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On a necessary condition for the principal circle groups to be discrete

Harushi FURUSAWA* and Tohru AKAZA

Department of Mathematics, Faculty of Science, Kanazawa University

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1. Let Γ be the principal circle group whose elements make the unit disc or the upper half plane invariant. Here we consider only the group whose elements act on the unit disc $D=\{z\,;|z|<1\}$. The unit circle $Q=\{z\,;|z|=1\}$ is called the principal circle.

Some results about the discreteness of the group of the linear transformations are known as in the following.

Proposition 1 (H. Shimizu [4]). *Let* $A(z)=(az+b)\,(cz+d)^{-1}$ *with* $a, b, c, d \in \mathfrak{C}$ *and* $ad-bc=1$. *Let* $T(z)=z+1$. *Then the group generated by A and T is not discrete if* $0<|c|<1$.

The following result is due to S. Lauritzen ([2]).

Proposition 2. *Let G be a nonabelian real group consisting of the identity and hyperbolic elements. Then G is discrete.*

But it seems open under what conditions the group containing the elliptic elements is discrete. So we shall give the necessary condition for the principal circle group with this property to be discrete.

2. The purpose of this paper is to prove the following theorem.

Theorem. *Let* $V(z)=(az+\bar{c})\,(cz+\bar{a})^{-1}$ *with* $a, c \in \mathfrak{C}$ *and* $|a|^2-|c|^2=1$ *be the transformation which preserves the unit disc D. Let* $F(z)=e^{\frac{2\pi i}{\nu}}z$ $(\nu\geqq 3)$ *be the elliptic transformation with order* ν. *Then the group* $\Gamma=<E, V>$ *generated by E and V is not discrete if* $0<|c|\leqq 1/\sqrt{15}$.

Remark 1. We shall find from this theorem the following fact: if the fuchsian

* Present address: Kanazawa Womnen's College.

group Γ' preserving the unit disc D contains the elliptic element $E(z)=e^{\frac{2\pi i}{\nu}} z$ $(\nu \geqq 3)$, the radii of the isometric circles of the elements of Γ' except the elements of the subgroup $\Gamma'(0)=\{V;\ V\in\Gamma',\ V(0)=0\}$ are smaller than $\sqrt{15}$.

The proof of this theorem is divided into two cases with respect to the order ν of E : (i) $\nu\geqq 7$, (ii) $3\leqq\nu\leqq 6$.

3. Proof of the case (i).

In this case Theorem is proved directly from the following lemma.

Lemma 1. *Let* $V(z)$, $E(z)$ $(\nu\geqq 7)$ *be the transformations defined in Theorem. Set* $V_0=V$ and $V_{n+1}=V_n\circ E\circ V_n^{-1}$ *for* $n\geqq 0$, $n\in Z$. *Then it holds* $V_n\to E$ *as* $n\to\infty$, *if* $0<|c|<\sqrt{(2\sin\frac{\pi}{\nu})^{-2}-1}$.

Proof. It is clear that the sequence $\{V_n\}$ is distinct. We write

$$V_n=\frac{a_n z+\bar{c}_n}{c_n z+\bar{a}_n},\ a_n,\ c_n\in\mathbb{C},\ |a_n|^2-|c_n|^2=1,$$

and we compute

$$\begin{aligned} V_{n+1}&=\begin{pmatrix} a_n, & \bar{c}_n \\ c_n, & \bar{a}_n\end{pmatrix}\begin{pmatrix} e^{i\pi\nu^{-1}}, & 0 \\ 0 &, e^{-i\pi\nu^{-1}}\end{pmatrix}\begin{pmatrix} \bar{a}_n, & -\bar{c}_n \\ -c_n, & a_n\end{pmatrix} \\ &=\begin{pmatrix} |a_n|^2 e^{i\pi\nu^{-1}}-|c_n|^2 e^{-i\pi\nu^{-1}}, & * \\ 2i\,a_n c_n \sin\frac{\pi}{\nu}, & * \end{pmatrix}\end{aligned}\tag{1}$$

We shall get from (1)

$$|c_n|^2(1+|c_n|^2)(2\sin\tfrac{\pi}{\nu})^2=|c_{n+1}|^2.\tag{2}$$

Now we can show by induction

$$(2\sin\tfrac{\pi}{\nu})^{-2}-1>|c_n|^2,\quad (n=0,1,2,\cdots).\tag{3}$$

It is trivial for n=0 from the assumption of Lemma. Suppose that (3) holds for $n=k$. Using (2) and (3) we have

$$|c_{k+1}|^2=|c_k|^2(1+|c_k|^2)(2\sin\tfrac{\pi}{\nu})^2<(2\sin\tfrac{\pi}{\nu})^{-2}-1.$$

Hence from (2) and (3) we have also

$$|c_n|>|c_{n+1}|\qquad (n=0,1,2,\cdots).\tag{4}$$

Thus we obtain from (2) and (4)

$$(5) \qquad |c_n|^2(1+|c_0|^2)\left(2\sin\frac{\pi}{\nu}\right)^2 > |c_{n+1}|^2.$$

Since $k_0=(1+|c_0|^2)\left(2\sin\frac{\pi}{\nu}\right)^2<1$, it is clear that $|c_n|\longrightarrow 0$ for $n\longrightarrow\infty$. Therefore we can find from (1) and $|a_n|^2-|c_n|^2=1$ that $\lim_{n\to\infty} V_n=E$. q.e.d.

Using this lemma we can easily prove the case (i). Since it holds $\frac{\sqrt{15}}{8}=0.4841\cdots > \sin\frac{\pi}{7}=0.4337\cdots$, we have $\sqrt{\left(2\sin\frac{\pi}{\nu}\right)^{-2}-1} > \sqrt{\left(2\sin\frac{\pi}{7}\right)^{-2}-1} > \frac{1}{\sqrt{15}}$.

4. Proof of the case (ii).

This is the case in which Γ contains an elliptic element $E(z)=e^{\frac{2\pi i}{\nu}}z$ of order $\nu=3, 4, 5, 6$. Suppose that the group $\Gamma=\langle E, V\rangle$ is discrete. Then it is well known that there exists an element $W=\begin{pmatrix}\tilde{a}, & \bar{\tilde{c}}\\ \tilde{c}, & \bar{\tilde{a}}\end{pmatrix}(|\tilde{a}|^2-|\tilde{c}|^2=1)$ of Γ such that it holds $|c|\geqq|\tilde{c}|>0$ for all element $U=\begin{pmatrix}a, & \bar{c}\\ c, & \bar{a}\end{pmatrix}$ of $\Gamma=\langle E, V\rangle$ which do not fix the infinity ([3], p. 19). The isometric circles $I(W)$ and $I(EWE^{-1})$ have the same radii $R_W=R_{EWE^{-1}}=1/|\tilde{c}|$ with centers $g_W=-\tilde{a}/\tilde{c}$ and $g_{EWE^{-1}}=(-\tilde{a}/\tilde{c})\,e^{\frac{2\pi i}{\nu}}$, respectively.

At first we shall prove that $I(W)$ and $I(EWE^{-1})$ intersect to each other.

If they do not intersect, it holds $R_W=|\tilde{c}|^{-1}\leqq\tan\frac{\pi}{\nu}$ $(3\leqq\nu\leqq6)$, (See Fig. 1). Hence we have $|c|\geqq|\tilde{c}|\geqq\left(\tan\frac{\pi}{\nu}\right)^{-1}\geqq\left(\tan\frac{\pi}{3}\right)^{-1}=\frac{1}{\sqrt{3}}=0.577\cdots$.

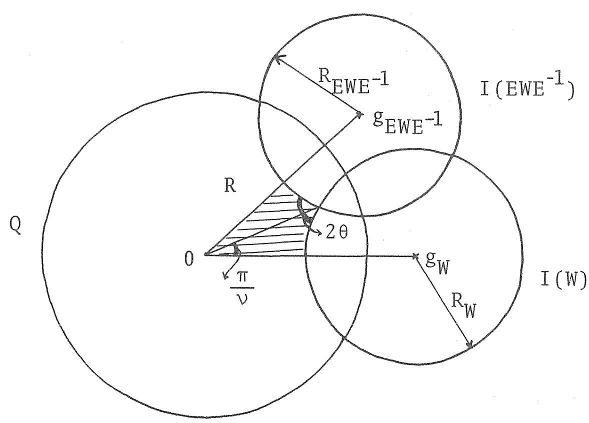

Fig. 1.

On the other hand we have from the hypothesis of Theorem $0<|c|\leqq\frac{1}{\sqrt{15}}=0.258\cdots$. This is the contradiction. Thus we may assume that $I(W)$ and $I(EWE^{-1})$ intersect to each other under $0<|c|\leqq\frac{1}{\sqrt{15}}$.

5. Denote by *Ext* $(I(W))$ the exterior of $I(W)$ and set $R=\{z\,;\ \arg(g_W)<\arg z \leqq \frac{2\pi}{\nu}+\arg(g_W)\} \cap \{Ext\, I(W)\} \cap \{Ext\, I(EWE^{-1}))\}$ (shadow part of Fig. 1). Further we denote by $\sigma(S)$ and $N(\Gamma)$ the Poincaré (non-euclidean) area of the measurable set S and the normal polygon of Γ, respectively. Then it is obvious that $\sigma(R) \geqq \sigma(N(\Gamma))$.

It is well known that the Poincaré area of the normal polygon N of the fuchsian group can be computed by using the *Gauss-Bonnet* formula ([1], [3]). Suppose that N has $2n$ sides with no free sides. Associate with each cycle a number l, which is 1 for an accidental cycle, the order of a generator of the cycle for an elliptic cycle and ∞ (that is, $1/l=0$) for a parabolic cycle. With this definition, the value of $\sigma(N)$ is given by

$$(6) \qquad 2\pi\left(n-1-\Sigma\frac{1}{l}\right),$$

the sum being extended over all cycles. Using (6), we have

$$(7) \qquad \sigma(N(\Gamma)) \geqq \begin{cases} \dfrac{\pi}{21}, & (\nu=3) \\[2ex] \dfrac{\pi}{10}, & (\nu=4,\ 5) \\[2ex] \dfrac{\pi}{6}, & (\nu=6) \end{cases}$$

Here the constants of the right hand side of the above inequality are the minimum values of $\sigma(N(\Gamma))$, when Γ has the elliptic element of order ν ([3]).

Denote by 2θ the intersecting angle of $I(W)$ and $I(EWE^{-1})$ (Fig. 1). Since $\sigma(R)=2\pi-\pi-\frac{2\pi}{\nu}-2\theta$, we obtain easily from (7) and $\sigma(R) \geqq \sigma(N(\Gamma))$

$$(8) \qquad \pi-\frac{2\pi}{\nu}-2\theta \geqq \begin{cases} \dfrac{\pi}{21}, & (\nu=3) \\[2ex] \dfrac{\pi}{10}, & (\nu=4,\ 5) \\[2ex] \dfrac{\pi}{6}, & (\nu=6) \end{cases}$$

Hence we have the following estimate:

$$(9) \qquad \theta \leqq \begin{cases} \dfrac{\pi}{7}, & (\nu=3) \\[2ex] \dfrac{\pi}{5}, & (\nu=4) \\[2ex] \dfrac{\pi}{4}, & (\nu=5,\ 6) \end{cases}$$

6. In order to develop the further discussion, we have to seek for the relation among the angles $\frac{\pi}{\nu}$, θ and the radius $R_W=\frac{1}{|\tilde{c}|}$. For this purpose we shall give the following lemma (See p. 510 of [5]).

Lemma 2. *There is a relation among the angles* $\frac{\pi}{\nu}$, θ *and the radius* $R_W=\frac{1}{|\tilde{c}|}$ as in the following:

$$(10) \qquad (R_W)^{-2} = -1+\frac{1+\cot^2 \frac{\pi}{\nu}}{1+\tan^2\theta}.$$

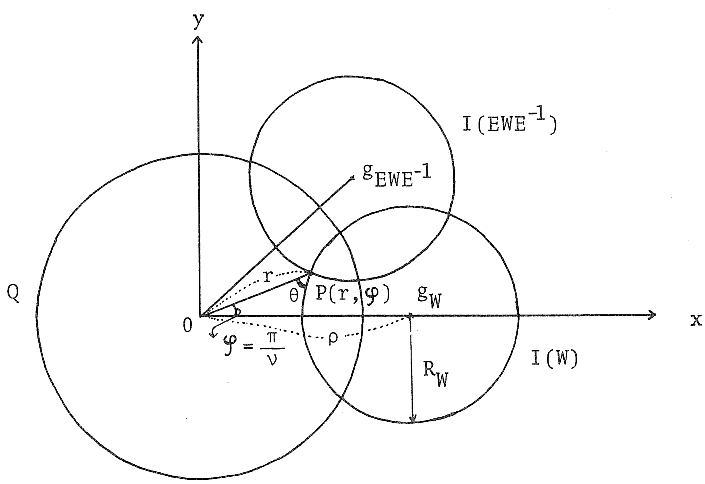

Fig. 2.

Proof. By a suitable linear transformation preserving the unit disc, we may assume that $I(W)$ is orthogonal to the real axis (Fig. 2). For the brevity we set $\varphi=\frac{\pi}{\nu}$, $a=R_W$ and $\rho=g_W$ (>0). Let $(x-\rho)^2+y^2=a^2$ be the equation of $I(W)$. Since $I(W)$ is orthogonal to $Q:|z|=1$, then it holds $\rho^2=a^2+1$, so that the equation of $I(W)$ in the polar coordinate is

$$(11) \qquad r^2-2r\rho\cos\varphi+1=0.$$

Hence we have from (11)

$$(12) \qquad \frac{dr}{d\varphi}=\frac{r\rho\sin\varphi}{\rho\cos\varphi-r}\quad.$$

Since θ is the angle between the tangent of $I(W)$ at P (r, φ) and the vector $\overrightarrow{\mathrm{OP}}$, we obtain from (12)

$$(13) \qquad \tan\theta= r/\frac{dr}{d\varphi}=\frac{\sqrt{\rho^2\cos^2\varphi-1}}{\rho\sin\varphi}$$

Hence from (13) and $\rho^2 = a^2 + 1$, we have

$$a^{-2} = -1 + \frac{1+\cot^2\varphi}{1+\tan^2\theta} \quad . \qquad \text{q.e.d.}$$

7. Now we can prove the case of (ii). We obtain from (9) and (10) the following estimate:

$$(14) \qquad (R_W)^{-2} = (\,|\tilde{c}|\,)^2 \geqq M_\nu > \frac{1}{15} \quad (3 \leqq \nu \leqq 6),$$

where $M_3 = -1 + 4/3(1+\tan^2\frac{\pi}{7})$, $M_4 = -1 + 2/(1+\tan^2\frac{\pi}{5})$, $M_5 = (1-\tan^2\frac{\pi}{5})/2\tan^2\frac{\pi}{5}$, $M_6 = 1$. Thus we can conclude that $\sqrt{15} > |\tilde{c}|^{-1}$, if Γ is discrete.

Remark 2. In the case of $\nu = 2$, Theorem does not necessarily hold. Take the sequence $\{x_n\}_{n=1}^{\infty}$ such that $0 < x_n < 1$, $\lim_{n\to\infty} x_n = 0$. We consider the group Γ generated by the following two elliptic transformations: $E(z) = -z$ and

$$A(z) = \frac{(1+x_n^2)z - 2x_n}{2x_n z - (1+x_n^2)}.$$

It is obvious Γ is discrete, but the reciprocal of the radius R_A of the isometric circle $I(A)$ is $|\,2x_n i/(1-x_n^2)\,|$ which tends to 0 as $n \to \infty$.

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Exact Solutions with Arbitrary Integer Deltas for Gravitational Fields of Spinning Masses

Masatoshi YAMAZAKI

Department of Physics, Kanazawa University

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Abstract A simple and systematic recipe is presented which gives for any integer δ (distortion parameter) the family of exact solutions for gravitational fields of spinning masses and which reduces to give the famous Kerr solution for $\delta=1$ and the Tomimatsu-Sato solutions for $\delta=2$, 3, and 4. Our family of solutions reduces to that of the Weyl metrics in the case of no rotation where the parameter q vanishes.

Kerr[1)] has discovered a solution ($\delta=1$ solution) for the gravitational field of a spinning mass. Ernst[2)] has formulated the axially symmetric gravitational field problem, obtained the differential equation

$$(\xi\xi^*-1)\nabla^2\xi = 2\xi^* \nabla\xi \cdot \nabla\xi,$$

and showed that the Kerr solution satisfies this equation. Tomimatsu and Sato[3)] have discovered a series of solutions ($\delta=2$, 3, and 4) for gravitational fields of spinning masses which reduce to the series of Weyl[4)] metrics in the limit of angular momentum parameter $q=0$.

We shall present a simple and systematic recipe to give exact solutions with arbitrary integer δ(distortion parameter) which are members of the series of Kerr and Tomimatsu-Sato solutions and which therefore reduce to the family of Weyl metrics[4)] in the limit of the parameter $q=0$.

We use notations in reference 3). We use prolate spheroidal coordinates x, y in place of cylindrical coordinates ρ, z and the notation $a=x^2-1$ and $b=y^2-1$. Ernst's complex functions ξ are written as $\xi=(u+iv)/(m+in)$. We use the notations $A=u^2+v^2-m^2-n^2$, $G=m^2+n^2$, $H=um+vn$, and $I=vm-un$. There are the distortion parameter δ and the angular momentum parameter q (and p such as $p^2+q^2=1$).

Our family of exact solutions with any integer δ can be expressed, besides px, qy, a^j, and b^j ($j=1, 2, 3, \cdots$), by functions $F(i)$ with $i=\delta^2-k_\delta$ and $k_\delta=0, 1, 2, \cdots, \delta$. The functions $F(i)$ are polynomials of a, b, p^2, and q^2. Polynomials $F(i)$ are homogeneous with degree i over a and b and simultaneously homogeneous with degree δ over p^2 and