

A Note on Mr. Komm's Theorems

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A Note on Mr. Komm's Theorems

By Toshio HIRAGUCHI

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§ 1. Mr. Horace Komm [1] has proved that (1) $\dim P_n'(E_n) = n$ provided $n \leq \aleph_0$ by showing that (2) $\dim P_n'(E_n) \leq n$ ($n \leq \aleph_0$) and (3) if P is a denumerable partially ordered set of dimension n , there exists a subset M_n of E_n such that $P \sim P_n'(M_n)$. Here E_n means the set of all sequences of real numbers $\{x_k\}$, $k=1, 2, \dots, n$ where $n \leq \aleph_0$ and $P_n'(M_n)$ for every $M_n \subseteq E_n$ a partially ordered system in which $x = \{x_k\} > \{y_k\} = y$ if and only if $x_k \geq y_k$ for all k and $x_i \neq y_i$ for some i . In the sequel we use the convenient abbreviation "poset" for partially ordered set and the notation $D[P]$ for $\dim P$. Now let us define a cardinal product of posets. Let X_s be a poset associated with each element s of a set S , $x = \{x_s | s \in S\}$ a set of x_s selected one at a time from each set X_s and X the set of all such x . Then

$$\mathbf{X} = \{(x, y) | (x_s, y_s) \in X_s \text{ for all } s \in S\}$$

is a partial order¹⁾ on the set X where \mathbf{X}_s is the partial order associated with each poset X_s . By the cardinal product $\prod_s X_s$ we mean the set X together with the partial order \mathbf{X} . When X_s are all isomorphic with a poset Y , $\prod_s X_s$ is the cardinal power Y^S which will also be denoted by Y^m where m is the number of elements of S . Thus if the chain of real numbers is denoted by R the three propositions of Mr. Komm will be formulated as follows. (1*) $D[R^m] = m$, $m \leq \aleph_0$ (2*) $D[R^m] \leq m$, $m \leq \aleph_0$ (3*) Every poset P with n (P)* $\leq \aleph_0$ and $D[P] = m$ is isomorphic with a subposet of R^m .

The purpose of this note is to prove that the propositions (1*), (2*) and (3*) hold in more general forms. That is

Theorem 1. Let $\prod_s X_s = X$ be a cardinal product of posets X_s , $s \in S$ and $\mathfrak{R}_s = \{M_{t(s)} | t(s) \in T_s\}$ a minimal realizer²⁾ of X_s . Then $D[X] \leq n[T]$ where $T = \bigcup_s T_s$. (A generalization of (2*).)

Theorem 2. If X_s is a chain for every $s \in S$, $D[\prod_s X_s] = n[S]$. (A generalization of (1*).)

Theorem 3. Every poset P with $D[P] = m$ is isomorphic with a subposet of some cardinal product of m chains whose dimension is m . (A generalization of (3*).)

Theorem 3 is a special case of

Theorem 4. If X_s is a poset which is d -reducible³⁾ to a poset which is isomorphic with 2^{T_s} for every $s \in S$, then $D[\prod_s X_s] = n[T]$ where $T = \bigcup_s T_s$.

*) $n [\]$ means the number of elements of a set written in the brackets.

1), 2), 3), 4) As to the terminologies refer to the author's previous paper [2].

§ 2. 1. *Proof of the theorem 1.* It may be assumed that S and T_s are well-ordered sets without loss of generality. Let $t_0(s)$ be the first element of T_s . Then

$$\begin{aligned} L_s^{t(s)} = & \{(x, x) | x \in X\} \cup \{(x, y) | x_s \neq y_s, (x_s, y_s) \in M_{t(s)}\} \\ & \cup \{(x, y) | x_s = y_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)} \text{ for the first } \sigma \text{ such that } x_\sigma \neq y_\sigma\} \end{aligned}$$

is a linear order⁴⁾ on the set X . That the reflexivity and antisymmetry hold is evident.

To show that the transitivity holds let $(x, y) \in L_s^{t(s)}$ and $(y, z) \in L_s^{t(s)}$. Then there are following four cases :

- (1) $x_s \neq y_s, (x_s, y_s) \in M_{t(s)}$;
 $y_s \neq z_s, (y_s, z_s) \in M_{t(s)}$.
- (2) $x_s \neq y_s, (x_s, y_s) \in M_{t(s)}$;
 $y_s = z_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $y_\sigma \neq z_\sigma$.
- (3) $x_s = y_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$;
 $y_s \neq z_s, (y_s, z_s) \in M_{t(s)}$
- (4) $x_s = y_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$;
 $y_s = z_s, (y_{\sigma'}, z_{\sigma'}) \in M_{t_0(\sigma')}$ for the first σ' such that $y_{\sigma'} \neq z_{\sigma'}$.

When one of the first three cases occurs it is evident that $(x_s, z_s) \in M_{t(s)}$. Hence $(x, z) \in L_s^{t(s)}$. When the last case occurs $x_s = z_s$. If $\sigma \leq \sigma'$, then σ is the first suffix such that $x_\sigma \neq z_\sigma$ and for this $(x_\sigma, z_\sigma) \in M_{t_0(\sigma)}$. Similarly if $\sigma' < \sigma$, then σ' is the first suffix such that $x_{\sigma'} \neq z_{\sigma'}$, and for this $(x_{\sigma'}, z_{\sigma'}) \in M_{t_0(\sigma')}$. Hence in either case $(x, z) \in L_s^{t(s)}$. Thus

the transitivity is verified. The chain $L_s^{t(s)}$ obtained by associating $L_s^{t(s)}$ with X is a linear extension of the poset X . In fact, if $(x, y) \in X$ and $x \neq y$, then either $x_s \neq y_s, (x_s, y_s) \in M_{t(s)}$ or $x_s = y_s, (x_\sigma, y_\sigma) \in M_{t_0(\sigma)}$ for the first σ such that $x_\sigma \neq y_\sigma$; hence $(x, y) \in L_s^{t(s)}$.

Moreover the system

$$\mathfrak{R} = \{L_s^{t(s)} | s \in S, t(s) \in T_s\}$$

of all linear extensions $L_s^{t(s)}$ is a realizer of the poset X . To show this let $x \not\leq y$ in X . Then either (1) $x_s \not\leq y_s$ in X_s for some $s \in S$ or (2) $x_s \neq y_s, (x_s, y_s) \in X_s$ for some s and $y_{s'} \neq x_{s'}, (y_{s'}, x_{s'}) \in X_{s'}$ for some $s' \neq s$. If (1), then $(x_s, y_s) \in M_{t(s)}$ for some $t(s) \in T_s$ and $(y_s, x_s) \in M_{t'(s)}$ for some $t'(s) \in T_s$; hence $(x, y) \in L_s^{t(s)}$ and $(y, x) \in L_s^{t'(s)}$. If (2), then $(x_s, y_s) \in M_{t(s)}$ and $(y_s, x_s) \in M_{t(s')}$; hence $(x, y) \in L_s^{t(s)}$ and $(y, x) \in L_s^{t(s')}$. Thus \mathfrak{R} is a realizer of X . Therefore $D[X] = n [T_s]$.

2. *Proof of the theorem 2.* By the theorem 1 we have the inequality $D[\prod_s X_s] \leq n[S]$. To have the inverse inequality it suffices to show that $\prod_s X_s$ contains

a subposet of dimension n [S]. But evidently 2^S is a subposet of $\prod_s X_s$. To prove that $D(2^S) = n[S]$ let \mathcal{Z} be $\{0, 1\}$, a_s the element of 2^S whose s -component is 1 and other components are 0 and b_s the element of 2^S whose s -component is 0 and other components are 1. Then the subposet of 2^S composed of all a_s and all b_s is isomorphic with the poset composed of all elements of S and their complements in S and ordered by the relation of set-inclusion. It has been known [3] that the dimension of the latter poset is $n[S]$. Hence we have $D(2^S) = n[S]$.

3. *Proof of the theorem 3.* Let a minimal realizer of P be $\mathfrak{R} = \{L_s \mid s \in S\}$, $n[s] = m$. Then P is isomorphic with a subposet of $\prod_s L_s$. In fact P is isomorphic with the subposet P^* of $\prod_s L_s$ composed of all elements such that all the components are equal to an element $x \in P$.

4. *Proof of the theorem 4.*

$D(\prod_s X_s) \leq n[T]$ is evident by the theorem 1. On the other hand

$$D(\prod_s X_s) \geq D(\prod_s 2^{T^s}) = D(2^T) = n[T].$$

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