Non-Existence of Non-Zero □′-Harmonic Forms on a Complete Foliated Manifold

Non-existence of non-zero $\square'$-harmonic forms on a complete foliated manifold

Fumihiko ENOMOTO

Abstract

We study Bochner type theorem for the $\square'$-harmonic (1,0)-form on a complete foliated Riemannian manifold with a bundle-like metric with respect to the foliation. We have the non-existence of non-zero $\square'$-harmonic (1,0)-forms under the certain assumption of the curvature of the Levi-Civita connection.

1. Introdution.

Let M be a connected, compact and orientable foliated manifold with a bundle-like metric with respect to the foliation. In [4]O.K.I.Vaisman showed Bochner type theorem for the foliated harmonic(1,0)-forms:

"There are not non-zero foliated harmonic (1,0)-forms on M with positive definite Ricci curvature of the second connection."

In [6] O.K. S, Yorozu showed Bochner type theorem for the foliated harmonic (1,0)-and (1, m)-forms under the certain assumption of the curvature of the Levi-Civita connection, where m is codimension of the foliation.

In this note, we shall prove, by H. Kitahara's method [2], Bochner type theorem for the $\square'$-harmonic (1,0)-forms under the certain assumption of the curvature of the Levi-Civita connection as M is a complete manifold.

The author expresses his hearty thanks to Professor K. Takamatsu for his valuable advice.

2. Foliated manifold

We shall be in C^∞-category. Latin indices run from 1 to n and Greek ones from $n+1$ to $n+m$, and the Einstein summation convention will be used.

Let M be an $n+m$ dimensional, connected and orientable manifold with a foliation $\mathscr{F}$ of codimension m. $\mathscr{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. Then we find about each point a coordinate neighbourhood U with coordinate $(x^1, \cdots, x^n, y^{n+1}, \cdots, y^{n+m})$ such that

(i) $|x^i| \leqq 1$ and $|y^\alpha| \leqq 1$,

(ii) The integral manifolds of E are given locally by $y^{n+1}=c^{n+1}, \cdots, y^{n+m}=c^{n+m}$ for constants c^α satisfying $|c^\alpha| \leqq 1$.

Such a coordinate neighbourhood U will be called flat.

We may assume that there exist in a flat neighbourhood U differential forms ω^i and vectors

Institute of Mathematical Science

v_α such that

(i) $\{\partial/\partial x^i\}$ forms the base for the space of cross-sections of E in U at each point,

(ii) $\{\omega^1, \cdots, \omega^n, dy^{n+1}, \cdots, dy^{n+m}\}$ and $\{\partial/\partial x^1, \cdots, \partial/\partial x^n, v_{n+1}, \cdots, v_{n+m}\}$ are dual bases for the cotangent and tangent spaces at each point of U respectively,

where $\omega^i = dx^i + A_\alpha{}^i dy^\alpha$ and $v_\alpha = \partial/\partial y^\alpha - A_\alpha{}^i \partial/\partial x^i$ (cf. [3]).

Throughout this note, all local expression for differential forms and vectors will be taken with respect to those bases. Furthermore we shall assume that M has a complete Riemannian metric g which is bundle-like with respect to $\mathcal{F}$. The metric g has the local expression :

$$g \mid_U = g_{ij}(x, y)\, \omega^i \otimes \omega^j + g_{\alpha\beta}(y) dy^\alpha \otimes dy^\beta,$$

where $g_{ij}(x, y) = g(\partial/\partial x^i, \partial/\partial x^j)$ and $g_{\alpha\beta}(y) = g(v_\alpha, v_\beta)$ (cf. [3]).

3. $\square'$-harmonic form

A form ϕ on M is of type (p, q) (or a (p, q)-form) if it is expressed locally as

$$\phi \mid_U = \frac{1}{p!q!}\, \phi_{i_1\cdots i_p \alpha_1\cdots\alpha_q}(x, y)\, \omega^{i_1}\wedge\cdots\wedge \omega^{i_p}\wedge dy^{\alpha_1}\wedge\cdots\wedge dy^{\alpha_q}.$$

Hereafter, we omit "$\mid_U$". Let $\Lambda^{p,q}$ denote the space of all (p, q)-forms on M and Λ^r the space of all r-forms on M. Then we have the decomposition of Λ^r, that is,

$$\Lambda^r = \sum_{p+q=r} \Lambda^{p,q}.$$

This decomposition induces a decomposition of the exterior derivative d, that is,

$$d = d' + d'' + d'''.$$

We here notice that $d' : \Lambda^{p,q} \to \Lambda^{p+1,q}$ and $(d')^2 = 0$ (cf. [4] , [5]).

Now we may give the local expression of $d'\phi$ and $*\phi$ for $\phi \in \Lambda^{p,q}$:

$$d'\phi = \frac{1}{(p+1)!q!p!}\, \delta^{ii_1\cdots i_p}_{k_1\cdots k_{p+1}} \left(\frac{\partial}{\partial x^i}\, \phi_{i_1\cdots i_p\alpha_1\cdots\alpha_q}\right) \omega^{k_1}\wedge\cdots\wedge \omega^{k_{p+1}}\wedge \mathrm{d}y^{\alpha_1}\wedge\cdots\wedge \mathrm{d}y^{\alpha_q},$$

$$*\phi = \frac{1}{(n-p)!\,(m-q)!p!q!}\, g^{i_1j_1}\cdots g^{i_pj_p} g^{\alpha_1\beta_1}\cdots g^{\alpha_q\beta_q} \eta_{j_1\cdots j_p k_1\cdots k_{n-p}\beta_1\cdots\beta_q\gamma_1\cdots\gamma_{m-q}}\, \phi_{i_1\cdots i_p\alpha_1\cdots\alpha_q}$$

$$\omega^{k_1}\wedge\cdots\wedge \omega^{k_{n-p}}\wedge \mathrm{d}y^{\gamma_1}\wedge\cdots\wedge \mathrm{d}y^{\gamma_{m-q}},$$

where $\delta^{ii_1\cdots i_p}_{k_1\cdots k_{p+1}}$ denote the generalized Kronecker symbol, (cf. [1]), (g^{ij}) and $(g^{\alpha\beta})$ denote the inverse matrices of (g_{ij}) and $(g_{\alpha\beta})$ respectively and $\eta_{i_1\cdots i_n\alpha_1\cdots\alpha_m}$ denote the components of the volume form ((n, m)-form) ;

$$dM = \sqrt{det \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}} \ \omega^1 \wedge \cdots \wedge \omega^n \wedge dy^1 \wedge \cdots \wedge dy^m.$$

We define $\delta' : \Lambda^{p,q} \to \Lambda^{p-1,q}$ by

$$\delta' = (-1)^{(n+m)(p+q)+(n+m)+1} * d' *.$$

We denote by $<, >$ the local scalar product on $\Lambda^{p,q}$. The global scalar product $\ll, \gg$ is defined by

$$\ll \phi, \psi \gg = \int_M < \phi, \psi > *1 = \int_M \frac{1}{p!q!} \phi_{i_1 \cdots i_p \alpha_1 \cdots \alpha_q} \psi^{i_1 \cdots i_p \alpha_1 \cdots \alpha_q} *1$$

for any ϕ, $\psi \in \Lambda_0^{p,q}$, where $\Lambda_0^{p,q}$ denotes the subspace of $\Lambda^{p,q}$ composed of forms with compact supports. Then we have

$$\ll d'\phi, \psi \gg = \ll \phi, \delta'\psi \gg,$$

for any $\phi \in \Lambda_0^{p,q}$ and any $\psi \in \Lambda_0^{p+1,q}$. Let $L_2^{p,q}$ be the completion of $\Lambda_0^{p,q}$ with respect to the scalar product $\ll, \gg$.

Similarly, by $<, >$ and $\ll, \gg$, we denote the local scalar product and the global scalar product for covariant tensor fields respectively. For example, if $T = T_{ij}\, \omega^i \otimes \omega^j$ and $S = S_{ij}\, \omega^i \otimes \omega^j$ are tensor fields with compact supports, we have

$$\ll T, S \gg = \int_M < T, S > *1 = \int_M \frac{1}{2!} T_{ij} S^{ij} *1.$$

Let ∇ denote the Levi-Civita connection on M. We set

$$\nabla_{X_A} X_B = \Gamma_{AB}^C X_C,$$

where each X_A denotes $\partial/\partial x^i$ or v_α. For example,

$$\nabla_{\partial/\partial x^i} \partial/\partial x^j = \Gamma_{ij}^k \partial/\partial x^k + \Gamma_{ij}^\alpha v_\alpha,$$

$$\nabla_{\partial/\partial x^i} v_\beta = \Gamma_{i\beta}^k \partial/\partial x^k + \Gamma_{i\beta}^\gamma v_\gamma,$$

and moreover, we set $\nabla_i = \nabla_{\partial/\partial x^i}$.

Lemma 3.1. $\Gamma_{ij}^k = \Gamma_{ji}^k$.

This lemma is proved by the direct calculation.

Let R denote the curvature tensor of ∇:

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

For example,

$$R\,(\partial/\partial x^k,\ \partial/\partial x^j)\ \partial/\partial x^i = R_{kji}{}^h\ \partial/\partial x^h + R_{kji}{}^\gamma\ v_\gamma,$$

$$R\,(\partial/\partial x^k,\ \partial/\partial x^j)\ v_\alpha = R_{kj\alpha}{}^h\ \partial/\partial x^h + R_{kj\alpha}{}^\gamma\ v_\gamma.$$

Definition 3.1. The operator $\Box'$ acting on $\Lambda^{p,q}$ is defined by $\Box' = d'\delta' + \delta' d'$. *A* (p, q)-form ϕ on M is called a $\Box'$-harmonic form if $\Box'\phi = 0$.

By Lemma 3.1, we have

Lemma 3.2. *For* $(1, 0)$-*form* $\phi = \frac{1}{1!0!}\phi_i\omega^i$,

(i) $d'\phi = \dfrac{1}{2!1!0!}\{\nabla_i\phi_j - \nabla_j\phi_i\}\ \omega^i \Lambda\ \omega^j$,

(ii) $\delta'\phi = -\nabla^i\phi_i$,

(iii) $\Box'\phi = \dfrac{1}{1!0!}\{-\nabla^i\nabla_i\phi_k + R'_k{}^h\phi_h\}\,\omega^k$,

where $R'_k{}^h = g^{ij}R_{kji}{}^h$, *which is a "partial" Ricci tensor.*

Remark : If all leaves in M are totally geodesic, the Ricci tensor with respect to the induced connection on each leaf agrees with $R'_k{}^h$.

4. Non-existence of $\Box'$-harmonic form.

We fix a point o in M, and for each point p in M, we denote by $\rho\ (p)$ the geodesic distance from o to p. We set

$$B\ (R) = \{p \in M\ ;\ \rho\ (p) < R\},$$

for any $R > 0$.

We consider a function $\mu(t)$ on $\mathbf{R}$ satisfying

(i) $0 \leqq \mu(t) \leqq 1$ on $\mathbf{R}$,

(ii) $\mu(t) = 1$ for $t \leqq 1$,

(iii) $\mu(t) = 0$ for $t \geqq 2$,

and we define functions w_R on M by

(4.1) $\quad w_R(p) = \mu\left(\dfrac{\rho(p)}{R}\right)$

for any $R > 0$. Then we have the following properties:

(4.2) $\quad 0 \leqq w_R \leqq 1$ on M,

(4.3) $\quad \operatorname{supp} w_R \subset \overline{B(2R)}$ (the closure of $B(2R)$), (4.4) $w_R(p)=1$ for $p \in B(R)$,

(4.5) $w_R(p) \to 1$ $(R \to \infty)$ for each $p \in M$,

(4.6) w_R are locally Lipschitz functions and of class C^∞ for almost all $p \in M$,

(4.7) $(d' w_R)_p \leqq \frac{C'}{R}$ for almost all $p \in M$,

where C' is a number dependig only on μ and the dimension $(n+m)$ of M.

By (4.7), we have

Lemma 4.1. (cf. [2]). *There exists a number* C *depending only on* μ *and the dimension* $(n+m)$ *of* M, *such that*

(i) $\| d' w_R \otimes \phi \|_{B(2R)} \leqq \frac{C}{R} \| \phi \|_{B(2R)}$,

(ii) $\| d' w_R \otimes * \phi \|_{B(2R)} \leqq \frac{C}{R} \| \phi \|_{B(2R)}$,

for any $\phi \in \Lambda^{1,0}$, where $\| \phi \|^2_{B(2R)} = \ll \phi, \phi \gg_{B(2R)} = \int_{B(2R)} < \phi, \phi > * 1$.

By the direct calculation, we have

(4.8) $$d'(w_R^2 \phi) = w_R^2\, d'\phi + 2 w_R\, d' w_R \wedge \phi,$$

(4.9) $$\delta'(w_R^2 \phi) = w_R^2\, \delta' \phi - * (2 w_R\, d' w_R \wedge * \phi),$$

for any $\phi \in \Lambda^{1,0}$.

Definition 4.2. For any (1,0)-form $\phi = \phi_i \omega^i$, we define $\nabla' \phi$ by

$$\nabla' \phi = \nabla_i \phi_j \omega^i \otimes \omega^j.$$

A (1,0)-form ϕ is called ∇'-parallel, if $\nabla' \phi = 0$.

Definition 4.3. A symmetric linear mapping $\mathscr{R}' : \Lambda^{1,0} \to \Lambda^{1,0}$ is defined by

$$\mathscr{R}' \phi = \frac{1}{1!0!} R'_k{}^h \phi_h \omega^k,$$

for any $\phi = \frac{1}{1!0!} \phi_i \omega^i$, where $R'_k{}^h = g^{ji} R_{kji}{}^h$. $\mathscr{R}'$ is called positive, if both $< \mathscr{R}' \phi, \phi >_p \geqq 0$ and $< \mathscr{R}' \phi, \phi >_p = 0$ iff $\phi_p = 0$ for each $p \in M$. $\mathscr{R}'$ is called non-negative (resp. non-positive), if $< \mathscr{R}' \phi, \phi >_p \geqq 0$ (resp. $\leqq 0$) for each $p \in M$.

Lemma 4.2. (cf. [2]). *For any* $\phi \in \Lambda^{1,0}$,

$$0 = \ll w_R \nabla'^2 \phi, w_R \phi \gg_{B(2R)} + 2 \| w_R \nabla' \phi \|^2_{B(2R)}, + 4 \ll d' w_R \otimes \phi, w_R \nabla' \phi \gg_{B(2R)},$$

where $(\nabla'^2\phi)=\nabla^j\nabla_j\phi_i\omega^i$.

Proof. For given ϕ, we consider a (1,0)-form Φ defined by

$$\Phi = \frac{1}{1!0!}(\nabla_i\phi_j)\,\phi^j\omega^i,$$

Since $*(w_R^2\Phi)$ is a $(n-1, m)$-form with compact support in $\overline{B(2R)}$, the Stokes' theorem gives the equality

$$\int_M d(*(w_R^2\Phi))=0.$$

On the other hand, we have $d(*(w_R^2\Phi))=-*(\delta'(w_R^2\Phi))$.

Since we have that

$$\begin{aligned} <d'w_R,\ \Phi> &= \frac{1}{1!0!}(d'w_R)^i(\nabla_i\phi_j)\,\phi^j \\ &= 2\,\frac{1}{2!}(d'w_R)^i\,\phi^j(\nabla_i\phi_j) \\ &= 2<d'w_R\otimes\phi,\ \nabla'\phi>, \end{aligned}$$

we have, by Lemma 3.2 and (4.9)

$$\delta'(w_R^2\,\Phi)=-<w_R\nabla'^2\phi,\ w_R\phi>-2\mid w_R\nabla'\phi\mid^2-4<d'w_R\otimes\phi,\ w_R\nabla'\phi>.$$

Therefore we have the assertion.

Let ϕ be a (1,0)-form on M. By the Schwarz inequality and Lemma 4.1, we have

(4.10) $\quad 2\mid\ll d'w_R\otimes\phi,\ w_R\nabla'\phi\gg_{B(2R)}\mid$

$$\begin{aligned} &\leqq 2\,\|\,d'w_R\otimes\phi\,\|_{B(2R)}\,\|\,w_R\nabla'\phi\,\|_{B(2R)} \\ &\leqq 2\,\frac{C}{R}\,\|\,\phi\,\|_{B(2R)}\,\|\,w_R\nabla'\phi\,\|_{B(2R)} \\ &\leqq \frac{C}{R}(\,\|\,\phi\,\|^2_{B(2R)}+\|\,w_R\nabla'\phi\,\|^2_{B(2R)}) \end{aligned}$$

and

(4.11) $\quad \mid\ll w_R\Box'\phi,\ w_R\phi\gg_{B(2R)}\mid\leqq\frac{1}{2}(\sigma\,\|\,w_R\Box'\phi\,\|^2_{B(2R)}+\frac{1}{\sigma}\,\|\,w_R\phi\,\|^2_{B(2R)})$

for every $\sigma>0$.

By Lemma 3.2, we have

(4.12) $\quad \mid\ll w_R\Box'\phi,\ w_R\phi\gg_{B(2R)}\mid\leqq-\ll w_R\nabla'\phi,\ w_R\phi\gg_{B(2R)}+\ll w_R\mathscr{R}'\phi,\ w_R\phi\gg_{B(2R)}.$

Thus from (4.11), (4.12), (4.13) and Lemma 4.2, we have, for every $\sigma>0$

$$\sigma\,\|\,w_R\Box'\phi\,\|^2_{B(2R)}+\frac{1}{\sigma}\,\|\,w_R\phi\,\|^2_{B(2R)}$$

$$\geqq 2 \mid \ll w_R \Box' \phi, w_R \phi \gg_{B(2R)} \mid$$
$$\geqq -2 \ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} + 4 \| w_R \nabla' \phi \|^2_{B(2R)} + 8 \ll d' w_R \phi, w_R \nabla' \phi \gg_{B(2R)}$$
$$\geqq 2 \ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} + 4 \left(1 - \frac{C}{R}\right) \| w_R \nabla' \phi \|^2_{B(2R)} - 4 \frac{C}{R} \| \phi \|^2_{B(2R)}.$$

In paticular, setting $\Box' \phi = 0$ and letting $\sigma \to \infty$, we have

Lemma 4.3. *For any* $\Box'$- *harmonic* (1,0)-*form* ϕ,

$$0 \geqq 2 \ll w_R \Box' \phi, w_R \phi \gg_{B(2R)} + 4 \left(1 - \frac{C}{R}\right) \| w_R \nabla' \phi \|^2_{B(2R)} - 4 \frac{C}{R} \| \phi \|^2_{B(2R)}.$$

Theorem 4.1. *Let M be an $(n+m)$ dimensional connected and orientable manifold with a foliation of codimension m, and M has a complete Riemannian metric being bundle-like with respect to the foliation.*

(i) *If $\mathscr{R}'$ is positive, then there are not non-zero $\Box'$-harmonic forms in $L_2^{1,0}$.*

(ii) *If $\mathscr{R}'$ is non-negative, then $\Box'$-harmonic forms in $L_2^{1,0}$ are ∇'-parallel.*

Proof. Let ϕ be a $\Box'$-harmonic (1,0)-form in $L_2^{1,0}$.

Noticing $\| \phi \|^2_{B(2R)} \to \| \phi \|^2 (R \to \infty)$ and Lemma 4.3, we have

$$0 \geqq \limsup_{R \to \infty} \left\{ 2 \ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} + 4 \left(1 - \frac{C}{R}\right) \| w_R \nabla' \phi \|^2_{B(2R)} \right\}.$$

On the other hand, we have, from the positivity of $\mathscr{R}'$

$$\ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} \geqq 0$$

for $R > 0$ and

$$\left(1 - \frac{C}{R}\right) \| w_R \nabla' \phi \|^2_{B(2R)} \geqq 0$$

for sufficiently large $R > 0$.

Thus the above three inequalities give

$$(*) \quad \limsup_{R \to \infty} \ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} = 0,$$

$$(**) \quad \limsup_{R \to \infty} \left(1 - \frac{C}{R}\right) \| w_R \nabla' \phi \|_{B(2R)} = 0.$$

Evidently it holds that for $0 < R_1 < R_2$

$$\ll w_{R_1} \mathscr{R}' \phi, w_{R_1} \phi \gg_{B(2R_1)} \leqq \ll w_{R_2} \mathscr{R}' \phi, w_{R_2} \phi \gg_{B(2R_2)}.$$

Thus form (*), we have for any $R > 0$,

$$\ll w_R \mathscr{R}' \phi, w_R \phi \gg_{B(2R)} = 0.$$

Since w_R is positive in $B(2R)$, we have for any $p \in M$,

$$<\mathcal{R}'\phi, \phi>_p = 0.$$

By the positivity of $\mathcal{R}'$, we have $\phi = 0$.

Taking account of the non-negativity of $\mathcal{R}'$ and (**), the case of (ii) is proved by the same way.

Example 4.1. Let S^2 denote the 2-dimensional sphere of constant sectional curvature and f a positive valued function on 1-dimensional Euclidean space $\mathbf{R}$ with the standard metric. Then the warped product $M = \mathbf{R} \times_f S^2$ is a connected, orientable and non-compact foliated manifold with $\{y\} \times S^2$ as leaves. And the metric of M is complete and bundle-like. If we take $f(y) = 2 + (\frac{1}{2})\sin y$, $\mathcal{R}'$ is positive.

Example 4.2. Let T^2 denote the 2-dimensional flat torus. We set $f(y) = e^{-y^2}$ for any $y \in \mathbf{R}$. Then the warped product $M = \mathbf{R} \times_f T^2$ is a connected, orientable and non-compact foliated manifold with $\{y\} \times T^2$ as leaves, whose metric is complete and bundle-like. And $\mathcal{R}'$ is non-positive. We define (1,0)-form ϕ on M by

$$\phi = \sqrt{f(y)}\ \omega^1.$$

Then ϕ is non-zero $\Box'$-harmonic form in $L_2^{1,0}$.

REFERENCES

[1] S. Goldberg, *Curvature and homology*, Academic Press, New York (1962).

[2] H. Kitahara, *Non-existence of non-trivial* $\Box''$*-harmonic 1-forms on a complete foliated riemannian manifold*, Trans. Amer. Math. Soc. 262 (1980), 429-435.

[3] B. L. Reinhart, *Foliated manifold with bundle-like metrics*, Ann. of Math. 69 (1959), 119-132.

[4] I. Vaisman, Variétés riemanniennes feuilletées, Czechoslovak. Math. J. 21 (1971), 46-75.

[5] ———, *Cohomology and differential forms*, Marcel Dekker, New York (1973).

[6] S. Yorozu, *Non-existence of non-zero foliated harmonic forms on a compact foliated manifold*, Ann. Sci. Kanazawa Univ. 18 (1981) 11-17.