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1. Introduction

Y. M. Chen [2] has proved that there exists a function of the class $L(\log^+\log^+L)^{1-\varepsilon}$ with the almost everywhere divergent Fourier series by construction similar to the well-known classical example.

The purpose of this paper is to show that there exists an everywhere divergent Fourier series of $L(\log^+\log^+L)^{1-\varepsilon}$, by Y. M. Chen's result [1] with respect to the best possibility of the order of the partial sum of Fourier series of $L(0, 2\pi)$.

2. The main theorem

We shall give the following theorem.

THEOREM. Let $\lambda(t)$, $\Phi(t)$ be defined for $t\geqq 0$ and satisfy the following conditions;

(i) $\lambda(t)$ is positive, decreasing to 0, convex, and $-t\lambda'(t)$ is slowly varying.

(ii) There exists a function $g(x)\in L(0, 2\pi)$ such that for every point $x\in[0, 2\pi)$,

$$s_n(g)(x) > \frac{1}{\lambda(n)} \quad \text{for infinitely many } n.$$

(iii) $\Phi(t)$ is non-negative and $t\Phi(t)$ is increasing and convex.

(iv) $$-\left(\frac{1}{x}\right)^2\lambda'\left(\frac{1}{x}\right)\Phi\left(\left(\frac{1}{x}\right)^{1+\delta}\right)\in L(0, 2\pi) \quad \text{for some } \delta>0.$$

Then, there exists a function $f(x)\in L\Phi(L)(0, 2\pi)$ such that the Fourier series diverges for all $x\in[0, 2\pi)$.

3. Proof of the theorem

We put

$$\varphi(x)=\sum_{j=0}^{\infty}\lambda(j)\cos jx$$

where this series converges for all $x\in(0, 2\pi)$ to a function $\varphi(x)\in L(0, 2\pi)$ and is the Fourier series of $\varphi(x)$, because $\lambda(j)$ decreases to 0 and is convex.

Now we consider the function

$$f(x)=(\varphi * g)(x)=\frac{1}{2\pi}\int_0^{2\pi}\varphi(x-t)g(t)dt.$$

We shall show that this function $f(x)$ is satisfactory.

(1) The Fourier series of $f(x)$ diverges for all $x\in[0, 2\pi)$.

Otherwise, there should be a $x_0\in[0, 2\pi)$ such that $s_n(f)(x_0)$ converges and let

$$s_n=s_n(f)(x_0)=\sum_{j=-n}^{n}\hat{\varphi}(j)\hat{g}(j)e^{ijx_0}\to s \qquad (n\to\infty)$$

where
$$\hat{\varphi}(j)=\begin{cases}\frac{1}{2}\lambda(|j|) & \text{for } j\neq 0\\ \lambda(0) & \text{for } j=0\end{cases}.$$

Then by Abel's transformation,

$$\hat{\varphi}(n)s_n(g)(x_0)=\hat{\varphi}(n)\sum_{j=1}^{n}\frac{1}{\hat{\varphi}(j)}(s_j-s_{j-1})+o(1)=s_n-\sum_{j=1}^{\infty}c_{n,j}s_j+o(1)$$

where
$$c_{n,j}=\begin{cases}\hat{\varphi}(n)\{\frac{1}{\hat{\varphi}(j+1)}-\frac{1}{\hat{\varphi}(j)}\} & \text{for } 1\leqq j\leqq n-1\\ 0 & \text{for } j\geqq n\end{cases}\geqq 0.$$

Since the matrix $(c_{n,j})$ satisfies the Toeplitz condition,

$$\hat{\varphi}(n)s_n(g)(x_0)\to s-s=0 \quad (n\to\infty).$$

On the other hand,

$$\hat{\varphi}(n)s_n(g)(x_0)>\hat{\varphi}(n)\frac{1}{\lambda(n)}=\frac{1}{2}\lambda(n)\frac{1}{\lambda(n)}=\frac{1}{2}$$

for infinitely many n. So we have the contradiction.

(2) $f(x)\in L\Phi(L)$.

In order to prove this, as $t\Phi(t)$ is increasing and convex, by Jensen's inequality we need only to show that

$$\varphi(x)\in L\Phi(L).$$

Since $\lambda(t)$ decreases to 0 and $-t\lambda'(t)$ is slowly varying, then

$$\varphi(x)\simeq-\frac{\pi}{2}\left(\frac{1}{x}\right)^2\lambda'\left(\frac{1}{x}\right) \qquad \text{as } x\to+0$$

(A. Zygmund [3], p. 189).

Therefore

$$\int_0^{2\pi} |\varphi(x)|\Phi(|\varphi|)(x)dx$$

$$\leqq \text{ const. } \int_0^{2\pi}(-\frac{\pi}{2})\ (\frac{1}{x})^2\lambda'(\frac{1}{x})\Phi((-\frac{\pi}{2})(\frac{1}{x})^2\lambda'(\frac{1}{x}))dx$$

$$\leqq \text{ const. } \int_0^{2\pi}(-\frac{\pi}{2})\ (\frac{1}{x})^2\lambda'(\frac{1}{x})\Phi(\text{const. } (\frac{1}{x})^{1+\delta})dx$$

$$\leqq \text{ const. } \int_0^{2\pi}(-1)(\frac{1}{x})^2\lambda'(\frac{1}{x})\Phi((\frac{1}{x})^{1+\delta})dx < \infty.$$

Q.E.D.

4. Corollaries

Corollary 1. Given any number ε $(0<\varepsilon\leqq 1)$, there exists a function of the class $L(\log^+\log^+L)^{1-\varepsilon}$ such that the Fourier series diverges everywhere.

Proof. We choose η such that $0<\eta<\varepsilon$. We put $\lambda(t)=(\log\log t)^{-(1-\eta)}$ for $t\geqq 3$ and extend it to be convex for $t\geqq 0$, and put $\Phi(t)=(\log^+\log^+t)^{1-\varepsilon}$. Then the conditions (i) and (iii) are satisfied. The condition (ii) is Y. M. Chen's remarkable result [1]. Since

$$-(\frac{1}{x})^2\lambda'(\frac{1}{x})\Phi((\frac{1}{x})^{1+\delta}) \leqq \text{ const. } \frac{1}{x}(\log\frac{1}{x})^{-1}(\log\log\frac{1}{x})^{-\{1+(\varepsilon-\eta)\}}$$

as $x\to+0$, the condition (iv) is satisfied.

Corollary 2. If for any given positive sequence $\mu_n=o(\log n)$ there exists a function $g(x)\in L$ such that for very point $x\in[0, 2\pi)$

$$s_n(g)(x)>\mu_n \quad \text{for infinitely many } n,$$

then for any given number $\varepsilon(0<\varepsilon\leqq 1)$, there exists a function of the class $L(\log^+L)^{1-\varepsilon}$ such that the Fourier series diverges everywhere.

The condition of this corollary is known as Zygmund's conjecture.

Proof. As before, we choose η such that $0<\eta<\varepsilon$. We put $\lambda(t)=(\log t)^{-(1-\eta)}$ for $t\geqq 2$ and extend it to be convex for $t\geqq 0$, and put $\Phi(t)=(\log^+t)^{1-\varepsilon}$. Since

$$-(\frac{1}{x})^2\lambda'(\frac{1}{x})\Phi((\frac{1}{x})^{1+\delta}) \leqq \text{ const. } \frac{1}{x}(\log\frac{1}{x})^{-\{1+(\varepsilon-\eta)\}}$$

as $x\to+0$, therefore the all conditions of the theorem are satisfied.

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