On Thullen' s Example of a Cousin-II Domain

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(Received 30 June 1964)

Introduction

A complex space in which any additive (or multiple) Cousin's distribution has a solution is called a *Cousin*-I (or *Cousin*-II) complex space. In 1935 P. Thullen [12] showed that $\{(z_1, z_2) ; |z_1|<1, |z_2|<1, (z_1, z_2)\neq(0,0)\}$ is not a Cousin-I domain but a Cousin-II domain. We shall remark in the present paper that this example possesses another important meaning.

Let $\mathfrak{O}^*$ be the sheaf of all germs of holomorphic mappings in $GL(1,C)$. A complex space X with $H^1(X, \mathfrak{O}^*)=0$ is a Cousin-II complex space. Thullen's example shows that the converse is not generally true. More generally, a Cousin-II domain E in C^2 with an isolated boundary point is a Cousin-II domain with $H^1(E, \mathfrak{O}^*)\neq 0$. Moreover, under the assumption of $H^1(D, Z)=0$ for the additive group Z of all integers, if a domain D in C^n is not a Cousin-I domain, D satisfies $H^1(D, \mathfrak{O}^*)\neq 0$.

1. Thullen's example of a Cousin-II domain.

LEMMA 1. *Let* $E=\{z=(z_1, z_2, \cdots, z_n) ; |z_1|<a ; |z_2|<a, \cdots, |z_n|<a, (z_1, z_2)\neq(0,0)\}$, $E_1=\{z ; z_1\neq 0, z\in E\}$, $E_2=\{z ; z_2\neq 0, z\in E\}$ *and* $\mathfrak{U}=\{E_1, E_2\}$ *for* $n\geqq 2$ *and* $a>0$. *Then* $e^{1/z_1z_2}\notin B^1(\mathfrak{U}, \mathfrak{O}^*)$.

PROOF. Suppose that $e^{1/z_1z_2}\in B^1(\mathfrak{U}, \mathfrak{O}^*)$. There exist $g\in H^0(E_1, \mathfrak{O}^*)$ and $h\in H^0(E_2, \mathfrak{O}^*)$ such that $e^{1/z_1z_2}=g/h$ in $E_1\cap E_2$. $g(e^u, z_2, z_3, \cdots, z_n)$ and $h(z_1, e^v, z_3, \cdots, z_n)$ are, respectively, holomorphic and different from zero in $\{(u, z_2, z_3, \cdots, z_n) ; \mathrm{Re}\, u<\log a, |z_2|<a, |z_3|<a, \cdots, |z_n|<a\}$ and $\{(z_1, v, z_3, \cdots, z_n) ; |z_1|<a, \mathrm{Re}\, v<\log a, |z_3|<a, \cdots, |z_n|<a\}$. Any branches of $\log g(e^u, z_2, z_3, \cdots, z_n)$ and $\log h(z_1, e^v, z_3, \cdots, z_n)$ are uniform and holomorphic there. If we take suitable branches of logarithmus, we have

$$\log g(e^u, e^v, z_3, \cdots, z_n)-\log h(e^u, e^v, z_3, \cdots, z_n)=e^{-u-v}$$

in $\{(u, v, z_3, \cdots, z_n) ; \mathrm{Re}\, u<\log a, \mathrm{Re}\, v<\log a, |z_3|<a, \cdots, |z_n|<a\}$. Hence we have

$$\log g(e^{u+2k\pi i}, z_2, z_3, \cdots, z_n)=\log g(e^u, z_2, z_3, \cdots, z_n)$$

in $\{(u, z_2, z_3, \cdots, z_n) ; \mathrm{Re}\, u<\log a, |z_2|<a, |z_3|<a, \cdots, |z_n|<a\}$ for any integer k and

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$$\log h(z_1, e^{v+2l\pi i}, z_3, \cdots, z_n) = \log h(z_1, e^v, z_3, \cdots, z_n)$$

in $\{(z_1, v, z_3, \cdots, z_n); |z_1|<a, \text{ Re } v<\log a, |z_3|<a, \cdots, |z_n|<a\}$ for any integer l. If we put

$$G(z_1, z_2, \cdots, z_n) = \log g(e^u, z_2, z_3, \cdots, z_n)$$

for $(z_1, z_2, \cdots, z_n) = (e^u, z_2, \cdots, z_n) \in E_1$ and

$$H(z_1, z_2, z_3, \cdots, z_n) = \log h(z_1, e^v, z_3, \cdots, z_n)$$

for $(z_1, z_2, z_3, \cdots, z_n) = (z_1, e^v, z_3, \cdots, z_n) \in E_2$. G and H are, respectively, uniform and holomorphic in E_1 and E_2. Therefore they are represented by absolutely and uniformly convergent series as follows;

$$G(z_1, z_2, \cdots, z_n) = \sum_{p, q=-\infty}^{+\infty} g_{pq}(z_3, z_4, \cdots, z_n) z_1^p z_2^q, \quad g_{pq} = 0 \text{ for } q<0$$

in $0<|z_1|<a, |z_2|<a$ where g_{pq} are holomorphic functions in $\{(z_3, z_4, \cdots, z_n); |z_3|<a, |z_4|<a, \cdots, |z_n|<a\}$.

$$H(z_1, z_2, \cdots, z_n) = \sum_{p, q=-\infty}^{+\infty} h_{pq}(z_3, z_4, \cdots, z_n) z_1^p z_2^q, \quad h_{pq} = 0 \text{ for } p<0$$

in $|z_1|<a, 0<|z_2|<a$ where h_{pq} are holomorphic functions in $\{(z_3, z_4, \cdots, z_n); |z_3|<a, |z_4|<a, \cdots, |z_n|<a\}$. There holds

$$G(z_1, z_2, \cdots, z_n) - H(z_1, z_2, \cdots, z_n) = 1/z_1 z_2$$

in $E_1 \cap E_2$. Hence we have

$$0 = g_{-1-1} - h_{-1-1} = 1.$$

But this is a contradiction. Thus we have $e^{1/z_1 z_2} \notin \mathrm{B}^1(\mathfrak{U}, \mathfrak{O}^*)$.

LEMMA 2. *Let D be an n-dimensional complex space ($n \geqq 2$) and f_1 and f_2 be holomorphic functions in D such that the rank of the functional matrix $(\partial f_j/\partial t_k)$ $(j=1, 2; k=1, 2, \cdots, n)$ is 2 at some uniformizable point x^o of $A=\{x; f_1(x)=f_2(x)=0, x \in D\}$ for local coordinates $t_1, t_2, \cdots, t_n$ of D at x^o. Then we have $\mathrm{H}^1(G, \mathfrak{O}^*) \neq 0$ for $G=D-A$.*

PROOF. We put $G_1=\{x; f_1(x) \neq 0, x \in D\}$ and $G_2=\{x; f_2(x) \neq 0, x \in D\}$ and $\mathfrak{U}'=\{G_1, G_2\}$. Suppose that $\mathrm{H}^1(G, \mathfrak{O}^*)=0$. Then we have $\mathrm{H}^1(\mathfrak{U}', \mathfrak{O}^*)=0$. Therefore we have $e^{1/f_1 f_2} \in \mathrm{B}^1(\mathfrak{U}', \mathfrak{O}^*)$. There exist $g \in \mathrm{H}^o(G_1, \mathfrak{O}^*)$ and $h \in \mathrm{H}^o(G_2, \mathfrak{O}^*)$ such that $e^{1/f_1 f_2} = g/h$ in $G_1 \cap G_2$. From the assumption there exists a biholomorphic mapping $t=(t_1, t_2, \cdots, t_n)=\tau(x)$ of a neighbourhood of x^o onto $E=\{t; |t_1|<a, |t_2|<a, \cdots, |t_n|<a\}$ such that $t_1=f_1(x)$, $t_2=f_2(x)$ and $(0, 0, \cdots, 0)=\tau(x^o)$. Let $E_1=\{t; t_1 \neq 0, t \in E\}$ and $E_2=\{t; t_2 \neq 0, t \in E\}$. Then we have $e^{1/t_1 t_2} = g \circ \tau^{-1}/h \circ \tau^{-1}$ in $E_1 \cap E_2$ for $g \circ \tau^{-1} \in \mathrm{H}^o(E_1, \mathfrak{O}^*)$ and $h \circ \tau^{-1} \in \mathrm{H}^o(E_2, \mathfrak{O}^*)$. But this contradicts to Lemma 1. Thus we have $\mathrm{H}^1(G, \mathfrak{O}^*) \neq 0$.

For the purpose of constructing a Cousin-II complex manifold G which satisfies the condition of Lemma 2, we dare prove the following well-known lemma making use of the prolongation of analytic sets due to P. Thullen [13].

LEMMA 3. *Let A be an $(n-1)$-dimensional irreducible analytic set in an n-dimensional Cousin*-II *complex manifold D. Let U be a subdomain of D such that $U \cap A \neq \emptyset$ is connected. Then $E=(D-A)\cap U$ is a Cousin*-II *complex manifold.*

PROOF. Let $\mathfrak{C}=\{(p_\lambda, V_\lambda) ; \lambda \in \Lambda\}$ be a multiple Cousin's distribution in E such that each p_λ is holomorphic in V_λ. We shall prove the uniqueness of the prolongation of $\mathfrak{C}$ to a subdomain W of D containing E, if the prolongation exists. Let $\{(P_i', U_i) ; i \in I\}$ and $\{(p_i'', U_i) ; i \in I\}$ be multiple Cousin's distributions in W such that p_i'/p_λ and $p_i''/p_\lambda \in H^0(V_\lambda \cap U_i, \mathfrak{O}^*)$ for $V_\lambda \cap U_i \neq \emptyset$. We consider U_i such that $A \cap U_i = \emptyset$. For any point x of U_i, there exists V_λ containing x. Then we have

$$p_i'/p_i'' = (p_i'/p_\lambda)/(p_i''/p_\lambda) \in H^0(V_\lambda \cap U_i, \mathfrak{O}^*).$$

Hence we have $p_i'/p_i'' \in H^0(U_i, \mathfrak{O}^*)$.

The set B of all irregular points of A is an at most $(n-2)$-dimensional analytic set in D. We consider U_i such that $A \cap U_i \neq \emptyset$. For any point $x \in U_i - B$, there exist, repsectively, subdomains W_1, $W_2, \cdots$ and $W_s \ni x$ of $U_{i_1}-B$, $U_{i_2}-B, \cdots$ and $U_{i_s}-B$ such that $A \cap W_1 \cap \partial U \neq \emptyset$ and $A \cap W_1$, $A \cap W_2, \cdots$ and $A \cap W_s$ are $(n-1)$-dimensional connected analytic sets in W_1, $W_2, \cdots$ and W_s and that $A \cap W_1 \cap W_2 \neq \emptyset$, $A \cap W_2 \cap W_3 \neq \emptyset, \cdots$ and $A \cap W_{s-1} \cap W_s \neq \emptyset$ where $U_{i_s} = U_i$. Since p_{i_1}'/p_{i_1}'' and p_{i_1}''/p_{i_1}' are holomorphic in $(W_1 \cap U) \cup (W_1 - A)$, the sets of all their singularities are empty or $(n-1)$-dimensional analytic sets in W_1. Since A is irreducible in W_1, they must be empty. Continuing this argument, we can prove that $p_{i_s}'/p_{i_s}'' \in H^0(W_s, \mathfrak{O}^*)$. Therefore $p_i'/p_i'' \in H^0(U_i - B, \mathfrak{O}^*)$. Since B is at most $(n-2)$-dimensional, we have $p_i'/p_i'' \in H^0(U_i, \mathfrak{O}^*)$. Thus we have proved the uniqueness of the prolongation of $\mathfrak{C}$.

Let $\widetilde{E}$ be the sum of all subdomains W of D with connected $W \cap A$ containing E to which $\mathfrak{B}$ is prolongable. From the uniqueness of the prolongation of $\mathfrak{C}$ there exists a prolongation $\{(p_i, U_i) ; i \in I\}$ of $\mathfrak{C}$ to $\widetilde{E}$ for any open covering $\{U_i ; i \in I\}$ of $\widetilde{E}$ with connected $\widetilde{E} \cap A$ such that each U_i is a Cousin-II complex manifold. We shall prove that $\widetilde{E} = D$.

Suppose that $\widetilde{E} \subsetneqq D$. $N = \bigcup_{i \in I} \{x ; p_i(x) = 0, x \in U_i\}$ is a pure $(n-1)$-dimensional analytic set in $\widetilde{E} \supset D - A$. From P. Thullen [13] the set of all essential singularities of N is empty or coincides with A. Since N is analytic in $A \cap U$, it is empty. There exists an $(n-1)$-dimensional analytic set L in D such that $L \cap \widetilde{E} = N$. We shall take a point x^0 of $D \cap \partial\widetilde{E}$ and an open connected neighbourhood W_0 of x^0 with connected $\widetilde{E} \cap W_0 \cap A$ such that there exist holomorphic functions f_1, $f_2, \cdots$ and f_s in W_0 satisfying the following conditions:

f_1, $f_2, \cdots$ and f_s are irreducible in each point of W_0. $W_0 \cap L$ is the union of irreducible sets $\{x ; f_a(x) = 0, x \in W_0\}$ in W_0 for $a = 1, 2, \cdots. s$. Without loss of generality we may suppose that each $\{x\,;f_a(x)=0,\ x\in W_o\cap U_i\}$ is connected in $W_o \cap U_i$ for any a and i.

We consider f_a such that $\{x\,;f_a(x)=0\,,x\in W_o\}\not\subset A$. We take U_{i_1} containing $x^1\in\{x\,;f_a(x)=0\,,x\in W_o\}-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$. We can take an integer α_a such that $p_{i_1}/f_a^{\alpha_a}$ is holomorphic and different from zero in some neighbourhood W' of x^1. Then $p_{i_1}/f_a^{\alpha_a}$ and $f_a^{\alpha_a}/p_{i_1}$ are holomorphic in $(U_{i_1}\cap W_o-\bigcup_{b=1}^{s}\{x\,;f_b(x)=0\,,x\in W_o\})\cup W'$. Since $\{x\,;f_a(x)=0\,,x\in U_{i_1}\cup W_o\}$ is connected and irreducible in each point of $U_{i_1}\cap W_o$, $p_{i_1}/f_a^{\alpha_a}$ and $f_a^{\alpha_a}/p_{i_1}$ are holomorphic in $U_{i_1}\cap W_o-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$. Let x be any point of $U_i\cap\{x\,;f_a(x)=0\,,x\in W_o\}-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$. There exist, respectively, subdomains $W_1, W_2,\cdots$ and $W_s\ni x$ of $W_o\cap U_{i_1}-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$, $W_o\cap U_{i_2}-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\},\cdots$ and $W_o\cap U_{i_s}-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$ such that $W_1\cap\{x\,;f_a(x)=0\,,x\in W_o\}$, $W_2\cap\{x\,;f_a(x)=0\,,x\in W_o\},\cdots$ and $W_s\cap\{x\,;f_a(x)=0\,,x\in W_o\}$ are connected in W_1, $W_2,\cdots$ and W_s and that $W_1\cap W_2\cap\{x\,;f_a(x)=0\,,x\in W_o\}\neq\emptyset$, $W_2\cap W_3\cap\{x\,;f_a(x)=0\,,x\in W_o\}\neq\emptyset,\cdots$ and $W_{s-1}\cap W_s\cap\{x;f_a(x)=0\,,x\in W_o\}\neq\emptyset$ where $U_{i_o}=U_i$. Since $p_{i_2}/f_a^{\alpha_a}$ and $f_a^{\alpha_a}/p_{i_2}$ are holomorphic in $(W_2-\{x\,;f_a(x)=0\,,x\in W_o\})\cap(W_1\cap W_2)$, they are holomorphic in W_2. Continuing this argument, we can prove that $p_i/f_a^{\alpha_a}\in\mathrm{H}^o(W_o,\mathfrak{O}^*)$. Thus we have proved the existence of an integer α_a such that $p_i/f_a^{\alpha_a}$ is holomorphic and different from zero in each $U_i\cap W_o-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$ for f_a with $\{x\,;f_a(x)=0\,,x\in W_o\}\not\subset A$.

For f_a with $\{x\,;f_a(x)=0\,,x\in W_0\}\subset A$, we take U_{i_1} with $U_{i_1}\cap\{x\,;f_a(x)=0\,,x\in W_o\}\cap\tilde{E}\neq\emptyset$. If we repeat the same argument as above for this U_{i_1}, we can prove the existence of an integer α_a such that $p_i/f_a^{\alpha_a}$ is holomorphic and different from zero in each $U_i\cap W_o-\bigcup_{b\neq a}\{x\,;f_b(x)=0\,,x\in W_o\}$ as $\tilde{E}\cap W_o\cap A$ is cnnected.

We put $p_o=\prod_{a=1}^{s}f_a^{\alpha_a}$. For U_i with $U_i\cap W_o\neq\emptyset$, p_i/p_o and p_o/p_i are holomorphic in $U_i\cap W_o$ except in an at most $(n-2)$-dimensional analytic set $\bigcup_{b\neq c}\{x\,;f_b(x)=f_c(x)=0\,,x\in U_i\cap W_o\}$. Therefore they are holomorphic in $U_i\cap W_o$. Hence we have $p_i/p_o\in\mathrm{H}^o(U_i\cap W_o,\mathfrak{O}^*)$. This means that $\{(p_i,U_i)\,;i\in I\}\cup\{(p_o,W_o)\}$ is a multiple Cousin's distribution in $\tilde{E}\cup W_o$ which is a prolongation of $\mathfrak{C}$ to $\tilde{E}\cup W_o$. But this is contradictory to the definition of $\tilde{E}$. Therefore we have $\tilde{E}=D$ and $\mathfrak{C}$ has a prolongation $\tilde{\mathfrak{C}}$ to D. Since D is a Cousin-II complex manifold, $\tilde{\mathfrak{C}}$ has a solution $\tilde{p}$ in D. Its restriction p to E is a solution of $\mathfrak{C}$.

Let $\mathfrak{C}=\{(m_\lambda, V_\lambda)\ ;\ \lambda\in\Lambda\}$ be any multiple Cousin's distribution E. If we take V_λ sufficiently small, m_λ is represented as a quotient p_λ/q_λ of holomorphic functions p_λ and q_λ which are mutually coprime in each point of V_λ. In this case $\{(p_\lambda, V_\lambda)\ ;\ \lambda\in\Lambda\}$ and $\{(q_\lambda, V_\lambda)\ ;\ \lambda\in\Lambda\}$ are multiple Cousin's distributions from S. Hitotumatu [5]. Let p and q be their solutions in E. Then p/q is a solution of $\mathfrak{C}$.

If E is a complex space with $\mathrm{H}^1(E, \mathfrak{O}^*)=0$, E is a Cousin-II complex space. But the converse is not generally true as the following proposition shows.

PROPOSITION 1. *Let A be an analytic set in an n-dimensional Cousin*-II *complex manifold D satisfying the condition of Lemma* 2. *Then $E=D-A$ is an example of Cousin*-II *complex manifold with* $\mathrm{H}^1(E,\mathfrak{O}^*)\neq 0$.

PROOF. We have $\mathrm{H}^1(E, \mathfrak{O}^*)\neq 0$ from Lemma 2. We put $B=\{x\ ;\ f_1(x)=0,\ x\in D\}$. Let B_j be irreducible components of B in D $(j=1, 2, 3, \cdots)$. We put $A_j=B_j\cap A$. From the proof of Lemma 3, we can prove by induction that any multiple Cousin's distribution in $D-\bigcup_{j=m}^{\infty} A_j$ is prolongable to $D-\bigcup_{j=m+1}^{\infty} A_j$ for $m=1, 2, 3, \cdots$ as B_m is irreducible in $D-\bigcup_{j=m+1}^{\infty} A_j$. Therefore any multiple Cousin's distribution in E is prolongable in D. Since D is a Cousin-II complex manifold, E is a Cousin-II complex manifold.

We have easily

COROLLARY. *If E is a Cousin*-II *domain in C^2 with an isolated boundary point, E is a Cousin*-II *domain with* $\mathrm{H}^1(E, \mathfrak{O}^*)\neq 0$.

For the sheaf $\mathfrak{O}$ of all germs of holomorphic functions, however, the author does not know whether there exists a Cousin-I domain E in C^n $(n\geqq 3)$ with $\mathrm{H}^1(E,\mathfrak{O})\neq 0$ or not.

2. Domain D with $\mathrm{H}^1(D, \mathfrak{O}^*)\neq 0$.

Next we shall seek a sufficient condition that a Cousin-II domain D, which is not a Cousin-I domain, satisfies $\mathrm{H}^1(D, \mathfrak{O}^*)\neq 0$. Let Z be the additive group of all integers.

LEMMA 4. *If a domain D in C^n satisfies* $\mathrm{H}^1(D, \mathfrak{O}^*)=\mathrm{H}^1(D, Z)=0$ $(n\geqq 1)$, *then we have* $\mathrm{H}^1(D, \mathfrak{O})=0$.

PROOF. Let $\mathfrak{V}=\{V_\lambda\ ;\ \lambda\in\Lambda\}$ be any open covering of D and $\{f_{\lambda\mu}\ ;\ \lambda, \mu\in\Lambda\}$ be any element of $\mathrm{Z}^1(\mathfrak{V}, \mathfrak{O})$, that is, a 1-*cocycle* of $\mathfrak{V}$ with value in $\mathfrak{O}$. There exists an open covering $\mathfrak{U}=\{U_i\ ;\ i\in I\}$ finer than $\mathfrak{V}$ such that each U_i is an open sphere in D. Let $p: I\to\Lambda$ be a projection. For brevity we put $f_{ij}=f_{p(i)p(j)}$ in $U_i\cap U_j$. Then we have $\{e^{f_{ij}}\ ;\ i, j\in I\}\in\mathrm{Z}^1(\mathfrak{U}, \mathfrak{O}^*)$. As $\mathrm{H}^1(D, \mathfrak{O}^*)=0$, we have $\mathrm{H}^1(\mathfrak{U}, \mathfrak{O}^*)=0$. Hence there exists $h_i\in\mathrm{H}^0(U_i, \mathfrak{O}^*)$ for any $i\in I$ such that $h_i/h_j=e^{f_{ij}}$ in $U_i\cap U_j\neq\varnothing$.

Since each U_i is an open sphere, any branch of log h_i is holomorphic and uniform in U_i. We take an arbitrary but fixed branch of log h_i for any $i\in I$. We put

$$n_{ij}=\frac{1}{2\pi\sqrt{-1}}(\log h_i-\log h_j-f_{ij})$$

in any $U_i\cap U_j\neq\emptyset$. Then each n_{ij} is a holomorphic function in the intersection of two spheres U_i and U_j and satisfies $e^{2\pi\sqrt{-1}\,n_{ij}}=1$ in $U_i\cap U_j$. Therefore n_{ij} is an integer and there holds $\{n_{ij}\,;\,i,j\in I\}\in Z^1(\mathfrak{U},Z)$. As $H^1(D,Z)=0$, we have $H^1(\mathfrak{U},Z)=0$. Hence there exists an integer n_i for any i such that $\{n_{ij}\,;\,i,j\in I\}$ is a coboundary of the 0-cochain $\{n_i,i\in I\}$. If we put

$$f_i=\log h_i-2\pi\sqrt{-1}\,n_i$$

in any U_i. Then we have $\{f_i,i\in I\}\in C^0(\mathfrak{U},\mathfrak{O})$ and $\{f_{ij}\,;\,i,j\in I\}$ is its coboundary. Thus we have proved that $H^1(D,\mathfrak{O})=0$.

From Lemma 4 we have

PROPOSITION 2. *Under the assumption of* $H^1(D,Z)=0$, *if D is a domain in C^2 with* $H^1(D,\mathfrak{O})\neq 0$, *then D satisfies* $H^1(D,\mathfrak{O}^*)\neq 0$.

Thullen's example is not a Cousin-I domain but a Cousin-II domain in C^2 with $H^1(D,Z)=0$. Hence it is an example of Cousin-II domain with $H^1(D,\mathfrak{O}^*)\neq 0$, from Proposition 2, too. More generally we have

COROLLARY. *If E is not a domain of holomorphy but a Cousin*-II *domain in C^2 with* $H^1(E,Z)=0$, *E is a Cousin*-II *domain with* $H^1(E,\mathfrak{O}^*)\neq 0$.

3. Intersection of Cousin-II domains.

In the previous paper [6], we have proved that for any $n\geqq 3$ there exist Cousin-I domains D_1 and D_2 in C^n such that $D_1\cap D_2$ is not a Cousin-I domain. Moreover, in the previous paper [8], we have proved that a domain D in Stein manifold M with a smooth boundary is holomorphically convex if and only if $D\cap G$ is a Cousin-I open set for any holomorphically convex subdomain G of M. We shall remark that an intersection of two Cousin-II domains is not necessarily a Cousin-II open set.

By P. Cousin [2] and K. Oka [10] it is well known that a polycylinder in C^n is a Cousin-II domain if each component is simply connected except one.

LEMMA 5. *Let B_1 be a relatively compact subdomains in a Riemann surface whose boundary consists of finitely many Jordan curves and points and B_2 be Stein manifold contractible to a point. Then* $H^1(D,\mathfrak{O}^*)=0$ *for* $D=B_1\times B_2$.

PROOF. There exists simply connected subdomains B_1' and B_1'' of B_1 such that each connected component of $B_1'\cap B_1''$ is simply connected and $B=B_1'\cup B_1''$. We put $D'=B_1'\times B_2$ and $D''=B_1''\times B_2$. From Riemann's mapping theorem, D' and D'' are Stein manifolds contractible to a point. Hence from H. Grauert [3] we have $H^1(D',\mathfrak{O}^*)=H^1(D'',\mathfrak{O}^*)=0$. We consider an open covering $\mathfrak{U}=\{D',D''\}$ of

D. Then we have $H^1(D, \mathfrak{O}^*)=H^1(\mathfrak{U}, \mathfrak{O}^*)$.

Let F be any element of $H^0(D'\cap D'', \mathfrak{O}^*)$. Since each connected component of $D'\cap D''$ is simply connected, $\log F$ is uniform and holomorphic in $D'\cap D''$ for any branch of logarithmus. Since $H^1(\mathfrak{U}, \mathfrak{O})=0$, there exist $g\in H^0(D', \mathfrak{O})$ and $h\in H^0(D'', \mathfrak{O})$ such that $\log F=g-h$ in $D'\cap D''$. If we put $G=e^g\in H^0(D', \mathfrak{O}^*)$ and $H=e^h\in H^0(D'', \mathfrak{O}^*)$, there holds $F=G/H$ in $D'\cap D''$. Thus we have proved that $H^1(D, \mathfrak{O}^*)=H^1(\mathfrak{U}, \mathfrak{O}^*)=0$.

Let

$$D_1=\{z=z_1, z_2,\cdots, z_n)\ ;\ 1/2<|z_1|<2,\ |z_2|<2,\ |z_3|<2,\cdots,|z_n|<2\}$$

and

$$D_2=\{z\ ;\ |z_1|<2,\ 1/2<|z_2|<2,\ |z_3|<2,\cdots,|z_n|<2\}.$$

for $n\geqq 2$. From Lemma 5 D_1 and D_2 are Cousin-II domains. But their intersection

$$D_1\cap D_2=\{(z_1, z_2)\ ;\ 1/2<|z_1|<2,\ 1/2<|z_2|<2\}\times\{(z_3, z_4,\cdots, z_n)\ ;\ |z_3|<2,$$
$$|z_4|<2,\cdots,\ |z_n|<2\}$$

is not a Cousin-II domain as the first component of the righthand side of the above equation is not a Cousin-I domain from T. H. Gronwall [4]. K. Oka [10] gave a similar example as T. H. Gronwall [4].

PROPOSITION 3. *For any $n\geqq 2$ there exist Cousin-*II *domains D_1 and D_2 in C^n such that $D_1\cap D_2$ is not a Cousin-*II *domain.*

It is more interesting to prove Proposition 3 for $D_1\cap D_2$ with vanishing fundamental group. We call a complex manifold *simply connected* if its fundamental group π_1 vanishes.

4. Sums of Cousin-II domains which unite themselves.

It is equivalent to the Levi problem that the sum $D_1\cup D_2$ of domains of holomorphy D_1 and D_2, which ***unite themselves***, that is, which satisfies $\overline{D_1-D_1\cap D_2}\cap\overline{D_2-D_1\cap D_2}=\varnothing$, is also a domain of holomorphy. In this way the Levi problem is affirmatively solved by H. J. Bremermann [1], F. Norguet [9] and K. Oka [11]. In the previous paper [6] we have remarked the existence of an n-dimensional complex manifold D which is a sum of subdomains D_1 and D_2 with the following properties for $n\geqq 3$; D_1 and D_2 are Stein manifolds which unite themselves. Therefore $H^1(D_1, \mathfrak{O})=H^1(D_2, \mathfrak{O})=0$. D is not, however, a Cousin-I complex manifold and therefore $H^1(D_1\cup D_2, \mathfrak{O})\neq 0$. It is more interesting if this is solved for a domain in C^n.

Concerning the Cousin-II problem, however, we can make a similar example in C^n. We put

$D=\{z=(z_1, z_2, \cdots, z^n)\ ;\ 1/2<|z_1|<2,\ 1/2<|z_2|<2,\ |z_3|<2,\ |z_4|<2, \cdots, |z_n|<2\}$,

$D_1=\{z\ ;\ \mathrm{Im}\ z_1>-1/4, z\in D\}$ and $D_2=\{z; \mathrm{Im}\ z_1<1/4,\ z\in D\}$.

Then $\mathrm{H}^1(D_1, \mathfrak{O}^*)=\mathrm{H}^1(D_2, \mathfrak{O}^*)=0$ from Lemma 5.

D_1 and D_2 unite themselves and $D=D_1\cup D_2$. From T. H. Gronwall [4], however, D is not a Consin-II domain. Hence we have $\mathrm{H}^1(D, \mathfrak{O}^*)\neq 0$.

PROPOSITION 4. *For* $n\geqq 2$ *there exist relatively compact domains* D_1 *and* D_2 *in* C^n *with the following properties*;

(1) $\mathrm{H}^1(D_1, \mathfrak{O}^*)=\mathrm{H}^1(D_2, \mathfrak{O}^*)=0$ (*therefore* D_1 *and* D_2 *are Cousin*-II *domains*).

(2) D_1 *and* D_2 *unite themselves.*

(3) $D_1\cup D_2$ *is not a Cousin*-II *domain* (*therefore* $\mathrm{H}^1(D_1\cup D_2, \mathfrak{O}^*)\neq 0$).

The above domain $D_1\cup D_2$ is a domain which is locally a Cousin-II domain. Therefore Proposition 4 gives a negative answer for the Levi problem for Cousin-II problems. It is more interesting to prove Proposition 4 for simply connected $D_1\cup D_2$.

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