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Remarks and Corrections to the Paper "($\frac{3}{2}$)-dimensional Measure of the Singular Sets of Some Kleinian Groups"

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In the previous paper [1] the writer gave the example of the Kleinian groups whose singular sets have positive ($\frac{3}{2}$)-dimensional Hausdorff measure. In the process of proving the existence of them there are some mistakes. One of them is that we used the inequality with respect to the subcomputing function. The upper and lower bounds of this function depend on the number of the boundary circles. The others are trivial ones, which are easily corrected. But our assertion remains valid.

To get out of the way of using the above inequality, we were obliged to treat the infinitely generated Kleinian groups with some properties. And we proved that the values of the subcomputing functions with respect to the outermost boundary circles of the fundamental domain tend to zero according as the number of the boundary circles increases and tends to the infinity. By using this fact instead of the above inequality, we could prove the existence of the desired groups completely. The essential parts of the proof are almost all the same as ones of the previous paper. The purpose of this paper is to correct the mistakes and report rapidly that our assertion is true. Here we state the results only without proofs and the details of the proof will be given in this Science Report.

1. At first we shall give the preliminaries and notations.

Let $\{K_j\}_{j=1}^{p}$ and $\{H_i, H_i'\}_{i=p+1}^{\infty}$ be an infinite number of circles external to one another in the extended complex plane $\widetilde{C}=\{z; |z| \leqq +\infty\}$, where $\{H_i, H_i'\}_{i=p+1}^{q}$ tends to only a finite point Q for $q\to\infty$. Let B be a domain bounded by these circles. Without loss of generality we may assume that these circles are contained in a closed disc $D_0=\{z; |z| \leqq \rho_0\}$ for some number ρ_0 (>0).

Let $\{T_j\}_{j=1}^{p}$ be the elliptic transformations with period 2 corresponding to $\{K_j\}_{j=1}^{p}$, each of which transforms the outside of K_j onto the inside of itself. Let $\{T_i\}_{i=p+1}^{\infty}$ be a system of hyperbolic or loxodromic transformations, each of which transforms the outside of H_i' onto the inside of H_i. Then the system $\mathcal{Y}=\{T_i,$

$T_i^{-1}\}_{i=1}^{\infty}$ $(T_i=T_i^{-1},\ 1\leqq i\leqq p)$ generates an infinitely generated discontinuous group denoted by G and we call $\mathcal{Y}$ the generator system of G, where T_i^{-1} denotes the inverse of T_i.

Take a positive integer $q(>p)$ and consider a subset $\mathcal{Y}_N=\{T_j\}_{j=1}^{p}\cup\{T_i, T_i^{-1}\}_{i=p+1}^{q}$ $(N=2q-p)$ of $\mathcal{Y}$. Then $\mathcal{Y}_N$ generates a finitely generated subgroup G_N of G. If we denote a domain bounded by $\{K_j\}_{j=1}^{p}\cup\{H_i, H_i'\}_{i=p+1}^{q}$ by B_N $(N=2q-p)$, it is well known that B_N coincides with a fundamental domain of G_N which is the $(2q-p)$-ply connected domain. We shall get G from G_N for $N\to\infty$.

2. Denote by $r(H)$ the radius of a circle $H\in\{H_i, H_i'\}_{i=p+1}^{\infty}$ and assume that there exists some positive constant K independent of H such that it holds

$$\text{(A)}\qquad \frac{r(H)}{l(H)}\leqq K,$$

where $l(H)=\inf|z-\zeta|$ and the infimum is taken for all point $z\in H$ and for all points ζ on any circle from $\{H_i, H_i'\}_{i=p+1}^{\infty}-\{H\}$.

Defining the product ST in G by $ST(z)=S(T(z))$, we can write any element U of G in the form

$$U=T_{i_n}\cdots T_{i_2}T_{i_1}\ (T_{i_j}\in\mathcal{Y}\ (1\leqq j\leqq n)\ ;\ T_{i_{j+1}}^{-1}\neq T_{i_j}).$$

We call the positive integer n the grade of U and for simplicity we use the notation $S_{(n)}$ to clarify the grade of U.

Since we can let the generator $T_i(\in\mathcal{Y})$ correspond to the boundary circle H_i, we shall denote by C_{T_i} and $C_{T_i^{-1}}$ the circles H_i and H_i', and further by D_{T_i} and $D_{T_i^{-1}}$ the closed discs bounded by C_{T_i} and $C_{T_i^{-1}}$, respectively. Then it is obvious that $C_{T_i}=T_i(C_{T_i^{-1}})$.

Now let us impose a restriction with respect to the accumulation of circles for G. Consider the circle $C_{T_i}: |z-a(T_i)|=r_{T_i}$ of radius r_{T_i} with center $a(T_i)$ for any T_i $(\in\mathcal{Y})$. Take some boundary circle $C_{T_j}(T_j\neq T_i)$ of B and denote the distance from $a(T_i)$ to C_{T_j} by $\rho_j(T_i)$, that is,

$$\rho_j(T_i)=\inf_{z\in C_{T_j}}|z-a(T_i)|.$$

We assume that there exists a positive constant $K_1(\alpha)$ depending only on some positive number α $(0<\alpha<2)$ satisfying

$$\text{(B)}\qquad W(T_j,\alpha)=\sum_{T_i\in\mathcal{Y}}{}'\left(\frac{r(T_i)}{\rho_j(T_i)^2}\right)^{\alpha}\leqq K_1(\alpha),$$

where $\sum'_{T_i\in\mathcal{Y}}$ denotes the sum with respect to all $T_i(\neq T_j)$. Then we can determine the unique number α_0 $(\geqq 0)$ such that

(1) $\alpha_0=\inf\{\alpha\,;\,K_1(\alpha)<+\infty\}$.

We note that α_0 is always equal to 0 for G_N. We shall call such discontinuous group with these properties (A) and (B) the Kleinian group with properties (A) and (B) and denote it by $G^*(\alpha_0)$ and the generator system by $\mathcal{Y}^*(\alpha_0)$.

3. Let $S_{(n)}=T_{i_n}\cdots T_{i_2}T_{i_1}$ $(T_{i_j}\in\mathcal{Y}_N)$ be any element of G_N and assume that $T_{i_1}^{-1}\neq T$ for a fixed element T $(\in\mathcal{Y}_N)$ and take any point $z\in D_T$. If we denote by $R_{S_{(n)}}$ the radius of the isometric circle of $S_{(n)}$, we obtain easily

$$\left|\frac{dS_{(n)}(z)}{dz}\right|^{\frac{\mu}{2}}=\left(\frac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu}\qquad(0<\mu<4),$$

where $S_{(n)}^{-1}$ denotes the inverse $(S_{(n)})^{-1}=T_{i_1}^{-1}\cdots T_{i_n}^{-1}$ of $S_{(n)}$. Here we note that $z\in D_T$ and $S_{(n)}^{-1}(\infty)\in D_{T_{i_1}^{-1}}$ $(T\neq T_{i_1}^{-1})$.

Forming the sum of $(N\text{-}1)^n$ terms with respect to all $S_{(n)}$ $(\in G_N)$ and $(N\text{-}1)^{n-1}$ terms with respect to all $S_{(n-1)}$ $(\in G_N)$ for $S_{(n)}=T_{i_n}S_{(n-1)}$ such that $T_{i_1}^{-1}\neq T$ and $T_{i_j}\neq T_{i_{j+1}}^{-1}$ $(1\leqq j\leqq n-1)$, respectively, we have the following two functions:

$$(2)\qquad\begin{cases}\chi_{n,N}^{(\mu\,;\,T)}(z)=\sum\limits_{S_{(n)}\in G_N}\left(\dfrac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu},\\[2ex] \chi_{n,N}^{(\mu\,;\,T,T_{i_n})}(z)=\sum\limits_{S_{(n-1)}\in G_N}\left(\dfrac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu},\ (S_{(n)}=T_{i_n}S_{(n-1)}),\end{cases}$$

and called $\chi_{n,N}^{(\mu\,;T)}(z)$ and $\chi_{n,N}^{(\mu\,;\,T,\,T_{i_n})}(z)$ the μ-dimensional computing function and subcomputing T_{i_n}-function of order n on T, respectively. The domain of definition of both functions is D_T ([1]).

Since each term in the sum of them is positive, $\chi_{n,N}^{(\mu\,;\,T)}(z)$ and $\chi_{n,N}^{(\mu\,;\,T,\,T_{i_n})}(z)$ have necessarily the unique limit containing the infinity for any $z\in D_T$, if N tends to the infinity. Then we can define the following functions:

$$(3)\qquad\begin{cases}\lim\limits_{N\to\infty}\chi_{n,N}^{(\mu\,;\,T)}(z)=\chi_{n,\infty}^{(\mu\,;\,T)}(z)=\sum\limits_{S_{(n)}\in G}\left(\dfrac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu},\\[2ex] \lim\limits_{N\to\infty}\chi_{n,N}^{(\mu\,;\,T,T_{i_n})}(z)=\chi_{n,\infty}^{(\mu\,;\,T,T_{i_n})}(z)=\sum\limits_{S_{(n-1)}\in G}\left(\dfrac{R_{S_{(n)}}}{|z-S_{(n)}^{-1}(\infty)|}\right)^{\mu},\end{cases}$$

and we shall call them the μ-dimensional limiting computing function and limiting subcomputing T_{i_n}-function of order n on T, respectively.

Now let us give the following definition ([4]).

DEFINITION. *Let* $\{\chi_{n,\infty}^{(\mu;T)}(z)\}$ $(n=1,2,\ldots)$ *be a sequence of the* μ*-dimensional limiting computing functions on* $T\in\mathcal{Y}$. *If it holds*

(4) $\lim\limits_{n\to\infty}\chi_{n,\infty}^{(\mu\,;\,T)}(z)=0$ (or ∞)

for some element $T \in \mathfrak{Y}$ *and some point* $z \in D_T$, *we call G the* μ*-convergent (or divergent) type. If it holds*

$$(5) \qquad 0 < \varliminf_{n\to\infty} \chi_{n,\infty}^{(\mu\,;\,T)}(z) \leqq \varlimsup_{n\to\infty} \chi_{n,\infty}^{(\mu;T)}(z) < +\infty$$

for some element $T\ (\in \mathfrak{Y})$ *and some point* $z \in D_T$, *we call G the* μ*-finite type.*

4. Now let us seek for the properties of computing and subcomputing functions and the relations between them.

PROPOSITION 1 ([1], [2]). *It holds the following relations between two computing and two subcomputing functions on the different elements of* $\mathfrak{Y}_N$:

$$(6) \qquad \begin{cases} \chi_{n+l,N}^{(\mu\,;\,T)}(z) > K(G_N,\ l,\ \mu)\ \chi_{n,N}^{(\mu;T_l)}(S_{(l)}(z)) \\ \chi_{n+l,N}^{(\mu\,;\,T,T_i)}(z) > k(G_N,\ l,\ \mu)\ \chi_{n,N}^{(\mu\,;\,T_l,T_i)}(S_{(l)}(z)), \end{cases}$$

where $S_{(n+l)} = S_{(n)}S_{(l)} = T_iT_{(n-1)}T_lS_{(l-1)}$ *and* $K(G_N,\ l\ \mu)$ *and* $k(G_N,\ l,\ \mu)$ *are constants depending only on* G_N, l *and* μ.

PROPOSITION 2 ([2]). *It holds for any elements T and* T_i $(\in \mathfrak{Y}_N)$ *and any two points z and* $z_0 \in DT$

$$(7) \qquad \begin{aligned} K_1(G_N,\mu)\ \chi_{n,N}^{(\mu\,;\,T)}(z_0) &\leqq \chi_{n,N}^{(\mu\,;\,T)}(z) \leqq K_2(G_N,\mu)\ \chi_{n,N}^{(\mu\,;\,T)}(z_0) \\ k_1(G_N,\mu)\ \chi_{n,N}^{(\mu;T,T_i)}(z_0) &\leqq \chi_{n,N}^{(\mu;T,T_i)}(z) \leqq k_2(G_N,\mu)\,\chi_{n,N}^{(\mu;T,T_i)}(z_0) \end{aligned}$$

where $K_i(G_N,\ \mu)$ *and* $k_i(G_N,\ \mu)$ $(i=1,\ 2)$ *are constants depending only on* G_N *and* μ.

There exist the following relations between the computing and subcomputing functions.

PROPOSITION 3 ([1]). *It holds for any T and* T_i $(\in \mathfrak{Y}_N)$ *and any point* $z \in D_T$

$$(8) \qquad \chi_{n,N}^{(\mu\,;\,T)}(z) = \sum_{i=1}^{N} \chi_{n,N}^{(\mu;T,T_i)}(z),$$

$$(9) \qquad \begin{aligned} K_3(G_N,\mu)\,\chi_{n,N}^{(\mu\,;T)}(z) &\leqq \chi_{n,N}^{(\mu;T,T_i)}(z) + \chi_{n+1,N}^{(\mu\,;\,T,T_i^{-1})}(z) \\ &\leqq K_4(G_N,\mu)\ \chi_{n,N}^{(\mu\,;T)}(z), \end{aligned}$$

where $K_i(G_N,\ \mu)$ $(i=3,\ 4)$ *are constants depending only on* G_N *and* μ.

REMARK 1. In the former paper [1], we gave the middle term in (9) as the form $\chi_{n,N}^{(\mu;T,T_i)}(z)+\chi_{n+1,N}^{(\mu;T,T_i)}(z)$. But in this case T_i is the elliptic transformation with period 2.

PROPOSITION 4. *It holds for any* T_i *and* T_j $(\in \mathcal{Y}_N)$

$$K_5(G_N, \mu)\ \chi_{n,N}^{(\mu\,;\,T_j,T_i)}(z) \leqq \chi_{n,N}^{(\mu\,;\,T_i^{-1},T_j^{-1})}(z^*), \tag{10}$$

where $K_5(G_N, \mu)$ *is a constant depending only on* G_N *and* μ *and* $z \in D_{T_j}$ *and* $z^* \in D_{T_i^{-1}}$.

5. By using the above propositions 1–4, we have the following theorems.

THEOREM 1 ([1], [2]). *The following five propositions are equivalent to each other:* (I) *It holds for some fixed element* T^* $(\in \mathcal{Y}_N)$ *and some point* $z_0 \in D_{T^*}$

$$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T^*)}(z_0)=\infty \qquad (or\ 0). \tag{11}$$

(II) *It holds for some* T^* *and* T $(\in \mathcal{Y}_N)$ *and some point* $z_0 \in D_{T^*}$

$$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T^*,T)}(z_0)=\infty \qquad (or\ 0). \tag{12}$$

(III) *It holds for any* T^* $(\in \mathcal{Y}_N)$

$$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T^*)}(z)=\infty \qquad (or\ 0) \tag{13}$$

uniformly on D_{T^*}.

(IV) *It holds for any* T^* and T $(\in \mathcal{Y}_N)$

$$\lim_{n\to\infty} \chi_{n,N}^{(\mu;T^*,T)}(z)=\infty \qquad (or\ 0) \tag{14}$$

uniformly on D_{T^*}.

(V) *It holds* $M_{\mu/2}(E_N)=\infty$ (*or* 0), *where* E_N *denotes the singular set of* G_N.

THEOREM 2 ([2]). *The following four propositions are equivalent to each other:* (i) *It holds for some* T^* $(\in \mathcal{Y}_N)$ *and some point* $z_0 \in D_{T^*}$

$$0 < \underline{\lim}_{n\to\infty}\ \chi_{n,N}^{(\mu;T^*)}(z_0) \leqq \overline{\lim}_{n\to\infty}\ \chi_{n,N}^{(\mu;T^*)}(z_0) < +\infty. \tag{15}$$

(ii) *It holds for some* T^* *and* T $(\in \mathcal{Y}_N)$ *and some point* $z_0 \in D_{T^*}$

$$0 < \underline{\lim}_{n\to\infty}\ \chi_{n,N}^{(\mu;T^*,T)}(z_0) \leqq \overline{\lim}_{n\to\infty}\ \chi_{n,N}^{(\mu;T^*,T)}(z_0) < +\infty. \tag{16}$$

(iii) $0<\mathrm{M}_{\mu/2}(E_N)<+\infty$.

(iv) *The Hausdorff dimension* $\mathrm{d}(E_N)$ *is equal to* $\frac{\mu}{2}$.

REMARK 2. It is easily proved that the above propositions (i) and (ii) hold for any T^*, T and T' ($\in \mathcal{Y}_N$) and any point $z_0 \in D_{T^*}$. We have from (16)

$$0 < k(G_N, \mu_N) < \varliminf_{n\to\infty} \frac{\chi_{n,N}^{(\mu_N;T^*,T')}(z)}{\chi_{n,N}^{(\mu_N;T^*,T')}(z)} \leqq \varlimsup_{n\to\infty} \frac{\chi_{n,N}^{(\mu_N;T^*,T')}(z)}{\chi_{n,N}^{(\mu_N;T^*,T')}(z)} < K(G_N, \mu_N) < +\infty \tag{17}$$

for $\mathrm{d}(E_N)=\frac{\mu_N}{2}$, where $K(G_N, \mu_N)$ and $k(G_N, \mu_N)$ are constants depending only on G_N and μ_N. We made a mistake in [1] as follows:

$$0 < k < \lim_{n\to\infty} \frac{\chi_{n,N}^{(\mu_N;T^*,T')}(z)}{\chi_{n,N}^{(\mu_N;T^*,T')}(z)} < K < +\infty \tag{18}$$

for some numbers K and k independent of G_N and μ_N. By using (18), we showed the existence of the desired Kleinian groups. Hence we must take another way to prove this fact.

6. In the former papers ([4], [5]), we obtained the following results with respect to the singular set E of some infinitely generated Kleinian groups $G^*(\alpha_0)$ with properties (A) and (B).

THEOREM 3 ([4]). (i) *$G^*(\alpha_0)$ is the μ-divergent type if and only if* $\mathrm{M}_{\mu/2}(E)=\infty$. (ii) *$G^*(\alpha_0)$ is the μ-convergent type if and only if* $\mathrm{M}_{\mu/2}(E)=0$.

THEOREM 4 ([4], [5]). *Let* $\mathrm{d}(E)=\frac{\mu_0}{2}$ *be the Hausdorff dimension of the singular set E of $G^*(\alpha_0)$. If* $\frac{\mu_0}{2} > \alpha_0$, *then it holds* $0<\mathrm{M}_{\mu_0/2}(E) <+\infty$ *and from this fact $G^*(\alpha_0)$ is the μ_0-finite type, that is,*

$$0 < \varliminf_{n\to\infty} \chi_{n,\infty}^{(\mu_0;\,T^*)}(z_0) \leqq \varlimsup_{n\to\infty} \chi_{n,\infty}^{(\mu_0;\,T^*)}(z_0) < +\infty \tag{19}$$

for some T^* ($\in \mathcal{Y}^*(\alpha_0)$) and some point $z_0 \in D_{T^*}$.

Now Theorem 4 contains an important meaning, that is, there exist positive constants $K^*(G^*(\alpha_0), T^*, \mu_0)$ and $k^*(G^*(\alpha_0), T^*, \mu_0)$ depending only on $G^*(\alpha_0)$, T^* and μ_0 such that it holds

$$(20)\qquad \begin{aligned} 0 &< k^*(G^*(\alpha_0), T^*, \mu_0) < \underline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu_0\,;\,T^*)}(z_0) \\ &\leqq \overline{\lim}_{n\to\infty} \chi_{n,\infty}^{(\mu_0\,;\,T^*)}(z_0) < K^*(G^*(\alpha_0), T^*, \mu_0) < +\infty \end{aligned}$$

for some T^* and some point $z_0 \in D_{T^*}$.

7. Now let us seek for the desired Kleinian groups whose singular sets have positive $(\frac{3}{2})$-dimensional measure in the subgroups of $G^*(\alpha_0)$. For this purpose we reconsider the example of the group given in [1]. Since the Hausdorff dimension of the Kleinian group is invariant under the linear transformation ([6]), we transforms that group by $z'=\frac{1}{z}$. Then it is obvious that the transformed group satisfies the property (A). Next we consider the property (B). Generally it is easily proved that $W(T_j, \alpha)$ converges for any T_j, if $W(T_{i_0}, \alpha)$ converges for some T_{i_0} and some α. If we put $\alpha=\frac{3}{2}$, then $W(T_{i_0}, \frac{3}{2})$ converges for some T_{i_0} in this example and we have $\alpha_0<\frac{3}{2}$. Hence this group also satisfies the condition (B).

For the convenience of the discussion, we shall consider $G^*(\alpha_0)$ as the original Kleinian group with the accumulation point at the infinity, which is obtained from the subgroup G_N in this example for $N\to\infty$.

The total number N of the boundary circles of the fundamental domain of G_N is equal to $N=3q(q+1)+1$ denoted by $N(q)$.

Assume that the Hausdorff dimension $d(E)=\frac{\mu_0}{2}$ of the singular set E of $G^*(\alpha_0)$ is greater than $\frac{3}{2}$ and sufficiently near to $\frac{3}{2}$. Take a large integer n and $S_{0,0}$ as T^* in (20) and let them be fixed. Then there exists $\lim_{q\to\infty} \chi_{n,N(q)}^{(\mu_0\,;\,S_{0,0})}(z_0)$ at the point z_0 ($\in D_{S_{0,0}}$) and it holds

$$(21)\qquad \begin{aligned} 0 &< k^*(G^*(\alpha_0), S_{0,0}, \mu_0) < \lim_{q\to\infty} \chi_{n,N(q)}^{(\mu_0\,;\,S_{0,0})}(z_0) \\ &< K^*(G^*(\alpha_0), S_{0,0}, \mu_0) < +\infty. \end{aligned}$$

Hence there is a positive integer q_0 depending only on small number ε (>0) so that it may hold for any positive integers p and q greater than q_0

$$(22)\qquad \left| \chi_{n,N(q)}^{(\mu_0;S_{0,0})}(z_0) - \chi_{n,N(p)}^{(\mu_0;S_{0,0})}(z_0) \right| < \varepsilon, \qquad (N(p)<N(q)).$$

Since $\chi_{n,N(q)}^{(\mu_0\,;\,S_{0,0})}(z_0) = \sum_{S_{i,j}\in\mathcal{Y}_{N(q)}} \chi_{n,N(q)}^{(\mu_0\,;\,S_{0,0},S_{i,j})}(z_0)$, we obtain from (22)

$$(23)\qquad \left| \sum_{S_{i,j}\in\mathcal{Y}_{N(q)}-\mathcal{Y}_{N(p)}} \chi_{n,N(q)}^{(\mu_0\,;\,S_{0,0},S_{i,j})}(z_0) \right| < \varepsilon.$$

If we can show that $\lim_{n\to\infty} \chi_{n,\infty}^{(\mu^*;S_{0,0})}(z_0)=+\infty$ for any $\frac{\mu^*}{2}$ $(\frac{3}{2}<\frac{\mu^*}{2}<\frac{\mu_0}{2})$ at some point z_0 ($\in D_{S_{0,0}}$), this means that our assertion to the existence of the desired group is valid. From the definition of the computing function we can modify $\chi_{n,N(q)}^{(\mu^*\,;\,S_{0,0})}(S_{(m)}(\infty))$ ($S_{(m)}=S_{0,0}S_{(m-1)}$) in the following:

(24)
$$\chi_{n,N(q)}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty))=\prod_{\nu=1}^{n}\left[\frac{\sum_{S_{(\nu)}} R^{\mu^*}_{S_{(\nu)}S_{(m)}}}{\sum_{S_{(\nu-1)}} R^{\mu^*}_{S_{(\nu-1)}S_{(m)}}}\right]$$
$$=\prod_{\nu=1}^{n}\left[\frac{\sum_{S_{(\nu-1)}} R^{\mu^*}_{S_{(\nu-1)}S_{(m)}}\ \{\chi_{1,N(q)}^{(\mu^*;S_{i,j})}(S_{(\nu-1)}S_{(m)}(\infty))\}}{\sum_{S_{(\nu-1)}} R^{\mu^*}_{S_{(\nu-1)}S_{(m)}}}\right],$$

where $S_{(\nu-1)}S_{(m)}=S_{i,j}S_{(\nu-2)}S_{(m)}$. Since $\frac{\mu_0}{2}$ is sufficiently near to $\frac{3}{2}$ and $\chi_{1,N(q)}^{(\mu^*;S_{i,j})}(z_0)$ is a continuous function of μ^* for fixed $N(q)$, $S_{i,j}$ and z_0, we can estimate the values of $\chi_{1,N(q)}^{(\mu^*;S_{i,j})}(S_{(\nu-1)}S_{(m)}(\infty))$ from below approximately by $\chi_{1,N(q)}^{(3;S_{i,j})}(S_{(\nu-1)}S_{(m)}(\infty))$. Hence we have the following estimation ([1]):

(25)
$$\chi_{1,N(q)}^{(\mu^*;S_{i,j})}(S_{(\nu-1)}S_{(m)}(\infty))\begin{cases} >1.129, & \text{when } S_{i,j} \text{ corresponds to the circles from 0-th to } q\text{-3 th rank,}\\ >1.069, & \text{when } S_{i,j} \text{ corresponds to the circles of } q\text{-2 th rank,}\\ >0.918, & \text{when } S_{i,j} \text{ corresponds to the circles of } q\text{-1 th rank,}\\ >0.5574, & \text{when } S_{i,j} \text{ corresponds to the circles of } q\text{-th rank.}\end{cases}$$

Substituting these values into (24), we have

(26)
$$\chi_{n,N(q)}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty))>$$
$$>\prod_{\nu=1}^{n}\left[\frac{1.129\sum^{(1)}_{S_{i,j}}\chi_{\nu-1,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))+1.069\sum^{(2)}_{S_{i,j}}\chi_{\nu-1,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))}{\ }\right.$$
$$\left.\frac{0.918\sum^{(3)}_{S_{i,j}}\chi_{\nu-1,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))+0.5574\sum^{(4)}_{S_{i,j}}\chi_{\nu-1,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))}{\sum_{k=1}^{4}(\sum^{(k)}_{S_{i,j}}\chi_{\nu-1,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))}\right]$$

where $\sum^{(k)}_{S_{i,j}}$ ($k=1, 2, 3, 4$) denote the sum of values of the subcomputing $S_{i,j}$-functions of order ν-1 on $S_{0,0}$ at $S_{(m)}(\infty)=S_{0,0}\ S_{(m-1)}(\infty)$ with respect to $S_{i,j}$ corresponding to the circles from 0-th to q-3 th rank, to q-2 th rank, to q-1 th rank and q-th rank in turn, respectively. If we denote each sum in the brace by $\sum^{(k)}_{S_{i,j}}\chi(\nu\text{-}1, N(q), S_{i,j}, \mu^*)$, we have

(27)
$$\chi_{n,N(q)}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty))>\prod_{\nu=1}^{n}\left[\frac{1.129}{1+\frac{\sum_{k=2}^{4}(\sum^{(k)}_{S_{i,j}}\chi((\nu-1),N(q),S_{i,j},\mu^*))}{\sum^{(1)}_{S_{i,j}}\chi((\nu-1),N(q),S_{i,j},\mu^*)}}\right]$$

Further we see easily from (21)

$$k^*(G^*(\alpha_0), S_{0,0}, \mu_0) < \chi_{n,\infty}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty)). \tag{28}$$

Hence we obtain from (27) and (28)

$$\chi_{n,N(q)}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty)) > \prod_{\nu=1}^{n}\left[1.129 / \{1+2\frac{\sum_{k=2}^{4}(\sum_{S_{i,j}}^{(k)}\chi((\nu-1), N(q), S_{i,j}, \mu^*)}{k^*(G^*(\alpha_0), S_{0,0}, \mu_0)}\}\right] \tag{29}$$

8. Take any large number M and any small number δ such that $0 < \delta < 0.129$. Then we can determine a small number ε (>0) satisfying

$$\frac{1.129}{1+\delta} > 1 + \frac{2\varepsilon}{k^*(G^*(\alpha_0), S_{0,0}, \mu_0)} \tag{30}$$

and further choose a positive integer n so that it may hold

$$(1+\delta)^n > M. \tag{31}$$

So we can determine the positive integer q in (23) such that

$$\sum_{S_{i,j}\in\mathcal{Y}_{N(q)}-\mathcal{Y}_{N(q-3)}} \chi_{\nu,N(q)}^{(\mu_0;S_{0,0},S_{i,j})}(S_{(m)}(\infty)) < \varepsilon, \ (\nu=1,2,\dots,n-1), \tag{32}$$

for the above fixed ε and n. Since the left hand side in (32) is the sum of the continuous functions of μ, we can choose a number μ^* ($\frac{3}{2}<\mu^*<\mu_0$) satisfying

$$\left| \sum_{S_{i,j}\in\mathcal{Y}_{N(q)}-\mathcal{Y}_{N(q-3)}} \chi_{\nu,N(q)}^{(\mu^*;S_{0,0},S_{i,j})}(S_{(m)}(\infty))\right| < \varepsilon, \ (\nu=1,2,\dots,n-1). \tag{33}$$

Thus we have from (29), (30), (31) and (33)

$$\chi_{n,N(q)}^{(\mu^*;S_{0,0})}(S_{(m)}(\infty)) > (1+\delta)^n > M. \tag{34}$$

We repeat the above method to the number M_1 ($>M$) for the above fixed δ and ε. Then we can also choose the numbers n_1 ($>n$), q_1 ($>q$) and μ^*_1 ($\frac{3}{2}<\mu^*<\mu^*_1$) so that it may hold

$$\chi_{n_1,N(q_1)}^{(\mu^*_1;S_{0,0})}(S_{(m)}(\infty)) > M_1. \tag{35}$$

Continuing this procedure successively, then we get the sequence of numbers for fixed δ and ε

$$\begin{aligned} &\mu^* < \mu^*_1 < \cdots < \mu^*_l < \cdots \longrightarrow \mu_0 \\ &N(q) < N(q_1) < \cdots < N(q_l) < \cdots \longrightarrow \infty \end{aligned} \tag{36}$$

such that it holds for any large number M_i (i=1, 2,...)

(37) $$\chi^{(\mu^*_i;\, S_{0,0})}_{n_i, N(q_i)}(S_{(m)}(\infty)) > M_i.$$

Since each function $\chi^{(\mu^*_i;\, S_{0,0})}_{n_i,\, N(q_i)}(S_{(m)}(\infty))$ is a monotone decreasing function of μ and a monotone increasing function of q_i, then we have for any n_i and M_i and fixed element $S_{0,0}$

(38) $$\chi^{(\mu^*;\, S_{0,0})}_{n_i, \infty}(S_{(m)}(\infty)) > M_i, \qquad (i=1,2,\ldots).$$

So we can find that $\lim_{n\to\infty} \chi^{(\mu^*;S_{0,0})}_{n,\infty}(S_{(m)}(\infty))=\infty$ and hence from our result [4] $M_{\mu^*/2}(E)=\infty$ for μ^* $(3<\mu^*<\mu_0)$. We have already found in [3] that the Hausdorff dimension $\frac{\mu_N}{2}$ of the singular set E_N of G_N increases strictly according as the increment of the boundary circles of B_N. Hence there exists a sequence of the Hausdorff dimensions $\{\frac{\mu_N}{2}\}$ of $\{E_N\}$ such that $\lim_{N\to\infty} \frac{\mu_N}{2}=\frac{\mu_0}{2}$ $(>\frac{3}{2})$. Thus we could show the existence of the Kleinian groups G_N whose singular sets have positive $(\frac{3}{2})$-dimensional measure.

REMARK 4. Observing the above method using an infinitely generated Kleinian group, we shall easily see that our conjecture in [1]

$$\lim_{n\to\infty} \frac{\chi^{(\mu;\, S_{0,0}, S_{k,l})}_{n,N}(z_0)}{\chi^{(\mu;\, S_{0,0}, S_{i,j})}_{n,N}(z_0)} = 1 \qquad (S_{i,j} \neq S_{k,l})$$

is not true.

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