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Abstract By the help of the high-temperature-expansion method we investigate the nonlinear critical relaxations in the kinetic Ising model. The associated exponents are estimated by means of the ratio method. The results are consistent with the dynamic-scaling predictions.

1. Introduction

The phenomenon of critical slowing down which means that the relaxation time of a physical quantity becomes much longer as the critical point is approached, is one of general behavior characteristic of the dynamic critical phenomena.[1]

In the past, the critical singularities of the relaxation times for the *linear* and *nonlinear* dynamic response had been asserted to be identical in ergodic systems by an intuitive expectation and a few verifications.[2] The intuitive expectation was that the relaxation time has to be determined mainly by the final (linear) stage of the relaxation, so that the difference between critical singularities of each relaxation time is negligeble, because an ergodic system approaches much more closely its equilibrium situation as time increases.[2] A few verifications for this were the results of the high-temperature--expansion method[2),3)] and the Monte Carlo method[4)] in the two-dimensional kinetic Ising model.[5)] These results showed that $\Delta_M^{(l.)} \simeq \Delta_M^{(n.l.)}$, where $\Delta_M^{(l.)}$ and $\Delta_M^{(n.l.)}$ are the critical exponents of the linear and nonlinear critical slowing down for the order parameter (magnetization M), respectively.

Recently, it was found that the singularities of the linear and nonlinear relaxation time can be different even in ergodic systems by using the mean field approximation (MFA) for the kinetic Ising model.[6)] That is, it was found that such a simple contrary evidence exists. The differences between $\Delta_M^{(l.)}$ and $\Delta_M^{(n.l.)}$ has been confirmed for other complicated cases[7),8),9)] (different lattices with short-range interactions). It was also reported that in the high-temperature series expansion[2)] for the two-dimensional kinetic Ising model there is an algebraic error and the estimate of $\Delta_M^{(n.l.)}$ is in error.[8),9)]

Furthermore, on the basis of scaling analysis[10),11)] it was shown that

$$\Delta_M^{(l.)} - \Delta_M^{(n.l.)} = \beta, \tag{1}$$

for purely dissipative systems, where β is the critical exponent of the order parameter. A simple physical interpretation of (1) is given in Refs. 11) and 12). It is natural that the difference $\Delta_M^{(l.)} - \Delta_M^{(n.l.)}$ had been overlooked in the two-dimensional kinetic Ising model, since β is very small in this case ($\beta=1/8$). The above-mentioned exponent estimates[7),8),9)] also support the scaling prediction (1). We can also consider the nonlinear critical relaxation of *the energy* in the kinetic Ising model. For the energy E, the scaling prediction becomes[11)]

$$\Delta_E^{(l.)} - \Delta_E^{(n.l.)} = 1-\alpha. \tag{2}$$

The exponent $\Delta_E^{(n.l.)}$ will be evaluated in this report. Here α denotes the critical exponent of the specific heat ($\alpha=0$ for the two-dimensional case).

Section 2 contains the formulation of the kinetic Ising model and the definition of the relaxation time. In § 3, we discuss briefly the high-temperature-expansion method and report the results for the nonlinear critical exponents calculated by the high-temperature-expansion method. Summary and discussions are given in § 4.

2. Kinetic Ising model and relaxation time

The kinetic Ising model is a model of spin system interacting with a heat bath that makes spins flip spontaneously. The system at time t is described by the probability $P(\sigma_1, \cdots, \sigma_N, t)$ to find the spins in the state $(\sigma_1, \cdots, \sigma_N)$, where $\sigma_n=\pm1$ are spin variables. The probability $P(\sigma_1, \cdots, \sigma_N, t)$ is assumed to be governed by the following master equation:

$$\frac{d}{dt}P(\sigma_1, \cdots, \sigma_N, t) = -\sum_j W_j(\sigma_j)P(\sigma_1, \cdots, \sigma_N, t) + \sum_j W_j(-\sigma_j)P(\sigma_1, \cdots, -\sigma_j, \cdots, \sigma_N, t), \tag{3}$$

where the transition probability $W_j(\sigma_j)$ is assumed to be, for zero external fields

$$W_j(\sigma_j) = (1/2)\,(1 - \sigma_j \tanh K\sum_k \sigma_k) \tag{4}$$

with $\sum_k$ denoting the sum over the nearest-neighbors of the j-th spin, and $K=J/kT$. Here the time scale is chosen to be unity.[3)] Eq. (4) is the simplest assumption consistent with the detailed balance condition[5)] for the Hamiltonian $H=-J\sum_{pair}\sigma_i\sigma_j$.

When the probability $P(\sigma_1, \cdots, \sigma_N)$ is determined by Eq. (3), the time evolution of a physical quantity $X=X(\{\sigma\})$ can be written as

$$X(t) = \sum_{\{\sigma\}} P(\sigma_1, \cdots, \sigma_N, t)\, X(\{\sigma\}), \tag{5}$$

where the sum $\sum_{\{\sigma\}}$ is over all possible configurations of spins. If the physical quantity X vanishes in the final equilibrium state, namely, $X(\infty)=0$, the nonlinear relaxation time is defined by[2)]

$$\tau_X^{(n.l.)}=\int_0^\infty \frac{X(t)}{X(0)}\,dt \tag{6}$$

with $X(0)$ being finite and sufficiently large.[2),6)] (In the case of the energy relaxation, we can regard $E(\infty)$ as the zero point energy.) If we take the limit $X(0)\longrightarrow 0$ in Eq. (6), we obtain the usual linear relaxation time :[6),10)]

$$\tau_X^{(l.)}=\lim_{X(0)\to 0}\int_0^\infty \frac{X(t)}{X(0)}\,dt=\int_0^\infty \frac{<X(t)X(0)>_\infty}{<X^2>_\infty}\,dt, \tag{7}$$

where $\langle\cdots\rangle_\infty$ is the equilibrium average in the final state ($t=\infty$). Thus the critical exponents appearing in Eqs. (1) and (2) are defined by

$$\tau_X^{(n.l.)}\sim \varepsilon^{-\Delta_X^{(n.l.)}}, \tag{8}$$

$$\tau_X^{(l.)}\sim \varepsilon^{-\Delta_X^{(l.)}} \tag{9}$$

with $\varepsilon=(T/T_c)-1$, T_c being the critical temperature. ($X=M$ for the order parameter, E for the energy, etc.) In the MFA, the exponents $\Delta_X^{(l.)}$, $\Delta_X^{(n.l.)}$ can be calculated directly from the evolution equation for the quantity X.[6)] In other cases such as the systems with short-range interactions, the high-temperature-expansion method[2),3),13)] is helpful. We will quote it briefly in the next section.

3. High-temperature series expansion

If we rewrite the master equation (3) as

$$-\frac{d}{dt}P(\sigma_1,\cdots,\sigma_N,t)=-\Gamma P(\sigma_1,\cdots,\sigma_N,t), \tag{10}$$

and introduce a function $\phi(t)$ by $P(\sigma_1,\cdots,\sigma_N,t)=\phi(t)\,P(\sigma_1,\cdots,\sigma_N,\infty)$, then we obtain

$$\phi(t)=e^{-Lt}\phi(0), \tag{11}$$

where $L\phi=P(\infty)^{-1}\Gamma\phi P(\infty)$, or more explicitly

$$L=\sum_j W_j(\sigma_j)\,(1-P_j) \tag{12}$$

with P_j being the spin-flip operator: $P_j\sigma_k=-\sigma_j\delta_{jk}+\sigma_k(1-\delta_{jk})$.

By the help of this Liouvillean-like operator L, the relaxation times (6) and (7) can be written as

$$\tau_X^{(n.l.)}=\frac{<L^{-1}X>_i}{<X>_i}, \tag{13}$$

$$\tau_X^{(l.)}=\frac{<XL^{-1}X>_\infty}{<X^2>_\infty}, \tag{14}$$

where $\langle\cdots\rangle_i$ denotes the average in the initial ensemble ($t=0$). Following Suzuki[2)], we consider the case where the initial state is completely ferromagnetic ($T=0$, or infinite external fields).

Since we can write the operator L as L_0-L^*, where L^* contains higher-order

terms with respect to $\varkappa=\tanh K$ ($\varkappa\to 0$ for $T\to\infty$), the term L^{-1} can be expanded in the resolvent form:

$$L^{-1} = L_0^{-1} + L_0^{-1}L^*L_0^{-1} + L_0^{-1}L^*L_0^{-1}L^*L_0^{-1} + \cdots, \tag{15}$$

Thus, the relaxation times are expanded in powers of $\varkappa$ as

$$\tau^{(n.l.)} = \sum_n a_n\varkappa^n, \quad \tau^{(l.)} = \sum_n b_n\varkappa^n. \tag{16}$$

If the coefficients a_n, b_n are known, we can estimate the approximate successive exponents Δ_n by using the ratio method (or other approximant technics).

In this report we consider the nonlinear-relaxation critical exponents for the order parameter[7),8)] (in 2 and 3 dimensions) and for the energy (in 2 dimensions). The calculation of the coefficients in Eq. (16), which reduces to counting problem on a lattice, is lengthy and tedious, and we report only the final results. The coefficients a_n and the relevant exponents Δ_n are given in Table I~III for each case. From these results, we conjecture that

$$\Delta_M^{(n.l.)} \sim 1.85 \text{ (for 2 dimensions)},$$
$$\Delta_M^{(n.l.)} \sim 1.05 \text{ (for 3 dimensions)},$$
$$\Delta_E^{(n.l.)} = 0.5\sim 1.0 \text{ (for 2 dimensions)}.$$

Table I. The coefficients a_n and the relevant exponents Δ_n in successive approximations for the order-parameter relaxation time on a square lattice. The critical temperature is given by $\varkappa_c=\sqrt{2}-1$.

n	a_n	Δ_n
0	1	
1	4	1.656
2	16	2.312
3	148/3	1.825
4	416/3	1.655
5	10444/27	1.775
6	433264/405	1.871

Table II. The coefficients a_n and the exponents Δ_n for the order-parameter relaxation time on a cubic lattice. $\varkappa_c\simeq 0.2182$.

n	a_n	Δ_n
0	1	
1	6	1.308
2	36	1.616
3	518/3	1.137
4	776	1.081
5	17606/5	1.054
6	2174324/135	1.017

Table III. The coefficients a_n and the exponents Δ_n for the energy relaxation time on a triangular lattice. $\kappa_c \simeq 0.2679$.

n	a_n	Δ_n
0	0.5	
1	2.5	1.340
2	14.0	2.001
3	52.0	0.986
4	167.0	0.442
5	561.167	0.501
6	2075.646	0.947

Table IV. The exponents of the linear and nonlinear relaxation time for the order parameter in the kinetic Ising model.

cases	2 dimensions ($\beta=1/8$)		
	Yahata and Suzuki[a]	Suzuki and Ikeda[b]	Rácz and Collins[c]
$\Delta_M^{(l.)}$	2.0 ± 0.05	———	2.125 ± 0.01
$\Delta_M^{(n.l.)}$	———	~1.85	1.95 ± 0.15

	2-D	3 dimensions ($\beta\simeq0.31$)		MFA ($\beta=1/2$)
	Stoll *et al.*[d]	Yahata[e], Ikeda[f]	Rácz and Collins[c]	Rácz[g]
$\Delta_M^{(l.)}$	1.85 ± 0.10	~1.4	1.32 ± 0.03	1
$\Delta_M^{(n.l.)}$	1.85 ± 0.10	~1.05	———	1/2

a) Ref.3). (high-temperature series expansion, ratio method)
b) Ref.8). (〃)
c) Ref.9). (high-temperature series expansion, Padé approximation)
d) Ref.4). (Monte Carlo method)
e) Ref.13). (high-temperature series expansion, ratio method)
f) the latter in Ref.7). (〃)
g) Ref.6). (exact calculation)

For the sake of clarity, we collect in Table IV the exponent estimates of the linear and nonlinear critical relaxation of the order parameter so far available. It is seen that in the kinetic Ising model the scaling prediction (1) satisfactorily holds true. In the two-dimensional case, however, it is difficult to conclude that $\Delta_M^{(l.)} > \Delta_M^{(n.l.)}$ and that the difference $\Delta_M^{(l.)} - \Delta_M^{(n.l.)}$ is certainly equal to β, because β is very small in this case. Namely, a definite conclusion may not be obtained in the two-dimensional case.[8),9)] The result for the 3-dimensional case supports more convincingly the scaling prediction (1).

Since the successive exponents for the energy relaxation show rather erratic behaviour, we can not definitely conclude that $\Delta_E^{(n.l.)} \sim 1.0$ ($1-\alpha=1$ and $\Delta_E^{(l.)} \simeq 2.0$ in this case[13)]). Then, we only mention that the result for $\Delta_E^{(n.l.)}$ may not be inconsistent

with (2).

4. Summary and discussions

We have investigated the high-temperature series expansions for the nonlinear relaxation time in the kinetic Ising model, and estimated the associated exponents. The estimates have been performed for the case of the order-parameter relaxation in two and three dimensions[7),8)] and of the energy relaxation in two dimensions. These results support the scaling predictions (1) and (2). However, in order to obtain a definite conclusion, especially for two-dimensional case ($\Delta_M^{(n.l.)}$ and $\Delta_E^{(n.l.)}$), it will be needed to calculate the high-temperature series expansions up to order higher than the order obtained so far.[8)] These calculations will be reported by other authors.[14)]

We next refer to the scaling theory for the nonlinear critical relaxation[10),11)], in which a *scaled* equation of motion has been solved in order to analyze the relaxation time. It should be remarked that in the scaled equation of motion[10),11)] there are no coupling terms between different wave vectors, for example, $\psi_{k=0}$ and $\psi_{k'\neq 0}$ with ψ being the order parameter. The effect of such coupling terms may not be clear at this stage.

In order to clarify the effect of coupling terms, and to obtain a definite conclusion in the two-dimensional case, the renormalization group approach[15)] will be helpful. We note that the *perturbational* renormalization group approach by Suzuki and Tanaka[16)] holds even in the nonlinear regime. This approach is now under consideration.

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