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Abstract. The ground state of the superradiant two level system is analyzed by means of the mean field theory. With the Emeljanov-Klimontovich Hamiltonian which involves the term quadratic in atomic dipoles, the ground state energy and the excitation spectrum in the superradiant phase are calculated and expressed in terms of the set of given parameters. The condensate photon density and the transverse electric polarization density are also obtained. It is emphasized that the superradiant phase is ferroelectric. The criterion for the stability of the superradiant ground state is estimated. It shows that to realize the equilibrium superradiance a rather high level of the density of atoms is required.

§ 1. Introduction

An interesting idea was first stated by Dicke[1] that in a system of identical atoms (molecules) coupled with a radiation field the cooperative motion of the transition dipole moments arises through the interaction with the spontaneous field, which, in turn, should give rise to an observable coherence of the field itself.

Later, Hepp and Lieb[2] analyzed the thermodynamics of the Dicke system with mathematical rigour, being based upon a somewhat trancated Hamiltonian i.e. so called the Dicke model. They proved that below a certain critical temperature the system undergoes a second order transition to the new "superradiant phase" which is accompanied with broken symmetries in the equilibilium state. Their analysis, though mathematically abstruct, was greatly simplified by making use of the truncated "rotating field" Hamiltonian, which made it possible to find the solutions for the thermodynamic problem being exact asymptotically in the limit of $N \to \infty$, $N/V = \rho$ kept constant.

Recently, attempts have been made to generalize the model to physically more realistic cases[3~6]. Here, we concentrate our attention to the model proposed by Emeljanov and Klimontovich. In this model the induction effect with an arbitrary value of the effective Lorentz field coefficient β is considered. With this extension the

Hamiltonian involves an additional dipole-dipole interaction, which drops out at the value of $\beta=1$, thus reduses to the H. L. model[2]. E.K.[6] discussed the dynamics of collective modes under equilibrium and non equilibrium conditions, emphasizing the appearence of the Goldstone mode in the superradiant phase.

In the present paper we investigate in detail the ground state of E.K. model[6] The superradiant ground state energy depends upon the densities of the condensate photons and the self-induced cooperative dipole moments through the renormalized effective coupling constant. The above densities are determined by the self-consistent energy minimum requirement and their expressions are found in terms of the set of the given parameters of the model. It is found that the superradiant state is energitically stable as long as the bare coupling is sufficiently strong. The renormalized coupling is proportional to the condensate field intensity multiplied by induction parameter β. If $\beta = 0$ the problem reduces to a weak coupling one because of the vanishing of the renormalized coupling. The excitation spectrum of the indivisual atom exhibits a Jahn-Teller type shift the amount of which depends on the condensate density. This brings out the net energy lowering of the superadiant state at the expence of the positive energy of the electric field.

There has been cotroversy about the time dependence of the condensate electric field. Elesin and Kopaev[7] studied this problem on the Dicke model and found that it is static. In the present work it is shown that this is true in our case. At the same time we see that the coherent atomic polarization is also static. Thus we confirm the ferroelectricity of superradiant phase as asserted by Elesin and Kopaev[7].

The present method of calculation is based on the u-v transformation to diagonalize the effective mean field Hamiltonian. The method has an advantage that it is simple to handle and at the same time makes it easy to draw a physical picture comparing with other Bose condensation problems such as the BCS or other. The diagonalization procedure is shown to be equivalent to the conventional variational calculation[8]. The quantities characterizing the superradiant state such as the averaged electric field and the atomic (off-diagonal) polarization should be interpreted as anomalous in the sense of well-known Bogoliubov's quasi-average.

A tentative estimation made for the superradiance criterion shows that a system with rather a high density of atoms is prefered in order to realize the equilibrium superradiance. The experiments thus far reported have been performed at lower densities where only non equiliblium emissions are interested.[9].

Our solution involves at the same time an additional trivial one where the quasi-averages are zero and corresponds to the normal state. It is energitically unstable in the region of strong coupling.

§ 2 Ground State of E.K. Model

Our starting point is the E.K. Hamiltonian[6]. The set of N identical atoms with

two discrete levels are contained within a volume V set in a resonant cavity. Only a single cavity mode with (k, e) is taken into account, where k is the wave vector and e is the direction of the polarization of the field. For simplicity the volume V is assumed to have a linear dimension much less than the wave length of the relevant field. Then, the system is described by the following Hamiltonian,

$$H = H_0 + H_{r-d} + H_{d-d}, \tag{1}$$

with

$$H_0 = \omega \sum_i \left(b_{i2}^+ b_{i2} - b_{i1}^+ b_{i1} \right) + cka^+ a,$$
$$H_{r-d} = -i \frac{g}{\sqrt{N}} \left(a - a^+ \right) \sum_i \left(d b_{i2}^+ b_{i1} + d^* b_{i1}^+ b_{i2} \right),$$
$$H_{d-d} = 2\pi (1-\beta) \frac{1}{V} \left\{ \sum_i \left(d b_{i2}^+ b_{i1} + d^* b_{i1}^+ b_{i2} \right) \right\}^2.$$

In (1), 2ω is the separation between the levels 2 and 1; $2\omega = E_2 - E_1 > 0$, b_{i1} and b_{i2} are the destruction operators of 1-and 2-level of an i-th atom for which the Fermi C.R. is assumed. H_{r-d} is the interaction between the field and the atoms in an electric dipole approximation, where g is a coupling constant; $g = (2\pi ck\rho)^{1/2}$, and d is the off-diagonal matrix element of the atomic transverse polarization; $d = (\mathbf{e} \cdot \mathbf{d}_{21})$, with $\mathbf{d}_{21} = \int \Psi_2^* \, e\boldsymbol{r} \, \Psi_1 d\tau$. The bare coupling constant of the problem is the product gd. Here, no rotating field approximation is assumed. H_{d-d} arises due to an induction effect for the field acting on each atom, where β is an arbitrary Lorentz type field coefficient. H_{d-d} is quadratic in the atomic polarization density,

$$\hat{P} = \frac{1}{V} \sum_i \left(d b_{i2}^+ b_{i1} + d^* b_{i1}^+ b_{i2} \right). \tag{2}$$

The Hamiltonian (1) reduces to the Dicke model[2] by trancating it to the rotating field interaction

$$H_{r-d} \longrightarrow -i \frac{g}{\sqrt{N}} \sum_i \left(a b_{i2}^+ b_{i1} - a^+ b_{i1}^+ b_{i2} \right) \tag{3}$$

and letting $\beta \to 1$, thus $H_{d-d} = 0$.

Now, if we rely on a mean field approximation, the Hamiltonian (1) is linearized into

$$H^{eff} = Nck \mid \alpha \mid^2 - 2\pi (1-\beta) V \mid P \mid^2$$
$$+ \omega \sum_i \left(b_{i2}^+ b_{i2} - b_{i1}^+ b_{i1} \right) + \lambda \sum_i \left(d b_{i2}^+ b_{i1} + d^* b_{i1}^+ b_{i2} \right), \tag{4}$$

where we have defined the renormalized coupling as

$$\lambda = 2Re \left\{ g\alpha + 2\pi (1-\beta) P \right\}, \tag{5}$$

and the quasi-averages as

$$\alpha = -i<a>/\sqrt{N} \quad , \quad P = <\hat{P}> \quad , \tag{6}$$

where $<\ >$ means the expectation value with respect to the ground state of H^{eff}.

The solution of (4) is easily obtained by means of the Bogoliubov transformation,

$$\beta_{i1} = ub_{i1} - vb_{i2} \quad , \quad \beta_{i2} = v^* b_{i1} + ub_{i2} \quad , \tag{7}$$

where u is chosen as real and

$$u^2 + |v|^2 = 1 \quad . \tag{8}$$

Then, the Hamiltonian (4) is written as

$$H^{\text{eff}} = H^{\text{d}} + H^{\text{od}} \quad , \tag{9}$$

with

$$H^{\text{d}} = \varepsilon \sum_i \left(\beta_{i2}^+ b_{i2} - b_{i1}^+ \beta_{i1}\right) \quad , \tag{10}$$

$$H^{\text{od}} = \left(-2\omega uv + \lambda d^* u^2 - \lambda d v^2\right) \sum_i \beta_{i1}^+ \beta_{i2} + h.c. \quad , \tag{11}$$

where

$$\varepsilon = \omega (u^2 - |v|^2) \quad . \tag{12}$$

In the same way the polarization density operator (2) is written as

$$\hat{P} = \frac{d}{V} \sum_i \left\{ 2uv \left(\beta_{i2}^+ \beta_{i2} - \beta_{i1}^+ \beta_{i1}\right) + u^2 \beta_{i2}^+ \beta_{i1} - v^2 \beta_{i1}^+ \beta_{i2} \right\} + h.c. \tag{13}$$

The parameters u and v are determined by the requirement to make the off-diagonal part of the Hamiltonian (11) vanish. This leads to

$$d = |d| e^{i\varphi} \text{ and } v = |v| h^{-i\varphi} \quad , \tag{14}$$

with an arbitrary phase φ, and

$$u^2 = \frac{1}{2} \left(1 + \omega / \sqrt{\omega^2 + \lambda^2 |d|^2}\right) \quad ,$$

$$|v|^2 = \frac{1}{2} \left(1 - \omega / \sqrt{\omega^2 + \lambda^2 |d|^2}\right) \quad , \tag{15}$$

where the positive roots should be taken for u and $|v|$. The arbitrary phase φ can be absorbed into the definition of the atomic wave function, so that from now on d and v will be taken real.

Substituting Eq (15) into Eq (12) we find the excitation spectrum of indivisual atom

$$\varepsilon = \sqrt{\omega^2 + (\lambda d)^2} \quad . \tag{16}$$

To calculate the ground state energy we note that within the subspace where the total number of atoms N is fixed, the following identities hold

$$\sum_i b_{i2}^+ b_{i2} = N - \sum_i b_{i1}^+ b_{i1} = \sum_i b_{i1} b_{i1}^+ \quad , \tag{17}$$

so that

$$\sum_i \left(b_{i2}^+ b_{i2} - b_{i1} b_{i1} \right) = -N + 2 \sum_i b_{i2}^+ b_{i2} \quad . \tag{18}$$

The same is true for β-operators. The ground state is defined as

$$\beta_{i2}^+ \beta_{i2} \mid E_s > \; = \beta_{i1} \beta_{i1}^+ \mid E_s > = 0 \quad . \tag{19}$$

Taking these equations into account the ground state expectation value of H^d is obtained as

$$E_s = N \left\{ ck \mid \alpha \mid^2 - 2\pi (1-\beta) \rho P^2 - \varepsilon \right\} \quad . \tag{20}$$

The next step is to minimize Eq (20) with respect to α and P. This leads to the following equations*

$$\left\{ \omega^2 + (\lambda d)^2 \right\}^{1/2} = \lambda d^2 g / ck \alpha \quad , \tag{21}$$

$$P = -\lambda d^2 \rho \left\{ \omega^2 + (\lambda d)^2 \right\}^{1/2} \tag{22}$$

It is also evident that α and P should be real. In view of Eq (15) we see that P given by Eq (22) is the same as the ground state expectation value of the operator (13), as it should be. At this stage we notice that Eq (21) is meaningless unless

$$\omega / 4\pi \rho d^2 \beta \equiv \tilde{\omega} \leqq 1 \quad . \tag{23}$$

is satisfied. Provided this condition is fulfilled Eqs (21) and (22) are solved to give

$$P = -ck \rho \alpha / g \quad , \tag{24}$$

and

$$\lambda = 2g \beta \alpha \quad . \tag{25}$$

The point $\beta = 0$ is very special because the two terms of r.s.h. of Eq. (5) are mutually cancelled out to zero. As a result the renormalized coupling vanishes so that the system remains normal at an arbitrary large value of the bare coupling gd. This is reason why the condition (23) can never be satisfied at $\beta = 0$.

Combining Eqs (24) and (25) with Eqs (21) and (22) we find the expressions for the

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* Note; The equation always contains another trivial solution that $\alpha = 0$, so that $u = 1$ and $v = 0$, corresponding the normal state.

condensate photon density and the electric polarization in terms of the set of the given model parameters as

$$\alpha^2=\frac{\omega}{2\beta ck}\frac{1-\tilde{\omega}^2}{\tilde{\omega}} \quad , \tag{26}$$

and

$$P=-\rho d\ (1-\tilde{\omega}^2)^{1/2}. \tag{27}$$

The right hand sides of Eqs (26) and (27) have meaning only in the region given by (23). Otherwise, only the trivial solution is permissible where $\alpha = 0$ and $P = 0$.

Now, by making use of Eqs (16), (24), (25) and (26) into Eq (20) one obtains the ground state energy. Here we write down the net energy lowering measured from the normal ground state energy $E_n=-N\omega$ as

$$\Delta = E_s-E_n=-N\omega\,(1-\tilde{\omega})^2/2\tilde{\omega} \quad , \tag{28}$$

which is negative in so far as the inequality of (23) holds. Thus we arrive at the conclusion that the superradiant ground state is stable if the bare coupling $gd \propto \sqrt{\rho d}$ is sufficiently strong and, otherwise, the system stays at the normal state.

Another physical content is the occupation number of the excited level 2. The population difference between the two levels is given by

$$W =\sum_i <b_{i2}^+ b_{i2}-b_{i1}^+ b_{i1}>=\sum_i (u^2-v^2)\ <\beta_{i2}^+\beta_{i2}-\beta_{i1}^+\beta_{i1}>. \tag{29}$$

By making use of Eqs (15), (21), and (25) and keeping in mind Eq(19) we have for (29)

$$W = -N\tilde{\omega} \quad , \tag{30}$$

hence, the occupation number of level 2

$$\sum <b_{i2}^+ b_{i2}> =\frac{1}{2}N\,(1-\tilde{\omega}),\ (>0) \tag{31}$$

Therefore, in the superradiant state we find the atoms excited at the level 2 with the fraction $\frac{1}{2}(1-\tilde{\omega})$.

§ 3 Variational Calculation for u and v

Now, let us return to the starting Hamiltonian (1). We apply the canonical transformation (7) on it to write it as

$$\begin{aligned} H =& cka^*a+\omega\sum_i\left\{(u^2-v^2)(\beta_{i2}^+\beta_{i2}-\beta_{i1}^+\beta_{i1})-2uv\,(\beta_{i1}^+\beta_{i2}+\beta_{i2}^+\beta_{i1})\right\} \\ &-i\frac{g}{\sqrt{N}}(a-a^+)\,d\sum_i\left\{2uv\,(\beta_{i2}^+\beta_{i2}-\beta_{i1}^+\beta_{i1})+(u^2-v^2)(\beta_{i1}^+\beta_{i2}+\beta_{i2}^+\beta_{i1})\right\} \\ &+2\pi(1-\beta)\frac{1}{V}\left[\sum_i\left\{2uv(\beta_{i2}^+\beta_{i2}-\beta_{i1}^+\beta_{i1})+(u^2-v^2)(\beta_{i1}^+\beta_{i2}+\beta_{i2}^+\beta_{i1})\right\}\right]. \end{aligned} \tag{32}$$

The ground state expectation value of (32) is

$$<H> = N\left\{ck\mid \alpha\mid^2-\omega(u^2-v^2)-2\mid d\mid \lambda(u)\,uv\right\}+O[N^0] \tag{33}$$

where

$$\lambda(u) = g(\alpha+\alpha^*)-4\pi(1-\beta)\rho\,uv. \tag{34}$$

Let us attempt to determine the parameters u and v by a variational method, that is,

$$\partial<H>/\partial u=0.$$

This leads to the following equation

$$2\omega\,uv+\left\{gd(\alpha+\alpha^*)-8\pi(1-\beta)\rho\,d^2uv\right\}(u^2-v^2)=0. \tag{35}$$

It is readily seen that the solution of Eq(35) is the same as Eq(15) and $\lambda(u)$ defined by Eq(34) is identical with λ given by Eq(5). Thus the present method of calculation is equivalent to the one in Sec. 2, as expected.

§ 4 Ferroelectricity of Superradiance

It was previously argued that in the superradiant phase the condensate electric field is static[7]. This interesting fact make it possible to interpret the superradiant state as a ferroelectricity. In addition to this interpretation we notice that the situation is very similar to the $2k^F$-phonon softening, so called the kohn anomaly, and the corresponding lattice distortion in the Peierls state observed in a (quasi-) one dimensional electron--phonon system[10], while the analogy of the mechanisms between the two cases is less clear.

Now, we examine that this is also the case in the present model. We show that not only the electric field but also the atomic polarization is static. The latter has been not yet explicitly mentioned. To this end let us consider the equation of motion for the operators a and P. Under the Hamiltonian (1) they lead to

$$i\dot{a} = cka + i\frac{g}{\sqrt{N}}VP\quad, \tag{36}$$

$$i\dot{\hat{P}} = 2\omega\,d\frac{1}{V}\sum_i\left(b_{i1}^{+}b_{i2}-b_{i2}^{+}b_{i1}\right) \tag{37}$$

On account of Eqs(6), (24) and (27) one finds the vanishing of the r.h.s. of Eq(36) and (37) at the ground state, hence

$$<\dot{a}> = 0 \quad\text{and}\quad <\dot{\hat{P}}> = 0, \tag{38}$$

while $<a>$ and $<P>$ themselves are non vanishing. Thus, the statement above mentioned is confirmed.

§ 5 Conclusion

In conclusion, we briefly examine the implication of the condition (23) for the superradiant state to be stable. The parameters in (23) are in the order of magnitude

$$\omega \sim e^2/a_0 \quad , \quad d \sim e\,a_0 \quad , \quad \beta \sim 0[1]$$

where a_0 is a length of an order of the atomic Bohr radius. Therefore. Eq(23) means that

$$4\pi\rho\, a_0^3 \quad \gtrsim \quad 1 \quad ; \tag{39}$$

that is, the linear dimension of the interatomic spacing must be of the order of or exceed the Bohr radius. At this point, one may argue that under such a level of density the direct interatomic interaction could not be negligible. We speculate, however, that it dose not alter the essential features above derived, whereas the condition (23) or (39) will be largely modified. The detailed investigation of this problem will be done in my future works.

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