On the Pointwise "o" Saturation Theorem for Positive Convolution Operators

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Kan-ichi SUZUKI*

Department of Mathematics, Faculty of Science, Kanazawa University
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Abstract　Saturation problems for positive convolution operators are well-known and widely investigated. In 1971, DeVore proved a pointwise "o" saturation theorem. DeVore's method yields somewhat more than the conclusion he obtained. It is shown that any sequence of positive convolution operators, which saturated, is pointwise "o" saturated.

1. Introduction

C^* denotes the space of 2π-periodic continuous functions with the supremum norm $\| \cdot \|$. Let (L_n) be a sequence of linear operators on C^*, given by the convolution formula

$$(1) \qquad L_n(f, x) = (f * d\mu_n)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) d\mu_n(t)$$

where $d\mu_n$ is a non-negative, even Borel measure on $[-\pi, \pi]$ with $\frac{1}{\pi} \int_{-\pi}^{\pi} d\mu_n(t) = 1$.

We consider the saturation of these operators. We say that (L_n) is satureted if there exitsts a sequence of positive numbers (ϕ_n) which converges to 0 such that

a)　for $f \in C^*$

$$\lim_{n \to \infty} \frac{\|f - L_n(f)\|}{\phi_n} = 0 \qquad \text{if and only if } f \text{ is constant};$$

b)　there exists a non-constant function $f_0 \in C^*$ such that $\|f_0 - L_n(f_0)\| = O(\phi_n)$.

The sequence (ϕ_n) is called the saturation order.

If we define the real Fourier-Stieltjes coefficients of $d\mu_n$ by $\rho_{k,n} = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos kt \, d\mu_n(t)$, the following theorem determines when (L_n) is saturated and its saturation order (DeVore [3], pp.56-58).

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* Present address : Turuoka Nishi Height School, Turuoka City, Yamagata Prefecture

THEOREM A.   *Let* (L_n) *be a sequence of operators of the form* (1). *A necessary and sufficient condition that* (L_n) *be saturated is that for some positive integer* m

(2) $$\lim_{n\to\infty} \frac{1-\rho_{k,n}}{1-\rho_{m,n}} = \psi_k > 0 \quad \text{for } k= 1, 2, 3,\cdots\cdots$$

In this case $(1-\rho_{m,n})$ *is a saturation order.*

In paticular, if (L_n) satisfies (2), we obtain that for $f \epsilon C^*$, $\| f- L_n(f) \| = o(1-\rho_{m,n})$ if and only if f is constant. DeVore [2] proved a pointwise "o" saturation theorem.

THEOREM B.   *Let* (L_n) *be a sequence of positive convolution operators of the form* (1), *which satisfies*

$$\lim_{n\to\infty} \frac{1-\rho_{k,n}}{1-\rho_{1,n}} = \psi_k > 0 \quad \text{for } k= 1, 2, 3,\cdots\cdots$$

If $f \epsilon C^*$ *then*

$$f(x)- L_n(f, x) = o_x(1-\rho_{1,n}) \quad \text{for each } x\epsilon[-\pi, \pi]$$

if and only if f *is constant on* $[-\pi, \pi]$.

DeVore's method of the proof of THEOREM B yields somewhat more than the conclusion he obtained. In fact we have next theorem.

THEOREM 1.   *Let* (L_n) *be a sequence of positive convolution operators of the form* (1), *where the Fourier coefficients of* $d\mu_n$ *satisfy* (2). *If* $f \epsilon C^*$ *then for each* $x\epsilon[-\pi, \pi]$

(3) $$f(x)- L_n(f, x) = o_x(1-\rho_{m,n})$$

if and only if f *is constant on* $[-\pi, \pi]$.

Combining THEOREM A and THEOREM 1, we can easily show next theorem.

THEOREM 2.   *Let* (L_n) *be a sequence of positive convolution operators of the form* (1), *which saturated with order* (ϕ_n). *If* $f \epsilon C^*$ *then for each* $x\epsilon[-\pi, \pi]$

$$f(x)- L_n(f, x) = o_x(\phi_n)$$

if and only if f *is constant on* $[-\pi, \pi]$.

2. Proof of THEOREM 1

The "if" part of the theorem is obvious. The proof of the "only if" part is based on a trigonometric analogue of the parabola technique of Bajsanski-Bojanic [1]. For this purpose we must prove two lemmas.

Let **m** be the set of all numbers $x\epsilon[0,2\pi]$, such that, for each neighbourhood I of x, we have $\int_I d\mu_n(t) \neq o(1-\rho_{m,n})$. If $\mathbf{m}=\phi$, with a compactness argument, we have

$\int_{-\pi}^{\pi} d\mu_n(t) = o(1 - \rho_{m,n})$. This contradicts to the hypothesis of $(d\mu_n)$. So **m** is nom-void.

LEMMA 1. *Let* $f \in C^*$ *be a function which satisfies* (3). *Then for each* $x_0 \in [-\pi, \pi]$, *with* $f(x_0) = \max \{f(x) ; x \in [-\pi, \pi]\}$, *and each* $t \in \mathbf{m}$, *we have* $f(x_0 + t) = f(x_0)$.

PROOF. Let assume $f(x_0 + x) < f(x_0)$ for some x_0, with $f(x_0) = \max \{f(x) ; x \in [-\pi, \pi]\}$, and some $x \in \mathbf{m}$, Then $I \equiv \{y \in [-\pi, \pi] ; f(x_0 + y) < \frac{1}{2}(f(x_0) + f(x_0 + x))\}$ is a neighbourhood of x and

$$\frac{1}{2}(f(x_0) - f(x_0 + x)) \int_I d\mu_n(t)$$

$$\leq \int_I (f(x_0) - f(x_0 + t)) \, d\mu_n(t)$$

$$\leq \int_{-\pi}^{\pi} (f(x_0) - f(x_0 + t)) \, d\mu_n(t)$$

$$= \pi (f(x_0) - L_n(f, x_0))$$

$$= o(1 - \rho_{m,n}).$$

This shows that $\int_I d\mu_n(t) = o(1 - \rho_{m,n})$ and thus this contradicts to $x \in \mathbf{m}$.

LEMMA 2. *If there exists a non-constant* $f \in C^*$, *which satisfies* (3), *then* **m** *is a finite set. Also, if* x *is any number in* **m** *then* $x = 2\pi\alpha$, *where* α *is some rational number.*

PROOF. First of all, we note some properties of numbers, Let (x) be the decimal place of a positive number x.

i) Let α be an irrational number. Then the set
$$\{(k\alpha) ; k = 1, 2, 3, \cdots \cdots\}$$
is dense in $[0,1]$.

ii) Let (α_n) be a sequence of distinct numbers in $[0,1]$, and converge to some rational number. Then the set
$$\{(k\alpha_n) ; n = 1, 2, 3, \cdots, k = 1, 2, 3, \cdots\}$$
is dense in $[0,1]$.

Suppose that (x_n) is a sequence of distinct numbers, each of which is in **m**. Choosing a subsequence, if necessary, we can assume $x_n \to x$ where $x \in [0, 2\pi]$. We write $x_n = 2\pi\alpha_n$ and $x = 2\pi\alpha$.

Let x_0 be any number in $[-\pi, \pi]$, where $f(x_0) = \max \{f(x) ; x \in [-\pi, \pi]\} \equiv M$. Then, by LEMMA 1, for each positive integers k and n, we have $f(x_0 + kx_n) = M$. By continuity of f, $f(x_0 + kx) = M$ for each positive integer k.

If α is irrational, the set $\{(k\alpha) ; k = 1,2,3,\cdots\cdots\}$ is dense in $[0,1]$ by i). Therefore the set of numbers kx taken modulo 2π, is dense in $[0,2\pi]$. Thus, in this case, $f = M$ on a set of numbers which is dense in $[x_0, x_0 + 2\pi]$ and therefore $f = M$ on $[x_0, x_0 + 2\pi]$. From periodidity we conclude that f is constant.

If α is rational, the set $\{(k\alpha_n)\,;\ n=1,2,3,\cdots,\ k=1,2,3,\cdots\}$ is dense in $[0,1]$ by ii). Therefore, we again obtain that f is constant.

This shows that $\mathbf{m}$ has no limit point in $[0,2\pi]$ and hence $\mathbf{m}$ must be a finite set.

Finally, if a number of the form $x=2\pi\alpha$ with α irrational were in $\mathbf{m}$ then, as we have mentioned before, $f(x_0+kx)=M$ for each positive integer k, so that $f=M$ on a set of points which is dense in $[x_0,\,x_0+2\pi]$. This gives that f is constant.

PROOF OF THEOREM B.

Let f be a function which satisfies (3) and suppose f is not constant. By subtracting a constant and considering the function $-f$ instead of f, if necessary, we can suppose that $f(-\pi)=f(\pi)=0$ and $M\equiv\max\,\{f(x)\,;\ x\in[0,2\pi]\}>0$. Then, it follows from LEMMA 2 that there is a positive integer N such that $\mathbf{m}\subseteq\{\dfrac{2k\pi}{N}\,;\ k=0,1,2,\cdots,N\}$. In addition let N be the smallest positive integer which satisfies the above. In this case, by using LEMMA 1, we see that for any real number x_0, with $f(x_0)=M$, and any integer k,

(4) $$f(x_0)=M=f(x_0+\frac{2k\pi}{N}).$$

The function $h_\alpha(x)\equiv-\alpha\sin^2\dfrac{N}{2}x+2M$, with $0<\alpha\leq M$, is $\geq f(y)$ for any $y\in[-\pi,\pi]$. For each $0<\varepsilon,\ \delta<\dfrac{\pi}{2N}$, we define

$$I_{\varepsilon,\delta}\equiv\bigcup_{k=-N}^{N}[\frac{2k\pi}{N}-\varepsilon,\ \frac{2k\pi}{N}+\delta]\cap[-\pi,\pi]$$

$$S_{\varepsilon,\delta}\equiv[-\pi,\pi]\backslash I_{\varepsilon,\delta}.$$

Now, we fix x_0. Then either

a) there exist some ε and some δ such that

$$A\equiv\max\,\{f(x_0+\frac{2k\pi}{N}-\varepsilon),\,f(x_0+\frac{2k\pi}{N}+\delta)\,;\ k=0,1,\cdots,N\}\,<M,$$

or

b) for each δ (or ε), there exists some integer k_0 such that

$$f(x_0+\frac{2k_0\pi}{N}+\delta)=M\ (\text{or}\ f(x_0+\frac{2k_0\pi}{N}-\varepsilon)=M).$$

CASE a)

Let $\alpha\equiv\min\,\{\dfrac{M-A}{\sin^2\dfrac{N}{2}\varepsilon+\sin^2\dfrac{N}{2}\delta},\ M\}$. By (4),

$$h_\alpha(\frac{2k\pi}{N})-f(x_0+\frac{2k\pi}{N})=2M-M=M.$$

On the other hand, by the definitions of A and α,

$$h_\alpha(\frac{2k\pi}{N}+\delta)-f(x_0+\frac{2k\pi}{N}+\delta)$$

$$= -\alpha \sin^2 \frac{N}{2}\delta + 2M - f(x_0 + \frac{2k\pi}{N} + \delta)$$

$$\geq -\frac{M-A}{\sin^2 \frac{N}{2}\varepsilon + \sin^2 \frac{N}{2}\delta} \sin^2 \frac{N}{2}\delta + 2M - A$$

$$> -(M-A) + 2M - A$$

$$= M.$$

Similarly

$$h_\alpha(\frac{2k\pi}{N} - \varepsilon) - f(x_0 + \frac{2k\pi}{N} - \varepsilon) > M.$$

These show that $C \equiv \min\{h_\alpha(x) - f(x_0 + x) ; x \in I_{\varepsilon,\delta}\}$ is assumed at a point y in the interior of $I_{\varepsilon,\delta}$. Therefore, for each $x \in I_{\varepsilon,\delta}$

$$h_\alpha(x) - C \geq f(x_0 + x)$$

and

$$h_\alpha(y) - C = f(x_0 + y).$$

Then

(5)
$$\int_I [h_\alpha(x) - h_\alpha(y)] \, d\mu_n(x-y)$$

$$\geq \int_{I_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)] \, d\mu_n(x-y).$$

Since the interior of $I_{\varepsilon,\delta} - y$ contains **m**, we obtain by the use of the compactness argument that

(6)
$$\int_{S_{\varepsilon,\delta}} d\mu_n(x-y) = \int_{S_{\varepsilon,\delta}-y} d\mu_n(x) = o(1 - \rho_{m,n}).$$

By (5), (6) and the fact

$$h_\alpha(x) - h_\alpha(y) = -\alpha \cos Ny \, \sin^2 \frac{N}{2}(x-y)$$

$$-\frac{\alpha}{2} \sin Ny \, \sin N(x-y)$$

we have

$$L_n(f, x_0 + y) - f(x_0 + y)$$
$$= \frac{1}{\pi}\int_{I_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)] \, d\mu_n(x-y)$$

$$+ \frac{1}{\pi}\int_{S_{\varepsilon,\delta}} [f(x_0+x) - f(x_0+y)] \, d\mu_n(x-y)$$

$$\leq \frac{1}{\pi} \int_{I_{\varepsilon,\delta}} [h_\alpha(x) - h_\alpha(y)] \, d\mu_n(x-y) + o(1-\rho_{m,n})$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} [h_\alpha(x) - h_\alpha(y)] \, d\mu_n(x-y) + o(1-\rho_{m,n})$$

$$= -\frac{\alpha}{\pi} \cos Ny \int_{-\pi}^{\pi} \sin^2 \frac{N}{2}(x-y) \, d\mu_n(x-y) + o(1-\rho_{m,n})$$

$$= -\frac{\alpha}{2} \cos Ny \ (1-\rho_{N,n}) + o(-\rho_{m,n}).$$

Finally, from (2), there exists some sequence $\{n_i\}_{i=1}^{\infty}$ such that

$$\lim_{i\to\infty} \frac{L_{n_i}(f, x_0+y) - f(x_0+y)}{1-\rho_{m,n_i}} \leq -\frac{\alpha}{2} \lim_{i\to\infty} \cos Ny \ \frac{1-\rho_{N,n_i}}{1-\rho_{m,n_i}}$$

$$= -\frac{\alpha}{2} \cos Ny \ \psi_N.$$

Since $\cos Ny > \cos \frac{\pi}{2} = 0, \alpha > 0$ and $\psi_N > 0$, we have

$$L_n(f, x_0+y) - f(x_0+y) \neq o(1-\rho_{m,n}).$$

This contradicts to (3) at the point $x = x_0 + y$.

CASE b)

From (4), we have for each integer k and each $\delta < \frac{\pi}{2N}$

$$f\left(x_0 + \frac{2k\pi}{N} + \delta\right) = M.$$

Therefore there exists some x_1 such that

$$f(x_1 + x) = M \quad \text{for each } x \in \bigcup_{k=0}^{N-1} \left[\frac{2k\pi}{N}, \frac{2k\pi}{N} + \frac{\pi}{2N}\right].$$

The other case of b), that is $f\left(x_0 + \frac{2k_0\pi}{N} - \varepsilon\right) = M$, we can also obtain the same conclusion.

Then we use $x_1 + \frac{\pi}{4N}$ in place of x_0 in the armument above. In the case a), we have contradiction. In the case b), there exists some x_2 such that

$$f(x_2 + x) = M \quad \text{for each } x \in \bigcup_{k=0}^{N-1} \left[\frac{2k\pi}{N}, \frac{2k\pi}{N} + \frac{3\pi}{4N}\right].$$

Now, we use $x_2 + \frac{3\pi}{8N}$ in place of x_0.

In generally, if case b) continues n times, there exists some x_n such that

$$f(x_n + x) = M \quad \text{for each } x \in \bigcup_{k=0}^{N-1} \left[\frac{2k\pi}{N}, \frac{(2k+1)\pi}{N} - \frac{\pi}{2^n N}\right].$$

Therefore, if case b) continues infinitely, there exists some $z \in [0, \frac{\pi}{N}]$ such that

$$f(x) = M \quad \text{for each } x \in I_0 \equiv \bigcup_{k=-N}^{N} [z + \frac{2k\pi}{N}, \; z + \frac{(2k+1)\pi}{N}] \cap [-\pi, \pi].$$

If we let $g(x) \equiv M - f(x)$,

$$g(x) \geqq 0 \quad \text{on } [-\pi, \pi]$$
$$g(x) = 0 \quad \text{on } I_0$$
$$g(\pm\pi) = M > 0.$$

Let $x' \in [-\pi, \pi]$ be $M' \equiv \max \{g(x); x \in [-\pi, \pi]\} = g(x')$. Now we can wark with g, M' and x', in places of f, M and x_0 in the argument above. Then either case a) is reached, or case b) continues infinitely. In the latter case, we can conclude

$$g(x) = 0 \quad \text{on } I_0$$
$$g(x) = M' \quad \text{on } [-\pi, \pi] \backslash I_0.$$

This contradicts to the continuity of g.

From the argument above, in any cases, we have the desired contradiction. So (3) implies $f \equiv$ constant.

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