{(2)}_{S_{(n_1)}}$, where $S_{(n_1)}=S_{(n_1-n_0)}\circ S_{(n_0)}$. Hence we find that there exist $S_{j,(n_1)}\in G^*$ ($j=1,\cdots,(2p-3)^{n_1-n_0}$) such that

$$\bigcup_{j=1}^{(2p-3)^{n_1-n_0}} B^{(2)}_{S_{j,(n_1)}}\supset\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_0)}}.$$

We have from the definition of Hausdorff measure

$$(3.39)\qquad M_{\frac{\mu}{2}}(\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_0)}})\leqq\sum_{j=1}^{(2p-3)^{n_1-n_0}}(2k^*R^2_{S_{j,(n_1)}})^{\frac{\mu}{2}}.$$

Here we note that it holds from (3.3)

$$(3.40)\qquad \begin{aligned}\sum_{j=1}^{(2p-3)^{n_1-n_0}}(R_{S_{j,(n_1)}})^{\mu}&=\sum_{S_{(n_1-n_0)}}\Big(\frac{R_{S_{(n_1-n_0)}\circ S_{(n_0)}}}{R_{S_{(n_0)}}}\Big)^{\mu}\times R^{\mu}_{S_{(n_0)}}\\&=\chi^{(\mu;T^*)}_{n_1-n_0}(S_{(n_0)}(\infty))\times R^{\mu}_{S_{(n_0)}}.\end{aligned}$$

Putting $S_{(n_0)}(\infty)=z_0$, we can easily see $z_0\in\mathrm{Int}(D_{T^*})$. Hence we obtain the following from (3.39) and (3.40)

$$M_{\frac{\mu}{2}}(\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_0)}})\leqq(2k^*R^2_{S_{(n_0)}})^{\frac{\mu}{2}}\lim_{n_1\to\infty}\chi^{(\mu;T^*)}_{n_1-n_0}(z_0).$$

This concludes $\lim_{n_1\to\infty}\chi^{(\mu;T^*)}_{n_1-n_0}(z_0)=\infty$.

q.e.d.

7. In order to show that (iii) implies (i) in the case of the limit 0, we have to prove the following lemma.

LEMMA 13. *Suppose that there exists a subsequence* $\{\chi^{(\mu;T^*)}_{n_i}(z)\}$ *of* $\{\chi^{(\mu;T^*)}_n(z)\}$ *with respect to some* $T^*\in\mathcal{G}$ *such that* $\lim_{i\to\infty}\chi^{(\mu;T^*)}_{n_i}(z_0)=\infty$ (or 0) *for some* $z_0\in\mathrm{Int}(D_{T^*})\cap\Lambda_a(G^*)$. *Then it holds* $\lim_{n\to\infty}\chi^{(\mu;T^*)}_n(z_0)=\infty$ (or 0).

Proof. At first let us prove this lemma in the case of the limit ∞. Replacing $\{\chi_n^{(\mu;T^*)}(z_0)\}$ with $\{\chi_{n_i}^{(\mu;T^*)}(z_0)\}$ in Lemma 8, we have $\lim_{i\to\infty}\chi_{n_i}^{(\mu;T^*)}(z)=\infty$ uniformly on Int (D_{T^*}). Then for any large number $M'>0$, there exists a positive integer $n_0'=n_0'(M')$ depending on M' such that $\chi_{n_0'}^{(\mu;T^*)}(z)>M'$ for any $z\in$ Int (D_{T^*}). From Lemma 6 there exists a positive number $k(3, \mu)$ depending only on μ such that

$$\chi_{n_0'+3}^{(\mu;T)}(z) \geqq k(3, \mu)\, \chi_{n_0'}^{(\mu;T^*)}(S_{(3)}(z))$$

for any $z\in$ Int (D_T) and any $S_{(3)}=T^*\circ T'\circ T$, where $T, T'\in\mathcal{G}$. Noting $S_{(3)}(z)\in$ Int (D_{T^*}) for any $T\in\mathcal{G}$ and any $z\in$ Int (D_T), we find that $\chi_{n_0'+3}^{(\mu;T)}(z)\geqq k(3, \mu)M'$ for any $T\in\mathcal{G}$ and $z\in$ Int (D_T). Here let us put $n_0=n_0'(M')+3$. Since we may take $M'>1$ sufficiently large such that $k(3, \mu)M'=M>1$, we obtain

$$\chi_{n_0}^{(\mu;T)}(z) \geqq M>1 \tag{3.42}$$

uniformly on Int (D_T) for any $T\in\mathcal{G}$.

Now let us consider the computing function $\chi_{qn_0}^{(\mu;T^*)}(z)$ at z_0, where q is a positive integer. For any sufficiently small $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$ depending only on ε such that it holds

$$\chi_{qn_0}^{(\mu;T^*)}(z_0) > \chi_{qn_0}^{(\mu;T^*)}(z)-\varepsilon$$

for any $z\in D(z_0, \delta(\varepsilon))\cap$ Int (D_{T^*}), where $D(z_0, \delta(\varepsilon))=\{z \mid |z-z_0|<\delta(\varepsilon)\}$. Take a sufficiently large integer $l>0$. Then there exists a $S_{(l)}\in G^*$ such that $S_{(l)}(\infty)\in D(z_0, \delta(\varepsilon))\cap$ Int (D_{T^*}) and so

$$\chi_{qn_0}^{(\mu;T^*)}(z_0) > \chi_{qn_0}^{(\mu;T^*)}(S_{(l)}(\infty))-\varepsilon'. \tag{3.43}$$

Now we have from (3.3)

$$\chi_{qn_0}^{(\mu;T^*)}(S_{(l)}(\infty))=\sum_{S_{(qn_0)}}(R_{S_{(qn_0)}\circ S_{(l)}})^{\mu}/(R_{S_{(l)}})^{\mu}. \tag{3.44}$$

Modifying the right hand side of (3.44), we obtain

$$\frac{\sum_{S_{(qn_0)}}(R_{S_{(qn_0)}\circ S_{(l)}})^{\mu}}{(R_{S_{(l)}})^{\mu}}=\prod_{j=1}^{q}\left[\frac{\sum_{S_{(jn_0)}}(R_{S_{(jn_0)}\circ S_{(l)}})^{\mu}}{\sum_{S_{((j-1)n_0)}}(R_{S_{((j-1)n_0)}\circ S_{(l)}})^{\mu}}\right], \tag{3.45}$$

where $S_{(0)}$ is the identity. Since

$$\frac{\sum_{S_{(n_0)}} (R_{S_{(jn_0)}} \circ S_{(l)})^{\mu}}{(R_{S_{((j-1)n_0)}} \circ S_{(l)})^{\mu}} = \chi_{n_0}^{(\mu\,;\,T_{\nu_i})}(S_{((j-1)n_0)} \circ S_{(l)}(\infty)),\ (j \geqq 1),$$

we have from (3.42)

$$(3.46) \qquad \chi_{n_0}^{(\mu\,;\,T_{\nu_i})}(S_{((j-1)n_0)} \circ S_{(l)}(\infty)) > M,\ (j \geqq 1),$$

where $S_{((j-1)n_0)} \circ S_{(l)} = T_{\nu_i} \circ S_{((j-1)n_0+l-1)}$, $T_{\nu_i} \in \mathcal{G}$.
If we apply (3.44), (3.45) and (3.46) to (3.43), then we obtain

$$\chi_{qn_0}^{(\mu\,;\,T^*)}(z_0) > M^q - \varepsilon.$$

Hence we can conclude

$$(3.47) \qquad \lim_{q\to\infty} \chi_{qn_0}^{(\mu\,;\,T^*)}(z_0) = \infty.$$

For any positive integer $m = qn_0 + r(q,\ r \in Z,\ q,\ r \geqq 0,\ r \leqq n_0 - 1)$, let us put $n = r$ and $l = qn_0$ in Lemma 7. Then we have from (3.7)

$$(3.48) \qquad k_1(n_0, \mu)\chi_{qn_0}^{(\mu\,;\,T^*)}(z_0) < \chi_m^{(\mu\,;\,T^*)}(z_0) < k_2(n_0, \mu)\chi_{qn_0}^{(\mu\,;\,T^*)}(z_0).$$

Hence from (3.47) and (3.48) we can conclude $\lim_{n\to\infty} \chi_n^{(\mu\,;\,T^*)}(z_0) = \infty$.

Next we shall show this lemma in the case of the limit 0. From Lemma 6 there exists a positive number $k(3, \mu) > 0$ depending only on μ such that

$$(3.49) \qquad \chi_{n_i}^{(\mu\,;\,T^*)}(z_0) > k(3, \mu)\ \chi_{n_i-3}^{(\mu\,;\,T)}(S_{(3)}(z_0))$$

for any $i = 1, 2, \cdots$ and any $S_{(3)} = T \circ T' \circ T^* \in G^*$, where $T,\ T' \in \mathcal{G}$. Since $S_{(3)}(z_0) \in \mathrm{Int}\,(D_T)$, we can see from (3.49)

$$\lim_{n_i\to\infty} \chi_{n_i-3}^{(\mu\,;\,T)}(S_{(3)}(z_0)) = 0$$

for any $T \in \mathcal{G}$. Hence, putting $n'_i = n_i - 3$, we obtain from Lemma 9 that for any small $\varepsilon > 0$ there exists a positive integer $n_0 = n_0(\varepsilon)$ depending only on ε such that

$$(3.50) \qquad \chi_{n_0}^{(\mu\,;\,T)}(z) < \varepsilon$$

for any $z \in \mathrm{Int}\,(D_T)$ and any $T \in \mathcal{G}$.

In the analogous way as in the case of the limit ∞, we obtain from (3.50) $\lim_{n\to\infty} \chi_n^{(\mu\,;\,T^*)}(z_0) = 0$.

q.e.d.

8. At last let us prove that (iii) implies (i) from the following lemma in the case of the limit 0.

LEMMA 14. *If* $M_{\frac{\mu}{2}}(\Lambda(G^*))=0$, *then it holds that* $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=0$ *for some* $T^*\in\mathcal{G}$ *and some* $z_0\in\mathrm{Int}\,(D_{T^*})$.

Proof. Assume the contrary. Since (i) and (ii) of Theorem 3 are equivalent to each other from Lemma 10, we obtain that there exist a subseqemce $\{\chi_{n_i}^{(\mu;T)}(z)\}$ of $\{\chi_n^{(\mu;T)}(z)\}$ and $0<\alpha\leqq\infty$ such that

$$\lim_{i\to\infty}\chi_{n_i}^{(\mu;T^*)}(z_0)=\alpha$$

for some $T^*\in\mathcal{G}$ and some $z_0\in\mathrm{Int}\,(D_{T^*})$.

If $\alpha=\infty$, then we can see from Lemma 13 that $\lim_{n\to\infty}\chi_n^{(\mu;T^*)}(z_0)=\infty$. Hence we have from Lemma 11 $M_{\frac{\mu}{2}}(\Lambda(G^*))=\infty$. This contradicts the assumption $M_{\frac{\mu}{2}}(\Lambda(G^*))=0$. So we may assume that $0<\alpha<\infty$. Then it holds

$$0<\liminf_{n\to\infty}\chi_n^{(\mu;T)}(z)\leqq\limsup_{n\to\infty}\chi_n^{(\mu;T)}(z)<\infty \tag{3.51}$$

for any $T\in\mathcal{G}$ and any $z\in\mathrm{Int}\,(D_T)$.

Now take a compact set K in $\mathrm{Int}\,(D_{T^*})$ and let it be fixed. Then there exist positive constants c_1, c_2 such that

$$0<c_1\leqq\liminf_{n\to\infty}\chi_n^{(\mu;T^*)}(z)\leqq\limsup_{n\to\infty}\chi_n^{(\mu;T^*)}(z)\leqq c_2<+\infty$$

for any $z\in K$. Taking a sufficiently small $\varepsilon>0$ $(\varepsilon<c_1)$, we can easily see that there exists a positive integer $n_0=n_0(\varepsilon, K)$ depending on ε and K such that

$$0<c_1-\varepsilon\leqq\chi_n^{(\mu;T^*)}(z)\leqq c_2+\varepsilon<+\infty \tag{3.52}$$

for any $z\in K$ and any $n\geqq n_0$. For any sufficiently large integer $n_1>n_0$ we can take $S_{(n_1)}=T^*\circ S_{(n_1-1)}\in G^*$ such that $B^{(2)}_{S^{-1}_{(n_1)}}\subset\mathrm{Int}\,(K)$. Let such $S_{(n_1)}$ be fixed. Then for any small $\delta>0$, there exist a positive integer $\tilde{n}_0=\tilde{n}_0(\delta)$ depending only on δ and closed discs $B^{(2)}_{S_{(m_1)}},\cdots,B^{(2)}_{S_{(m_Q)}}\in F^{\delta/k_0}_{\tilde{n}_0}$ such that $m_j>n_1$ $(j=1,\cdots,Q)$ and $\mathrm{Int}\,(K)\supset\bigcup_{j=1}^{Q}B^{(2)}_{S_{(m_j)}}\supset\Lambda_a(G^*)\cap B^{(2)}_{S_{(n_1)}}$. Here we can take a natural number n^* so large that it may hold $n^*-m_j\geqq n_0$ for $j=1,\cdots,Q$. Then we get from (3.52)

$$(3.53)\qquad c_1-\varepsilon \leqq \sum_{S_{(n^*-m_j)}} \left(\frac{R_{S_{(n^*-m_j)}\circ S_{(m_j)}}}{R_{S_{(m_j)}}}\right)^{\mu} \leqq c_2+\varepsilon .$$

It holds from (3.53)

$$(3.54)\qquad \begin{aligned} R^{\mu}_{S_{(m_j)}} &\geqq \frac{1}{c_2+\varepsilon} \sum_{S_{(n^*-m_j)}} (R_{S_{(n^*-m_j)}\circ S_{(m_j)}})^{\mu} \\ &= \frac{1}{c_2+\varepsilon} \sum_{S_{(n^*-m_j)}} (R_{S_{(n^*)}})^{\mu} \end{aligned}$$

for $j=1, \cdots, Q$. Hence we have from (3.54) the following inequality:

$$(3.55)\qquad \sum_{j=1}^{Q} (R_{S_{(m_j)}})^{\mu} > \frac{1}{c_2+\varepsilon} \sum_{S_{(n^*-n_1)}} (R_{S_{(n^*)}})^{\mu} .$$

Since

$$\begin{aligned} \sum_{S_{(n^*-n_1)}} (R_{S_{(n^*)}})^{\mu} &= \sum_{S_{(n^*-n_1)}} \left(\frac{R_{S_{(n^*-n_1)}\circ S_{(n_1)}}}{R_{S_{(n_1)}}}\right)^{\mu} \times (R_{S_{(n_1)}})^{\mu} \\ &= \chi^{(\mu\,;\,T^*)}_{n^*-n_1} (S_{(n_1)}(\infty)) \times (R_{S_{(n_1)}})^{\mu}, \end{aligned}$$

we have from (3.55)

$$(3.56)\qquad \sum_{j=1}^{Q} (R_{S_{(m_j)}})^{\mu} > \frac{1}{c_2+\varepsilon} \chi^{(\mu\,;\,T^*)}_{n^*-n_1} (S_{(n_1)}(\infty)) \times (R_{S_{(n_1)}})^{\mu} .$$

Since $n^*-n_1 \geqq n_0$, we have from (3.52) the following:

$$(3.57)\qquad \sum_{j=1}^{Q} (R_{S_{(m_j)}})^{\mu} > \frac{c_1-\varepsilon}{c_2+\varepsilon} (R_{S_{(n_1)}})^{\mu} .$$

Hence we obtain from Corollary 2 the following relation:

$$\begin{aligned} &N\ (k^*k_0)^{-\frac{\mu}{2}} M_{\frac{\mu}{2}}(\Lambda(G^*)\cap B^{(2)}_{S_{(n_1)}}) \\ &\geqq \lim_{\delta\to 0} \Big[\inf_{\{F^{\delta/k_0}_{\tilde{n}_0}\}} \Big\{ \sum_{B^{(2)}_{S_{(m_j)}} \in F^{\delta/k_0}_{\tilde{n}_0}} (2R^2_{S_{(m_j)}})^{\frac{\mu}{2}} \Big\} \Big] \\ &= 2^{\frac{\mu}{2}} \cdot \frac{c_1-\varepsilon}{c_2+\varepsilon} R^{\mu}_{S_{(n_1)}} > 0. \end{aligned}$$

This contradicts the assumption $M_{\frac{\mu}{2}}(\Lambda(G^*))=0$.

q.e.d.

Then we could prove the main Theorem 3.

§4. Hausdorff dimension of $\Lambda(G^*)$ and Poincaré dimension of G^*.

1. In this section we shall consider the relation between the Hausdorff dimension of $\Lambda(G^*)$ and the Poincaré dimension of G^*. Now let us give the definitions.

Let Γ be a Kleinian group. Assume that $\infty\in\Omega(\Gamma)$ and ∞ is not fixed by any element of Γ other than the identity. The Hausdorff dimension $d(\Lambda(\Gamma))$ of $\Lambda(\Gamma)$ is defined as

$$(4.1) \qquad d(\Lambda(\Gamma)) = \inf \{\frac{\mu}{2} \mid M_{\frac{\mu}{2}}(\Lambda(\Gamma))=0\} .$$

The Poincaré dimension of Γ is

$$(4.2) \qquad P(\Gamma) = \inf \{\mu \mid \sum_{S\in\Gamma-\{id\}} (R_S)^{\mu} < +\infty\}$$

At first we shall prove the following theorem.

THEOREM 4. *Put $d(\Lambda(G^*))=\frac{1}{2}\mu^*$. Then it holds*

$$0 < M_{\frac{1}{2}\mu^*}(\Lambda(G^*)) < +\infty$$

Proof. If $M_{\frac{1}{2}\mu^*}(\Lambda(G^*))=\infty$, then we have from Theorem 3 that for any large $M>2$ there exists a positive integer $n_0=n_0(M)$ depending only on M such that

$$(4.3) \qquad \chi_{n_0}^{(\mu;T^*)}(z) > M$$

for any $T\in\mathscr{G}$ and any $z\in\mathrm{Int}(D_T)$. Since the function $\chi_{n_0}^{(\mu;T)}(z)$ is continuous for μ with respect to fixed n_0, z and T, we have from (4.3)

$$(4.4) \qquad \chi_{n_0}^{(\mu^*+\delta,T^*)}(z) > \frac{1}{2}M > 1$$

for any $T\in\mathscr{G}$ and any $z\in\mathrm{Int}(D_T)$. We have hrom (3.45)

$$\chi_{qn_0}^{(\mu^*+\delta,T)}(S(\infty)) > (\frac{1}{2}M)^q$$

for any $S(\infty)\in\mathrm{Int}(D_T)$ ($S\in G^*-\{id\}$) and any positive integer q. Hence we obtain $\lim_{q\to\infty}\chi_{qn_0}^{(\mu^*+\delta,T)}(S_{(l)}(\infty))=\infty$. It concludes $M_{\frac{1}{2}(\mu^*+\delta)}(\Lambda(G^*))=\infty$ from Theorem 3. This contradicts that $\frac{1}{2}\mu^*$ is the Hausdorff dimension. Hence we have $M_{\frac{1}{2}\mu^*}(\Lambda(G^*))<+\infty$.

If $M_{\frac{1}{2}\mu^*}(\Lambda(G^*))=0$, then for any small number $\varepsilon>0$ $(\varepsilon<\frac{1}{2})$, there exists a positive integer $n_0=n_0(\varepsilon)$ depending only on ε such that

$$(4.5)\qquad \chi_{n_0}^{(\mu^*;T)}(z) < \varepsilon$$

for any $T\epsilon\mathscr{G}$ and any $z\epsilon\mathrm{Int}\,(D_T)$. In the analogous way as the case of ∞, we can choose a positive number $\delta=\delta\,(\varepsilon)$ depending only on ε such that

$$(4.6)\qquad \chi_{n_0}^{(\mu^*-\delta,T)}(z) < 2\varepsilon < 1$$

for any $T\epsilon\mathscr{G}$ and any $z\epsilon\mathrm{Int}\,(D_T)$. By the use of (3.45) we have

$$\chi_{qn_0}^{(\mu^*-\delta,T)}(S(\infty)) < (2\varepsilon)^q$$

for any positive integer q, where $S(\infty)\epsilon\mathrm{Int}\,(D_T)$. Then we have $M_{\frac{1}{2}(\mu^*-\delta)}(\Lambda(G^*))=0$. This contradicts that $\frac{1}{2}\mu^*$ is the Hausdorff dimension. Hence we have $M_{\frac{1}{2}\mu^*}(\Lambda(G^*))>0$.

q.e.d.

2. At last we shall prove an important result.

TEOREM 5. *It holds* $d(\Lambda(G^*))=\frac{1}{2}P(G^*) < 2$.

In order to prove this theorem we need the following lemma.

LEMMA 15. *If* $M_{\frac{\mu}{2}}(\Lambda(G^*))=0$, *then* $\sum_{S\epsilon G^*-\{id\}}(R_S)^\mu<+\infty$.

Proof. Since $M_{\frac{\mu}{2}}(\Lambda(G^*))=0$, we have from Theorem 3 $\lim_{n\to\infty}\chi_n^{(\mu;T)}(z)=0$ uniformly on $\mathrm{Int}\,(D_T)$ for any $T\epsilon\mathscr{G}$. For any $\varepsilon>0$ there exists a positive integer $n_0=n_0(\varepsilon)$ depending only on ε such that $\chi_{n_0}^{(\mu;T)}(z)<\varepsilon$ for any $T\epsilon\mathscr{G}$ and any $z\epsilon\mathrm{Int}\,(D_T)$. If we take any integer $m_0>0$, then we have $\chi_{n_0}^{(\mu;T)}(S_{(m)}(\infty))<\varepsilon$ for any $m\geqq m_0$. Hence it holds from (3.3)

$$(4.7)\qquad \sum_{S_{(n_0)}}(R_{S_{(n_0)}\circ S_{(m)}})^\mu < \varepsilon\,(R_{S_{(m)}})^\mu.$$

Let us denote by L_m^μ the sum of terms $(R_{S_{(m)}})^\mu$ of all elements of the grade m of G^*. It can be easily seen

$$\sum_{S\in G^*-\{id\}} (R_S)^{\mu} = \sum_{m=1}^{\infty} L_m^{\mu} = \sum_{m=1}^{\infty} (\sum_{j=1}^{(2p-2)(2p-3)^{m-1}} (R_{S_{j,(m)}})^{\mu}).$$

In order to show the convergence of the series $\sum_{S\in G^*-\{id\}} (R_S)^{\mu}$, it suffices to prove the convergence of the series $\sum_{m=m_0}^{\infty} L_m^{\mu}$. We get from (4.7)

$$\sum_{m=m_0}^{\infty} L_m^{\mu} = \sum_{q=1}^{\infty} \sum_{m=m_0+(q-1)n_0}^{m_0+qn_0-1} L_m^{\mu}$$

$$\leqq (\sum_{m=m_0}^{m_0+n_0-1} L_m^{\mu}) \sum_{t=0}^{\infty} (\varepsilon)^t$$

$$=\frac{1}{1-\varepsilon} \sum_{m=m_0}^{m_0+n_0-1} L_m^{\mu}.$$

Since $\varepsilon>0$ is sufficiently small, we can conclude $\sum_{S\in G^*-\{id\}} (R_S)^{\mu} = \sum_{m=1}^{\infty} L_m^{\mu} < +\infty$. This completes the proof of this lemma.

q.e.d.

In [4] , Beardon and Maskit have proved the following important result.

PROPOSITION 4. *If Γ is a geometrically finite Kleinian group, then it holds* $d(\Lambda(\Gamma))\leqq \frac{1}{2}P(\Gamma)\leqq 2$.

3. Now let us give the proof of Theorem 5.

Proof of Theorem 5. From Proposition 4 and Lemma 15, it can be easily seen $d(\Lambda(G^*))=\frac{1}{2}P(G^*)\leqq 2$. It is well known that, if $\Omega(\Gamma)\neq\phi$, then $\sum_{S\in\Gamma-\{id\}} (R_S)^4<+\infty$ (see [3]). If $d(\Lambda(G^*))=2$, then Theorem 4 yields $0<M_2(\Lambda(G^*))<\infty$. But this contradicts the fact $\sum_{S\in G^*-\{id\}} (R_S)^4<\infty$. Thus we complete the proof of Theorem.

q.e.d.

References

[1] T. Akaza, Poincaré theta-series and singular sets of Schottky groups, Nagoya Math. J., **24** (1964), 43-65.

[2] T. Akaza, Local properties of the singular sets of some Kleinian groups, Tôhoku Math. J., **25** (1973), 1-22.

[3] A. F. Beardon and P. J. Nicholls, On classical series associated with Kleinian groups, J. London Math.

Soc., **5** (1972), 645-655.
[4] A. F. Beardon and B. Maskit, Limit points of Kleinian groups and finite-sided fundamental polyhedra, Acta Math., **132** (1974), 1-12.
[5] L. R. Ford, *Automorphic functions*, 2nd ed. Chelsea, New York, 1951.
[6] W. Magnus, *Non-euclidean tesselation and their groups,* Academic press. 1974.
[7] S. J. Patterson, The limit set of a Fuchsian group, Acta Math., **136** (1976), 241-273.
[8] T. Shimazaki and N. Ogi, On the limit sets of Inversion groups, Science Report of Kanazawa Univ., **13** (1968), 1-7.
[9] D. Sullivan, The density at infinity of a discrete group of hyperbolic motions, Inst. Hautes Etudes Sci. Publ. Math., **50** (1979), 171-202.

Appendix

Let G be a finitely generated and geometrically finite Kleinian group. The conjecture $d(\Lambda(G))=\frac{1}{2}P(G)$ was already stated in the Introduction of this paper.

Recently D. Sullivan announces in [1] that this is true for G and the proof will appear in [2] . And also it is reported that the Japanese mathematician M. Nakada proved this conjecture for G by the different method from Sullivan's.

[1] . D. Sullivan, Discrete conformal groups and measurable dynamics, Bulletin of the Amer. Math. Soc., **6** (1982), 57-73.
[2] . D. Sullivan, Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups, Acta Math. (to appear).