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A Remark on the Riemann-Sum

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1. Introduction

In the present note let $f(t)$, $-\infty<t<+\infty$, be Lebesgue integrable on the interval (0,1) and periodic with period 1. Then we put, for every positive integer k,

$$F_k(t, f)=\frac{1}{k}\sum_{v=1}^{k} f(t+\frac{v}{k})$$

which is known as the Riemann-sum of the function $f(t)$.

In [2] B. Jessen has proved that if $n_k|n_{k+1}$, then we have, for almost all t,

$$\lim_{k\to\infty} F_{n_k}(t, f)=\int_0^1 f(t)dt. \tag{1. 1}$$

Since then various sufficient conditions of (1. 1) for an increasing sequence $\{n_k\}$ without assuming any arithmetical property have been given, but in the case where $f(t)\in L_p$ (0, 1) the essential point is to deduce from them the convergence of the series $\sum_k\int_0^1|F_{n_k}(t, f)-\int_0^1 f(t)dt|^p dt$ (c. f. for example [5]). Therefore under these conditions we have, for almost all t,

$$\lim_{k\to\infty} F_{n_k}(t+h_k, f)=\int_0^1 f(t)dt,$$

where $\{h_k\}$ is any sequence of real numbers. (c. f. [1]).

On the other hand it is not known whether or not there exists a function of the class L_2 (0, 1) such that its Riemann-sum does not converge to the integral almost everywhere.

The purpose of this note is to prove the following

Theorem. Let $\{n_k\}$ be any increasing sequence of prime numbers. Then there exist a sequence of real numbers $\{h_k\}$ and a function $f(t)$ of the class $L_p(0, 1)$ for any $p\geqq 1$ such that the translated Riemann-sum $F_{n_k}(t+h_k, f)$ diverges almost everywhere in t as $k\to+\infty$ **).

2. Proof of the theorem

In this paragraph we prove the theorem. Let us put, for $k=1, 2,\cdots\cdots$,

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**) In the same way we can prove this theorem for the case where $\{n_k\}$ is any increasing sequence of positive integers. (c. f. [4]).

(2. 1) $$a_k=(2k^{-1}\log^{-2}(k+1))^{1/2},$$
and

(2. 1′) $$N_k=\prod_{m=1}^{k} n_m.$$

Then by the above definitions, the series $\sum_{m=1}^{\infty} a_m \cos 2\pi N_m t$ converges to a function $f(t)$ for almost all t which belongs to the class $L_p(0, 1)$ for any $p\geqq 1$ (c. f. [6]). On the other hand since n_k is a prime number, we have

(2. 2) $$F_{n_k}(t, f)=\sum_{m=k}^{\infty} a_m \cos 2\pi N_m t.$$

Further let us consider the sequence $\{F_{n_k}(t_0+x,f)\}$ for any fixed t_0, $0\leqq t_0\leqq 1$, and expand all real numbers x, $0\leqq x\leqq 1$, as follows;

$$x=\sum_{k=1}^{\infty}\varphi_k(x)\ N_k^{-1} \qquad (\varphi_k(x)=0, 1, \cdots\cdots, n_k-1),$$

and put

(2. 3) $$\theta_m(x)=\varphi_{m+1}(x)n^{-1}_{m+1},$$

(2. 3′) $$H_k(t_0, x)=\sum_{m=k}^{\infty} a_m[\cos 2\pi\{\theta_m(x)+N_m t_0\}-\int_0^1 \cos 2\pi\{\theta_m(x)+N_m t_0\}dx].$$

Then by (2. 1), (2. 3) and the relation

$$|\cos 2\pi N_m(t_0+x)-\cos 2\pi\{\theta_m(x)+N_m t_0\}|\leqq 2\pi n_{m+1}^{-1}\leqq 2\pi(m+1)^{-1},$$

we have,

(2. 4) $$|H_k(t_0,x)-F_{n_k}(t_0+x,f)|\leqq Ak^{-1/2}$$
and, for $k>1$,

(2. 4′) $$\left|\int_0^1 H_k^2(t_0, x)dx-\log^{-1} k\right|\leqq Ak^{-1/2},$$

where A is a constant independent of t_0. Since $\{\varphi_k(x)\}$ is a sequence of independent functions, by (2. 3) $[a_m \cos 2\pi\{\theta_m(x)+N_m t_0\}]$ is also a sequence of independent functions. Applying the lemma of A. N. Kolmogoroff [3] to $H_k(t_0,x)$, we obtain

(2. 5) $$\sum_k |(0\leqq x\leqq 1,\ H_k(t_0, x)>1)|=+\infty.$$

By (2. 4) and (2. 5), we have for any ε $(0<\varepsilon<1)$

(2. 6) $$\sum_k |(0\leqq x\leqq 1,\ F_{n_k}(t_0+x,f)>1-\varepsilon)|=+\infty.$$

Next let $\{t_k(\omega)\}$ be a sequence of independent random variables on a probability space (Ω, B, P) and for each k

(2. 7) $$P\{t_k(\omega)<x\}=x, \qquad \text{for } 0\leqq x\leqq 1.$$

By (2. 6) and (2. 7), we have

$$\sum_k P\{F_{n_k}(t_0+t_k(\omega))>1-\varepsilon\}=+\infty,$$

and this implies, by the Borel-Cantelli's lemma,

(2. 8) $$P\{\overline{\lim_{k\to\infty}}\ F_{n_k}(t_0+t_k(\omega))>1-\varepsilon\}=1.$$

Since t_0 in (2. 8) is arbitrary, there exists a point ω_0 such that

$$(2.\ 9) \qquad |(0\leqq t\leqq 1,\ \overline{\lim_{k\to\infty}}\ F_{n_k}(t+t_k(\omega_0))>1-\varepsilon)|=1.$$

If we put $h_k=t_k(\omega_0)$, then by (2. 2) (2. 9) and the fact that $\lim_{k\to\infty}\int_0^1 F^2_{n_k}(t+h_k, f)dt=0$, we can prove the theorem.

We can not seen that there exist a bounded function $f(t)$ and a sequence of real numbers $\{h_k\}$ such that $F_k(t+h_k, f)$ does not converge to $\int_0^1 f(t)dt$ almost everywhere in t as $k\to+\infty$.

References

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