Sci. Rep. Kanazawa Univ.,
Vol. 15 No. 2, 61-69
December 1970

On the Minimum of Some Subharmonic Function (II)

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(Received November 11, 1970)

1. Introduction

In our former paper [2] we got a result on the minimum problem of some subharmonic function. The purpose of this paper is to extend this result.

Let U denote a closed disc and $p_i \notin U$ $(i=1, \cdots\cdots n)$ and $p \in U$ be n fixed points and amoving point, respectively. We consider the following function

$$(1) \qquad F_n(p)=\sum_{i=1}^{n}\frac{1}{\overline{pp_i}^{\,\alpha}}=\sum_{i=1}^{n}\frac{1}{|z-z_i|^{\alpha}} \qquad (\alpha>0),$$

where $\overline{pp_i}$ denotes the distance between p and p_i and z and z_i represent complex numbers corresponding to p and p_i. Since each term of (1) is an absolute value of the regular function, it is obvious that $F_n(p)$ is subharmonic in U. We gave the result about the minimum of (1) in the case that $\alpha=3$ and n points are located in the special situations ([1]). Here we treat such similar problem in the case of any number α (>0) with some condition.

2. Problem

Let $D_{0,0}$ be a closed unit disc bounded by the unit circle $C_{0,0}$. Next we describe the six circles C_{1,j_1} $(j_1=1,\cdots,6)$ with equal radii 1 so that C_{1,j_1} $(j_1=1,\cdots,6)$ are tangent externally with each other around $C_{0,0}$ and hence the segments, which join the centers of $C_{1,j}$, successively, constitute a regular hexagon R_1, where the center of $C_{1,1}$ has the coordinate (2,0) with respect to rectangular coordinate system. Further we describe the twelve circles C_{2,j_2} $(j_2=1,2,\cdots,12)$ with equal radii 1 so that C_{2,j_2} $(j_2=1,2,\cdots,12)$ are tangent externally with each other around C_{1,j_1} and hence the segments, which join the centers of C_{2j_2} successively, constitute a regular hexagon R_2, where the center of $C_{2,1}$ has the coordinate (4,0). We continue such procedure by turns. Generally, we describe the $6n$ circles C_{n,j_n} $(j_n=1,2,\cdots,6n)$ with equal radii 1 so that C_{nj_n} $(j_n=1,2,\cdots,6n)$ are tangent externally with each other around $C_{n-1,j_{n-1}}$ and

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hence the segments, which join the centers of C_{n,j_n} successively, constitute a regualr hexagon R_n, where the center of $C_{n,1}$ has the coordinate $(2n,0)$. It is obvious that the total number of circles $C_{0,0}$ and C_{i,j_i} $(i=1,2,\cdots,n\ ;\ j_i=1,2,\cdots,\ 6i)$ is equal to $3n(n+1)+1$. Let us denote the center of C_{i,j_i} by z_{i,j_i}. Let p be a moving point in $D_{0,0}$, which has the coordinate (x,y), $z=x+iy$. We consider the following subharmonic function

$$(2)\qquad F_n(z)=\sum_{i=1}^{n}\sum_{j_i=1}^{6i}\frac{1}{|z-z_{i,j_i}|^{\alpha}}\qquad(\alpha>o),\qquad \forall_z\in D_{0,0}.$$

Our problem is to determine the point at which $F_n(z)$ attains its minimum in $D_{0,0}$. This problem occured in the investigation whether there exist or not Kleinian groups whose singular sets have positive $\left(\frac{3}{2}\right)$-dimensional measure ([1]). In the case of $\alpha=3$ in (2) we have already solved this problem in [2]. With some restriction about a number α, we shall solve this problem and extend the result gotten in [2].

3. Theorems

Now we shall give the main theorem.

THEOREM A. *If α is a positive number such that*

$$(3)\qquad \left(\frac{2}{3}\right)^{\alpha}+2\left(\frac{2\sqrt{3}}{3}\right)^{\alpha}>3$$

is satisfied, then $F_n(z)$ attains its minimum at the origin.

For the proof we prepare the following theorem as lemma.

THEOREM B. *Let $P_1=P(1,\pi)$, $P_2=P(1,-\frac{\pi}{3})$ and $P_3=P(1,\frac{\pi}{3})$ be fixed points on the unit circle $|z|=1$ in the complex z-plane, and $P=P(r,\theta)$ be a moving point on the fixed closed disc U_R : $|z|\ \leqq R$ $(\leqq(3+\sqrt{5})/2)$, where (r,θ) denotes the polar coordinate. Then for any positive number α the function*

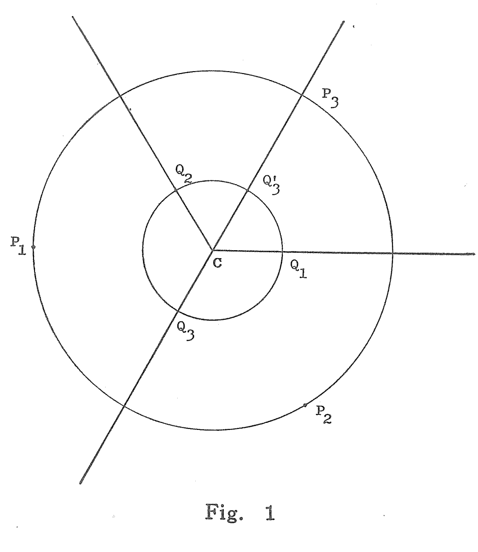

Fig. 1

$$(4) \qquad f_\alpha(P)=\sum_{j=1}^{3}\frac{1}{\overline{PP_j}^{\alpha}}=\sum_{j=1}^{3}\frac{1}{|z-z_j|^{\alpha}}, \quad z_j=e^{\frac{(2j-5)}{3}\pi i}, \ (j=1,2,3)$$

attains its minimum 3 *at the origin C when* $3<(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$, *or its minimum* $(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$ *at* Q_j $(j=1,2,3)$ *when* $3>(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$, *where* $\overline{PP_j}$ $(j=1,2,3)$ *denotes the the distance between* P *anb* P_j *and* z *andz*$_j$ *represent complex numbers corresponding to* P *and* P_j *and* Q_j $(j=1,2,3)$ *is the intersectng point of the circumference of* U_R *with the extension of the line segment* P_jC *toward the center* C *of* U_R (see Fig. 1).

4. Proof of THEOREM A

Suppose that THEOREM B establishes. We put $R=\frac{1}{2}$ in THEOREM B. Then it implies that $(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}=\left(\frac{2}{3}\right)^{\alpha}+2\left(\frac{2\sqrt{3}}{3}\right)^{\alpha}$. Take a positive number α such that $\left(\frac{2}{3}\right)^{\alpha}+2\left(\frac{2\sqrt{3}}{3}\right)^{\alpha}>3$. Then for such α THEOREM A is easily proved from THEOREM B. For the function (2) is decomposed into pieces consisting of three terms, each of which corresponds to centers of three circles so that such centers are vertices of a equilateral triangle with centroid at the origin C. We can apply THEOREM B to each piece, since we can consider that the distances from the origin to the fixed points and the radius of U_R in THEOREM B are relative.

Therefore if each piece attains its minimum at the origin, then it is easily seen that the sum of pieces attains also its minimum at the origin.

REMARK. We shall investigate the condition (3) of THEOREM A. We know easily that $\left(\frac{2}{3}\right)^{\alpha}+2\left(\frac{2\sqrt{3}}{3}\right)^{\alpha}$ is greater or smaller than 3 when α is 1.5 or 1. Hence a root of th əh the equation $\left(\frac{2}{3}\right)^{\alpha}+2\left(\frac{2\sqrt{3}}{3}\right)^{\alpha}=3$ is in the interval (1, 1.5). If α is any positive number greater than or equal to 1.5, THEOREM A establishes.

5. LEMMAS for the proof of THEOREM B

For the proof we shall prepare the folowing lemmas.

LEMMA 1. *Let* $\rho(>0)$ *and* θ_0 $(\geqq 0)$ *be fixed numbers satisfying the following inequality* :

$$(5) \qquad 0\leqq \rho\cos\theta_0\leqq 1.$$

Consider the function of θ $g(\theta)=\{\rho-\cos(\theta+\theta_0)\}\{\rho-\cos(\theta-\theta_0)\}$. *Then* $g(\theta)$ *takes its maximum at* $\theta=0$, *that is*, $\max\limits_{|\theta|\leqq\theta_1} g(\theta)=g(0)$, *if*

$$(6) \qquad |\theta|\leqq\theta_1(\rho)=|\cos^{-1}(2\rho\cos\theta_0-1)|.$$

Proof. Consideering the difference

$$g(0)-g(\theta)=(1-\cos\theta)\{\cos\theta-(2\rho\cos\theta_0-1)\},$$

we can easily prove this lemma. q.e.d.

For the later use we shall give some remarks.

REMARK (i). Put $\theta_0=\frac{\pi}{6}$ and $\rho=(r/\sqrt{3}+\sqrt{3}/r)/2$. If we suppose that $1\leqq r\leqq 3$, then we obtain easily from (6)

$$\cos\theta_1 = \frac{r^2-2r+3}{2r}. \tag{7}$$

REMARK (ii). Put $\theta_0=\frac{\pi}{3}$ and $\rho=(1/r+r)/2$. Then the sufficient condition for $\theta_1\geqq\pi/3$ is the following inequality :

$$\frac{(3-\sqrt{5})}{2} \leqq r \leqq \frac{(3+\sqrt{5})}{2}. \tag{8}$$

LEMMA 2. *The function*

$$h(\theta) = \sin^{\alpha}(\theta+\frac{\pi}{6})\ \{(2\sin\theta)^{-\alpha}+2\}, \qquad (\alpha>0) \tag{9}$$

is positive in $0<\theta<\frac{5}{6}\pi$ *and takes its minimum and maximum only once at* $\theta=\frac{\pi}{6}$ *and* $\theta=\theta^*(\alpha)$ $(\frac{\pi}{6}<\theta^*(\alpha)<\frac{\pi}{3})$, *respectively, where* $\theta^*(\alpha)$ *is th value depending only on* α.

Proof. Defferentiating $h(\theta)$ with respect to θ, we obtain

$$h'(\theta) = \frac{\alpha\ \sin^{\alpha-1}(\theta+\frac{\pi}{6})}{(3\ \sin\ \theta)^{\alpha+1}}\ \{2\ \cos(\theta+\frac{\pi}{6})\ (2\sin\theta)^{\alpha+1}-1\}. \tag{10}$$

Since the factor $\alpha\ \sin^{\alpha-1}(\theta+\frac{\pi}{6})/(2\ \sin\ \theta)^{\alpha+1}$ is positive in (10), the sign of $h'(\theta)$ coincides with one of $h_1(\theta)=2\ \cos\ (\theta+\frac{\pi}{6})\ (2\ \sin\ \theta)^{\alpha+1}-1$, that is,

$$\text{Sign}\ [h'(\theta)] = \text{Sign}\ [h_1(\theta)],\ 0<\theta<\frac{5}{6}\pi.$$

Further differentiating $h_1(\theta)$ with respect to θ, we have

$$h_1'(\theta) = 2(\alpha+2)\ (2\ \sin\ \theta)^{\alpha}\left\{\cos(2\theta+\frac{\pi}{6})+\frac{\sqrt{3}\,\alpha}{2(\alpha+2)}\right\}. \tag{11}$$

Since $0<\frac{\sqrt{3}\,\alpha}{2(\alpha+2)}<\frac{\sqrt{3}}{2}$ for $0<\alpha<\infty$, $h_1'(\theta)=0$ $(0<\theta<\frac{5}{6}\pi)$ has only two roots θ_1^* and θ_2^* which satisfy the following inequalities :

$$\frac{\pi}{6}<\theta_1^*<\frac{\pi}{3},\quad \frac{\pi}{2}<\theta_2^*<\frac{3}{3}\pi,$$

respectively. It is obvious that

$$(12)\qquad \begin{cases} h_1(0) = -1\,, & h_1'(0) = 0 \\ h_1\left(\frac{\pi}{6}\right) = 0\,, & h_1'\left(\frac{\pi}{6}\right) > 0 \\ h_1(\theta) < 0 \text{ in } \frac{\pi}{3} < \theta < \frac{5}{6}\pi\,. \end{cases}$$

Then we have the following table :

θ	0	$\frac{\pi}{6}$	θ_1^*	θ^*	$\frac{\pi}{3}$	θ_2^*
$h_1'(\theta)$	0	$+$	0	$-$	$-$	0
Sign $[h_1(\theta)]$	$-$	0	$+$	0	$-$	$-$

where

$$(13)\qquad \frac{\pi}{6} < \theta_1^* < \theta^* < \frac{\pi}{3} < \theta_2^* < \frac{5}{6}\pi\,.$$

Further we have $\lim_{\theta\to 0} h(\theta)=\infty$ and $\lim_{\theta\to\frac{5}{6}\pi} h(\theta)=0$. q.e.d.

6. Proof of THEOREM B

Now let us prove THEOREM B. From the symmetricity of the figure (Fig. 1), it is enough to prove the theorem in the closed sector $\bar{D}$ bounded by two line segments CQ_1 and CQ_3' and a circular arc $\widehat{Q_1Q_3'}$, respectively.

The proof is divideded into two parts (I) and (II).

(I) At first we shall show the minimum of $f_\alpha(P)$ lies on the line segment CQ_1.

6.1. The case $R \leqq \sqrt[3]{2}$.

Let us take P_1 as the pole of polar coordinate system and denote the coordinates of P_2, P_3 and P by $(\sqrt{3}, -\frac{\pi}{6})$, $(\sqrt{3}, \frac{\pi}{6})$ and (r,θ), respectively. Let r $(1 \leqq r \leqq R)$ be fixed. Putting $g_1(\theta)=(\overline{PP_2}\cdot\overline{PP_3})^2$, we obtain easily

$$(14)\qquad \begin{aligned} g_1(\theta) &= \left\{r^2+3-2\sqrt{3}r\cos\left(\theta+\frac{\pi}{6}\right)\right\}\left\{r^2+3-2\sqrt{3}r\cos\left(\theta-\frac{\pi}{6}\right)\right\} \\ &= (2\sqrt{3}r)^2\left\{\rho-\cos\left(\theta+\frac{\pi}{6}\right)\right\}\left\{\rho-\cos\left(\theta-\frac{\pi}{6}\right)\right\}, \end{aligned}$$

where $\rho=(r/\sqrt{3}+\sqrt{3}/r)/2$.

Take a point P in $\bar{D}$ and rotate the radius vector $\overrightarrow{P_1P}$ around the pole P_1 and denote by P' the intersecting point of $\overrightarrow{P_1P}$ with the line segment CQ_3'. Denoting by $\theta_2(r)$ the argument of the radius vector $\overrightarrow{P_1P'}$ that is, the angle which $\overrightarrow{P_1P'}$ and the polar contain, we obtain

(15) $$\cos\theta_2 = \frac{3+\sqrt{4r^2-3}}{4r}.$$

Because from the rule of sine $\sin\left(\frac{2}{3}\pi\right)\Big/ r = \sin\left(\frac{2}{3}\pi+\theta_2\right)$, we have $\cos\theta_2=(3\pm\sqrt{r^2-3})/4r$ and we see easily that θ moves from $\frac{\pi}{3}$ to $\frac{\pi}{2}$ for $\cos\theta=(3-\sqrt{4r^2-3})/4r$ and on the other hand from 0 to $\frac{\pi}{6}$ for $\cos\theta=(3+\sqrt{4r^2-3})/4r$ when r varies from 1 to $\sqrt{3}$ $(>\sqrt[3]{2})$.

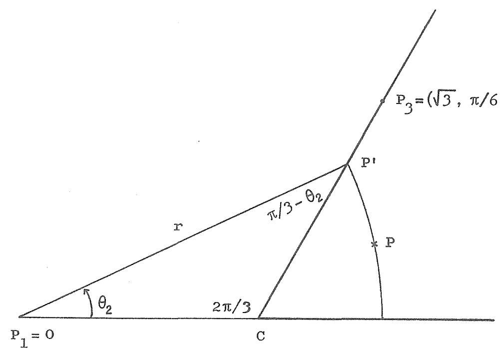

Fig. 2

Then it holds from (7) in REMARK (i) of No. 5, (15) and the assumption $1\leqq r\leqq R$ that

(16)
$$\cos\theta_2-\cos\theta_1 = \frac{3+\sqrt{4r^2-3}}{4r}-\frac{r^2-2r+3}{2r}$$
$$= \frac{(r-1)\{2-(r-1)^3\}}{r\{2(r-1)^2+\sqrt{4r^2-3}+1\}} \geqq 0\ .$$

Since θ_2 is less than $\frac{\pi}{3}$, we obtain from (16)

(17) $$0\leqq\theta_2\leqq\theta_1\ .$$

Thus we have from (14), (17) and LEMMA 1

(18) $$\max_{0\leqq\theta\leqq\theta_2}(\overline{PP_2}\cdot\overline{PP_3})^2 = \max_{0\leqq\theta\leqq\theta_2}\frac{1}{2\sqrt{3r}}\,g(\theta) = g_1(0).$$

Therefore we could proved that (18) holds in the closed domain $D'=D\cap K(P_1,\ 1+\sqrt[3]{2})$, where $K(P_1,\ 1+\sqrt[3]{2})$ denotes the closed disc of radius $1+\sqrt[3]{2}$ with center at P_1.

6. 2. The case $\sqrt[3]{2}<R<\dfrac{3+\sqrt{5}}{2}$.

In this case we take C as the pole. Then P_k $(k=1,2,3)$ and P have the following polar coordinates $(1,\ (2k+1)\pi/3)$ $(k=1,2,3)$ and $(r,\ \theta)$, respectively (Fig. 3).

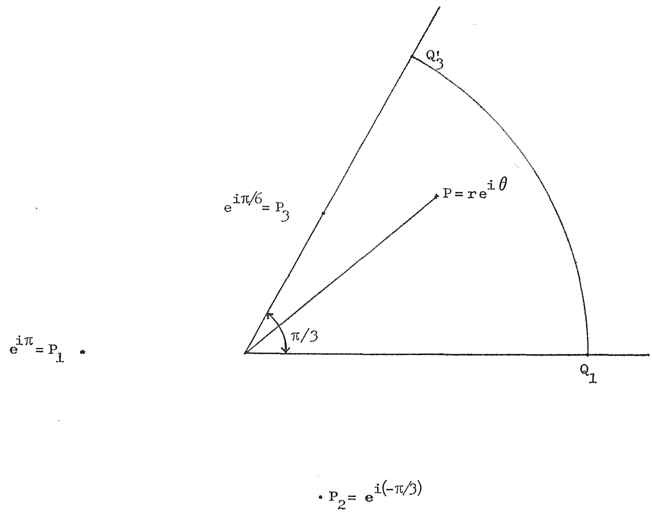

Fig. 3

Let r $(\sqrt[3]{2}<r\leqq R)$ be fixed. Putting $g_2(\theta)=(\overline{PP_2}\cdot\overline{PP_3})^2$, we have

$$(19)\qquad \begin{aligned} g_2(\theta)&=\{r^2+1-2r\cos(\theta+\frac{\pi}{3})\}\{r^2+1-2r\cos(\theta-\frac{\pi}{3})\}\\ &=(2r)^2\{\rho-\cos(\theta+\frac{\pi}{3})\}\{\rho-\cos(\theta-\frac{\pi}{3}\}, \end{aligned}$$

where $\rho=(1/r+r)/2$.

Now we want to get the sufficient condition for θ_1 defined in Lemma **1** to be greater than $\frac{\pi}{3}$. If such condition is gotten, $g_2(\theta)$ in (19) takes its maximum at $\theta=0$ under this condition. It is obvious that $\frac{\pi}{3}<\theta_1$ is equivalent to $\cos\theta_1=2\rho\cos\frac{\pi}{3}-1<\cos\frac{\pi}{3}$. Since $\rho=(1/r+r)/2$, we have

$$\frac{1}{2}(\frac{1}{r}+r)-1<\frac{1}{2},$$

and hence we obtain

$$(20)\qquad \frac{3-\sqrt{5}}{2}<r<\frac{3+\sqrt{5}}{2},$$

which is the desired condition (see Remark (ii) in No. **5**).

Suppose that $(3-\sqrt{5})/2<\sqrt[3]{2}<r\leqq R<(3+\sqrt{5})/2$. Then we get $\frac{\pi}{3}<\theta_1$ and hence the assumption (6) in Lemma **1** is satisfied in the case $\theta_0=\frac{\pi}{3}$. Thus we have

$$\max_{0\leqq\theta\leqq\frac{\pi}{3}} (\overline{PP_2}\cdot\overline{PP_3})^2 = \max_{0\leqq\theta\leqq\frac{\pi}{3}} \left(\frac{1}{2r}\right)^2 g_2(\theta) = g_2(0)\ . \tag{21}$$

Since

$$\overline{PP_1}^2 = r^2 + 1 + 2r\cos\theta, \tag{22}$$

$\overline{PP_1}^2$ attains its maximum on the line segment CQ_1 for fixed r.

6.3. Arranging the results (18), (21) and (22), we can easily see that $\overline{PP_1}$ and $\overline{PP_2}\cdot\overline{PP_3}$ attains their maximums on the line segment CQ_1 in both cases 6.1 and 6.2, respectively.

If P_1 or C is taken as the pole according to the cases 6.1 and 6.2, and r is fixed, then $f_\alpha(P)$ is written in the from of the function of θ, tha tis, $f_\alpha(P)=F(\theta)$. Hence we have from LEMMA 1 the following inequality:

$$\begin{aligned} F(0) &\geqq \min_\theta F(\theta) \geqq \min_\theta(\overline{PP_1})^{-\alpha} + \min_\theta\{(\overline{PP_2})^{-\alpha}+(\overline{PP_3})^{-\alpha}\} \\ &\geqq \min_\theta(\overline{PP_1})^{-\alpha} + 2\min_\theta\{(\overline{PP_2}\cdot\overline{PP_3})^{-\frac{\alpha}{2}}\} \\ &= \{\max_\theta(\overline{PP_1})\}^{-\alpha} + 2\{\max_\theta(\overline{PP_2}\cdot\overline{PP_3})\}^{-\frac{\alpha}{2}} \\ &= \{\overline{P(0)P_1}\}^{-\alpha} + 2\{\overline{P(0)P_2}\cdot\overline{P(0)P_3}\}^{-\frac{\alpha}{2}}, \end{aligned} \tag{23}$$

where $\max\limits_\theta$ and $\min\limits_\theta$ mean the maximum and minimum in $0\leqq\theta\leqq\frac{\pi}{6}$ or $0\leqq\theta\leqq\frac{\pi}{3}$ according to 6.1 or 6.2 and $P(0)$ denotes the point on the line segment CQ_1.

Since $\overline{PP_2}=\overline{PP_3}$, if P moves on the line segment CQ_1, it holds

$$2\{\overline{P(0)P_2}\cdot\overline{P(0)P_3}\}^{-\frac{\alpha}{2}} = \{\overline{P(0)\cdot P_2}\}^{-\alpha} + \{\overline{P(0)\cdot P_3}\}^{-\alpha}, \tag{24}$$

and hence from (23) and (24)

$$F(0) \geqq \min_\theta F(\theta) \geqq F(0)\ . \tag{25}$$

Thus we could prove that $f_\alpha(P)$ attains its minimum on the line segment CQ_1.

7. (II) Next we shall prove that the function $f_\alpha(P)$ takes its minimum at the point C and its maximum at some point P^* on the line segment CQ_1, when P moves on the half line from P_1 to the direction of C.

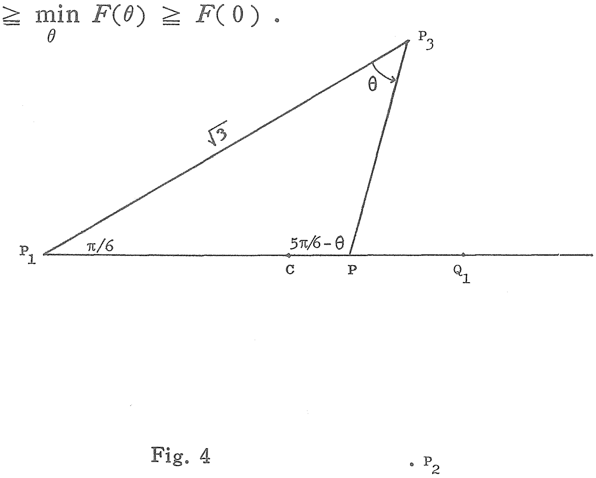

Fig. 4

Let θ denote the angle

which two sides P_1P_3 and PP_3 contain (Fig. 4).
It is easily found from the rule of sine that

$$\frac{\sin\theta}{PP_1}=\frac{\sin(\frac{5}{6}\pi-\theta)}{\sqrt{3}}=\frac{\sin(\frac{\pi}{6})}{PP_3} \quad (0<\theta<\frac{5\pi}{6}). \tag{26}$$

Then we have

$$\begin{cases} \dfrac{1}{PP_1}=\dfrac{\sin(\theta+\frac{\pi}{6})}{\sqrt{3}\,\sin\theta} \\[2ex] \dfrac{1}{PP_3}=\dfrac{2\,\sin(\theta+\frac{\pi}{6})}{\sqrt{3}}\left(=\dfrac{1}{PP_2}\right). \end{cases} \tag{27}$$

Thus the function $f_\alpha(P)$ is written in the following form:

$$f_\alpha(P)=(\frac{2}{\sqrt{3}})^\alpha \sin^\alpha(\theta+\frac{\pi}{6})\,\{(2\,\sin\theta)^{-\alpha}+2\}.$$

By using $h(\theta)$ in LEMMA 2 we have

$$f_\alpha(P)=(\frac{2}{\sqrt{3}})^\alpha h(\theta).$$

Hence from the result of LEMMA we can conclude that $f_\alpha(P)$ takes its minimun 3 at $P=C$ when θ is $\frac{\pi}{6}$, and its maximum at $P=P^*$ when θ is θ^*. It is obvious that $f_\alpha(P)$ takes the boundary value $(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$ at three points Q_k $(k=1.2.3)$ on the circumference of U_R (Fig. 1). Therefore $f_\alpha(P)$ attains its minimum 3 at the origin C when $3<(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$, or its minimum $(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$ at Q_j $(j=1.2.3)$ when $3>(R+1)^{-\alpha}+2(R^2-R+1)^{-\frac{\alpha}{2}}$ in the closed disc $U_R: |z|\leqq R\ (\leqq(3+\sqrt{5})/2)$. Thus our proof is completed.

q. e. d.

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[2] T. AKAZA and N. OGI, On the minimum of some subharmonic function, Science Reports of Kanazawa Univ., 14, No. 1 (1969), 1-11.