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メタデータ	言語: English 出版者: 公開日: 2017-10-03 キーワード: 作成者: Akaza, Tohru, 赤座, 暢 メールアドレス: 所属:
URL	https://doi.org/10.24517/00011505

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Note on the Subregions Bounded by the Level Curves of the Green's Function

By

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(Received February 9, 1955)

In this paper we shall extend *Walsh's* theorem ([1]) in §§ 1, 2 and investigate the linearization of $T(w)$ with property T in a domain D in § 3 (see § 1 for definition) and prove the existence of non-linear one-to-one conformal mapping of D onto itself in § 4.

§1. On the extension of *Walsh's* theorem.

If $w' = T(w)$ is a given one-valued, schlicht analytic function of w , we say that a region D of the extended w -plane has property T provided that the point $w' = T(w)$ lies in D , whenever w lies in D .

In [1], *J. L. Walsh* extended the results of *Radó*, *Seidel* and *Ford* [2] and proved the following theorem.

Suppose that a domain D has property T , where $T(w)$ is a linear transformation of the form;

$$(1) \quad T : \frac{w'}{w'-\alpha} = A \frac{w}{w-\alpha}, \quad (\alpha \neq 0, A \neq 1, 0 < |A| \leq 1).$$

If the origin $w=0$ lies in D and Green's function $G(w)$ for D with the pole at the point $w=0$ exists, then every subregion $D_r ; \{w ; G(w) > -\log r, 0 < r < 1\}$ has also property T .

Further he proved the following : Let $T(w)$ be a one-valued, analytic function in D with $T(0)=0$ and suppose that a region D has property T and that there exists Green's function of D . Then it holds that

$$(2) \quad |T'(0)| \leq 1.$$

In the case where $|T'(0)|=1$, or when one point and its image $T(w)$ simultaneously lies on the boundary of some D_r , $T(w)$ becomes a schlicht function which maps D onto itself.

§2. Now we shall prove the following.

Theorem 1. Suppose that a region D of the extended w -plane has property T , where $T(w)$ is a one-valued schlicht analytic function with $T(0)=0$. If $w=0$ lies in D and Green's function $G(w)$ for D with the pole at $w=0$ exists, then every subregion D_r has also property T .

Proof. By *Walsh's* method, our theorem is proved as follows. Let D' denote the image of D by $w' = T(w)$. By hypothesis $D' = T(D) \subseteq D$. Since Green's function $G'(w')$ for D' with the pole at $w=0$ is the transform of $G(w)$, we have $G'(w') \equiv G[T^{-1}(w')]$.

If we consider the difference $u(w') = G(w') - G'(w')$, then $u(w')$ is harmonic in D' and ≥ 0 for every boundary point w' of D' . By the maximum principle, we have $G(w') \geq G'(w')$ in D' . Consequently, whenever w lies in D , we have $G[T(w)] \geq G(w)$. Hence, if w lies in D , that is, if $G(w) > -\log r$, we have $G[T(w)] > -\log r$ and hence $T(w)$ lies in D_r . q. e. d.

If we restrict ourselves to the function $T(w)$ with the special condition in Theorem 1, we have the following theorem.

Theorem 2. In Theorem 1, if $T(w)$ is a function which is transformed into a linear transformation $S(z)$ by a suitably defined analytic function $z = f(w)$, that is,

$$(3) \quad S(z) = fTf^{-1}(z)$$

becomes a linear transformation, $S(z)$ is of the form

$$(4) \quad S : \frac{z'}{z' - \alpha} = A \frac{z}{z - \alpha}, \quad (\alpha \neq 0, A \neq 1, 0 < |A| \leq 1);$$

the case $\alpha = \infty$ being to be interpreted as $z' = Az$.

If we take the identical transformation instead of $f(w)$ in (3), the above theorem becomes the Walsh's theorem in itself.

Proof. From (2), we have

$$|S'(0)| = |f'(0)| |T'(0)| \left| \frac{1}{f'(0)} \right| \leq 1.$$

Since $S'(0)$ denotes the multiplier of a linear transformation, it is sufficient to prove the theorem in the case $|T'(0)| = 1$ where $T(w)$ is a one-to-one conformal mapping of D onto itself. Because in the other case $T(w)$ becomes the linear transformation which is obtained from the linear transformation corresponding to the one-to-one conformal mapping $T(w)$ of D onto itself by multiplying a multiplier.

The trivial case where $T(w)$ is the identity w is excluded.

A region D_r bounded by the level curve of Green's function has meaning only if the pole $w=0$ of Green's function lies in D .

Since every D_r has property T with $T(0)=0$, we find at first $w=0$ must be a fixed point of $T(w)$.

If a region D has property T , it has also properties T^2, T^3, \dots .

Since, now, $T(w)$ is a one-to-one conformal mapping of D onto itself, then the functions

$$f(w) = T^n(w), \quad (n=0, 1, 2, \dots),$$

which are generated from $T(w)$ and the inverse $T^{-1}(w)$ by the iteration, are also one-to-one conformal mappings of D onto itself. Hence the set $\{f_n(w)\}$ of all those transformations form a finite or an infinite cyclic group G .

Now we divide G into two cases.

i) If G is an infinite cyclic group and moreover contains no infinitesimal transformations, that is, G is the properly discontinuous group, there exist two or one *singular points*³⁾, which remain invariant under $T(w) \in G$, in the boundaries of D .

If there exist two *singular points*, one of them always coincides with $w=0$, and

the other can be transformed into infinity without loss of generality. Since $w=0$ is an isolated boundary point from the property of D , it is removable from the hypothesis of $T(w)$. Then a neighborhood of $w=0$ contains a point which is congruent to any point of the extended plane and the region, which remains invariant under $T(w) \in G$, coincides with the extended plane. This consequence is obviously inconsistent with the property of D .

Even if there exists only one singular point, from the reason analogous to the above we can lead to the contradiction.

Therefore G has no singular points and is a finite group.

ii) If G is an infinite cyclic group and contain the infinitesimal transformations, G is a continuous group and there exist infinite *singular points*. We note that $w=0$ is a fixed inner point of D , hence it is not a *singular point*.

After all G is a finite or a special infinite group.

When the universal covering surface D^∞ of D is mapped onto the unit circle $|z| < 1$ by the polymorphic function $z=f(w)$, so that $w=0$ corresponds to the origin of $|z| < 1$, every element of G corresponds to a linear transformation SL , which holds $|z| < 1$ invariant while L moves over the Fuchsian group Γ_o of the automorphic function $w=w(z) = f^{-1}(w)$. All such transformations SL form a Fuchsian group Γ , in which Γ_o is contained as the invariant subgroup. Therefore we have the relation $G \cong \Gamma/\Gamma_o$, that is, G is isomorphic with the factor group Γ/Γ_o which is a elliptic cyclic group⁴.

Thus we obtain the relation $S = fTf^{-1}$ and $S(z)$ has the form (4) in Theorem 2. This is the consequence to be proved. q. E. D.

§ 3. On the linearization of the one-to-one conformal mapping

$T(w)$ of D onto itself.

The subject of this paragraph means that, if we can decide a suitably defined analytic function in D , $T(w)$ can be made linear by it. In other words, it is the problem to decide a conformal mapping function $z=f(w)$ from D onto a schlicht domain D' of the extended z -plane, so that the transformed function $s=fTf^{-1}$ may be linear, and of the form

$$(5) \quad z' = S(z) = Az \quad (|A| = 1).$$

Let m be a closed set of the boundary points of D . We can suppose that the set m contains the points which belong only to the *transfinite kernel*,⁵ for the other point, for instance, the isolated point, is removable. Generally, the boundary consists of continums and the points of the discrete set.

In the case, when G is an infinite group, the problem is difficult. But it is not difficult in § 2. Without loss of generality, we can place two fixed points at $w=0$ and infinity.

We draw a Jordan curve k and the images k_ν of k obtained by T^ν , that is,

$$(6) \quad k_\nu = T^\nu(k), \quad (\nu = 1, 2, \dots, n-1).$$

If we denote the angular region bounded by k_ν and $k_{\nu+1}$ with $R_\nu(k_\nu, k_{\nu+1})$, so $\sum_{\nu=0}^{n-1} R_\nu$ cover the extended w -plane without overlappings and gaps. Suppose that m_o is the subset of

m in R_0 and $m = \sum_{\nu=0}^{n-1} T^\nu(m_0)$. If m_0 is surrounded by a finite number of closed Jordan curves

$$(7) \quad C_{\mu,n}^{(0)}, \quad (\mu=1,2,\dots,n)$$

in $D \cap R_0$, m_ν is also surrounded by the images

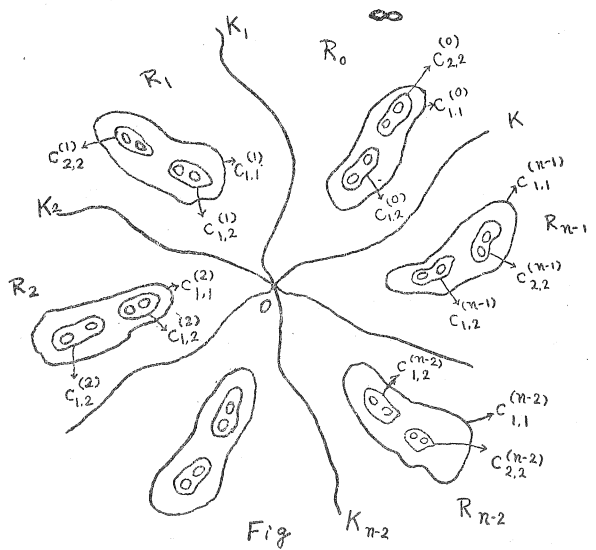
$$(8) \quad C_{\mu,n}^{(\nu)}, \quad (\mu=1,2,\dots,n)$$

of (7) in $D \cap R_\nu$. (c. f. Fig.)

By the conformal mapping

$$(9) \quad z_n = f_n(w),$$

a subregion D_n , which are formed from D' by removing (7), (8) and their inside, can be mapped conformally onto a region B_n of the z -plane bounded by a circle with radius r_n . Therefore $T(w)$ is transformed into $S_n(z)$ by $f_n(w)$. As $S_n(z)$ is a schlicht conformal mapping



of B_n onto itself and is linear, so it is transformed into the form

$$(10) \quad z'_n = A_n z_n, \quad (|A_n| = 1).$$

Next we consider the infinite sequence of the system of curves

$$(11) \quad C_{\mu,n}^{(0)}, \quad \begin{matrix} (\mu=1,2,3,\dots,n) \\ (n=1,2,3,\dots) \end{matrix}$$

which satisfy the following conditions;

- 1). The curves of the system $C_{\mu,n+1}^{(0)}$ is inside of the curves of the system $C_{\mu,n}^{(0)}$.
- 2). Any point w of D belongs to D_n for sufficiently large n .

From the infinite sequence $\{D_n\}$, we obtain $D = \lim_{n \rightarrow \infty} D_n$.

Now we normalize the infinite sequence $\{f_n(z)\}$, so that in any given point w_0 of D $|f'_n(w_0)| = 1$ may establish for all n . The functions $f_n(w)$, which map schlicht conformally any given compact subregion of D for sufficiently large n , are bounded by Koebe's distortion theorem. Thus $\{f_n(z)\}$ form a normal family and hence for a suitably chosen subsequence there exists a non-constant, schlicht analytic limit function in D

$$(12) \quad \lim_{n \rightarrow \infty} f_{\nu_n}(w) = f(w).$$

Therefore $T(w)$ is transformed into a linear transformation

$$(13) \quad Z' = AZ \quad (|A| = 1)$$

by $f(w)$, where $A = \lim_{n \rightarrow \infty} A_{\nu_n}$. Thus our theorem is completely proved.

§ 4. On the existence of the non-linear one-to-one conformal mapping of a domain D onto itself.

If the boundaries of D consist of continuums only, the existence of the above transformation is obvious. In the case where the boundaries are formed entirely by a discrete point set, the existence of such a transformation can be proved in the following way.

We begin an elliptic transformation

$$S: \quad z' = Az, \quad (|A| = 1)$$

and we take a discrete point set m of 2-dimensional positive outer measure in the angular region R_0 :

$$(14) \quad \frac{2\pi\nu}{n} \leq \arg z < \frac{2\pi(\nu+1)}{n}, \quad (\nu=0,1,2,\dots,n-1)$$

Then the 2-dimensional outer measure of the image

$$(15) \quad m_\nu = S^\nu(m) \quad (\nu=1,2,\dots,n-1)$$

is also positive.

Now let $w=f(z)$ be a regular analytic function in the extended z -plane without the set m deleted onto a schlicht region D' , the boundaries of which are a discrete point set m' of 2-dimensional measure zero.

Then the transformed function $T=f S f^{-1}$ is obviously nonlinear. Because by such a function the set m' corresponds with the set $f(m)$ of 2-dimensional positive outer measure.

References

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- [2] L. R. Ford, *On the properties of regions which persists in the subregions bounded by the level curves of the Green's function*, *Duke Math. Jour.*, vol. 1 (1935) pp 103-104.
- [3] We call a point z of the plane a *regular point* of the group G , if there exists a neighborhood $U(z)$ of z , so that there may exist only a finite number of images of $U(z)$, which have common points with $U(z)$, under the images of $U(z)$ transformed by G . All other points of the plane are called *singular points*.
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