Certain Hypersurfaces of a Manifold with (f,g,u,v,λ) -Structure

by

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Abstract

The (f,g,u,v,λ)-structure has been defined by K. Yano and M. Okumura [5] in even dimensional manifold. It is well known that the submanifold of codimension 2 of an almost Hermitian manifold and a hypersurface of an almost contact metric manifold admit the (f,g,u,v,λ)-structure under certain conditions. A hypersurface is said to be invariant if the tangent hyperplane is invariant by the action of the tensor f. The invariant hypersurface of a manifold with (f,g,u,v,λ)-structure was investigated by K. Yano and M. Okumura [6]. In the present paper we shall study the hypersurface such that the vector fields u, v are tangent to the hypersurface of a manifold with (f,g,u,v,λ)-structure and the invariant hypersurface of a manifold with (f,g,u,v,λ)-structure satisfying certain conditions.

1. (f,g,u,v,λ)-structure

Let M be a $(2n+2)$-dimensional differentiable manifold of class C^∞. If there exist in M a tensor field f of type (1, 1), a Riemannian metric g, two vector fields u, v and a function λ, such that

$$
\begin{aligned}
&f_j{}^i f_i{}^h = -\delta_j^h + u_j u^h + v_j v^h,\\
&f_j{}^t f_i{}^s g_{ts} = g_{ji} - u_j u_i - v_j v_i,\\
&f_j{}^i u_i = \lambda v_j, \quad f_j{}^i v_i = -\lambda u_j,\\
&f_i{}^h u^i = -\lambda v^h, \quad f_i{}^h v^i = \lambda u^h,\\
&u_i u^i = v_i v^i = 1-\lambda^2, \quad u_i v^i = 0,
\end{aligned}
\tag{1}
$$

where $u_i = g_{ji} u^j$, $v_i = g_{ji} v^j$, then the structure is called (f, g, u, v, λ)-structure.

Putting $f_{ji} = f_j{}^t g_{ti}$, we can easily see that

$$f_{ji} = -f_{ij}.$$

Next, we consider a product manifold $M \times R^2$, where R^2 is a 2-dimensional Euclidean space. Then we can define in it an almost complex structure F with local components $F_\lambda{}^\mu$ given by

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$$(F_\lambda{}^\mu)=\begin{pmatrix} f_j{}^i & u^i & v^i \\ -u_j & 0 & -\lambda \\ -v_j & \lambda & 0 \end{pmatrix} \tag{2}$$

The local components of Nijenhuis tensor N of F, are written as (Suzuki [3])

$$\begin{aligned}
S_{ij}{}^h &= f_j{}^m \nabla_m f_i{}^h - f_i{}^m \nabla_m f_j{}^h - f_m{}^h(\nabla_j f_i{}^m - \nabla_i f_j{}^m) + u^h(\nabla_j u_i - \nabla_i u_j) + v^h(\nabla_j v_i - \nabla_i v_j),\\
N_{ji}^{(1)} &= -f_j{}^m \nabla_m u_i + f_i{}^m \nabla_m u_j + u_m(\nabla_j f_i{}^m - \nabla_i f_j{}^m) - \lambda(\nabla_j v_i - \nabla_i v_j),\\
N_{ji}^{(2)} &= -f_j{}^m \nabla_m v_i + f_i{}^m \nabla_m v_j + v_m(\nabla_j f_i{}^m - \nabla_i f_j{}^m) + \lambda(\nabla_j u_i - \nabla_i u_j),\\
N_{(1)i}{}^h &= (\pounds_u f)_i{}^h + v^h \nabla_i \lambda,\\
N_{(2)i}{}^h &= (\pounds_v f)_i{}^h - u^h \nabla_i \lambda,\\
N_{(1)i}^{(1)} &= -(\pounds_u u)_i - \lambda \nabla_i \lambda,\\
N_{(1)i}^{(2)} &= -(\pounds_u v)_i - f_i{}^m \nabla_m \lambda,\\
N_{(2)i}^{(1)} &= -(\pounds_v u)_i + f_i{}^m \nabla_m \lambda,\\
N_{(2)i}^{(2)} &= -(\pounds_v v)_i - \lambda \nabla_i \lambda,\\
N_{(1)(2)}{}^h &= [u, v]^h,\\
N_{(1)(2)}^{(1)} &= -\pounds_u \lambda,\\
N_{(1)(2)}^{(2)} &= -\pounds_v \lambda,
\end{aligned} \tag{3}$$

where $\pounds_u$ denotes the operator of Lie derivation with respect to u^h, and ∇_j denotes the operator of covariant differentiation with respect to g_{ji}. If it satisfies $S_{ji}{}^h=0$, then the (f, g, u, v, λ)-structure is called to be normal. On normal structures, K. Yano and M. Okumura [5] proved the following

Theorem A. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying*

$$\nabla_j v_i - \nabla_i v_j = 2f_{ji}, \tag{4}$$

then we have

$$f_j{}^t \nabla_h f_{ti} - f_i{}^t \nabla_h f_{tj} = u_j(\nabla_i u_h) - u_i(\nabla_j u_h) + v_j(\nabla_i v_h) - v_i(\nabla_j v_h). \tag{5}$$

Theorem B. *Let M be a manifold with normal (f, g, u, v, λ)-structure satisfying* (4) *and*

$$\nabla_j u_i - \nabla_i u_j = 2\tau f_{ji}, \tag{6}$$

τ being a certain function. If the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero, then we have

$$\nabla_j \lambda = u_j - \tau v_j, \tag{7}$$

and τ is a constant.

On the other hand, the (f,g,u,v,λ)-structure is said to be quasi-normal if it satisfies (5). On quasi-normal structure, K. Yano and U-Hang Ki [4] proved the following

THEOREM C. *In a manifold with quasi-normal (f, g, u, v, λ)-structure such that the function $\lambda(1-\lambda^2)$ is almost everywhere non-zero and* (4) *is satisfied, we have* (6).

Taking account of Theorem A and the definitions of quasi-normal (f,g,u,v,λ)-structure, we can see that a normal (f,g,u,v,λ)-structure satisfying (4) is quasi-normal (f,g,u,v,λ)-structure. Therefore, we find that in a normal (f,g,u,v,λ)-structure satisfying (4), from Theorem C, the condition (6) is satisfied. Consequentely, we can state the following

REMARK. *Theorem B establishes without condition* (6).

2. Hypersurface of M with (f,g,u,v,λ)-structure

We consider a $(2n+1)$-dimensional differentiable manifold V and the immersion i: $V\to M$ as a hypersurface $i(V)$ of M. The hypersurface $i(V)$ is represented parametrically by the equation

$$x^h = x^h(y^a),$$

where $\{x^h\}$, $\{y^a\}$ be local coordinates of M and $i(V)$ respectively, and the indices a, b, c, etc. run over the range $\{1, 2, \cdots, 2n+1\}$.

If we put $B_a{}^h = \partial_a x^h$, $(\partial_a = \partial/\partial y^a)$, the induced Riemannian metric $\tilde{g}_{ca}$ on $i(V)$ is given

$$\tilde{g}_{cb} = g_{ji} B_c{}^j B_b{}^i.$$

Choosing a unit normal vector N^h to the hypersurface $i(V)$ in such way that the vectors $B_1{}^h, B_2{}^h, \cdots, B_{2n+1}{}^h$, N^h form the positive orientation of M, then we have

$$g_{ji} B_b{}^j N^i = 0, \quad g_{ji} N^j N^i = 1.$$

The transform $f_i{}^h B_b{}^i$ of $B_b{}^i$ by $f_i{}^h$ and the transform $f_i{}^h N^i$ of N^i by $f_i{}^h$ are respectively given by

$$f_i{}^h B_b{}^i = \phi_b{}^a B_a{}^h + w_b N^h,$$

$$f_i{}^h N^i = -w^a B_a{}^h, \tag{8}$$

where $\phi_b{}^a$ is a tensor field of (1, 1)-type, w_b is a 1-form in $i(V)$ and $w^a = w_b \tilde{g}^{ba}$. The vector fields u^h and v^h have the form, along $i(V)$

$$u^h = B_a{}^h u^a + \alpha N^h,$$

$$v^h = B_a{}^h v^a + \beta N^h, \tag{9}$$

where u^a, v^a are the vector fields of $i(V)$, and α, β are the functions of $i(V)$.

Now, applying $f_h{}^k$ to both members of (8) and (9) respectively and using (9), we find

$$
\begin{aligned}
&\phi_b{}^c\phi_c{}^a=-\delta_b^a+u_bu^a+v_bv^a+w_bw^a,\\
&\phi_c{}^au^c=-\lambda v^a+\alpha w^a,\\
&\phi_c{}^av^c=\lambda u^a+\beta w^a,\\
&\phi_c{}^aw^c=\alpha u^a+\beta v^a,\\
&u_au^a=1-\alpha^2-\lambda^2,\ v_av^a=1-\beta^2-\lambda^2,\\
&w_aw^a=1-\alpha^2-\beta^2,\\
&u_av^a=-\alpha\beta,\ w_au^a=-\lambda\beta,\ w_av^a=\lambda\alpha.
\end{aligned}
\tag{10}
$$

3. The hypersurface such that u, v are tangent to the hypersurface

We now assume that the vector fields u^h, v^h are tangent to the hypersurface $i(V)$, then we have

$$u^h=u^eB_e{}^h,\quad v^h=v^eB_c{}^h, \tag{11}$$

that is, we have $\alpha=0$, $\beta=0$ in (9).

Therefore, (10) becomes

$$
\begin{aligned}
&\phi_b{}^c\phi_c{}^a=-\delta_b^a+u_bu^a+v_bv^a+w_bw^a,\\
&\phi_e{}^au^e=-\lambda v^a,\ \phi_e{}^av^e=\lambda u^a,\\
&\phi_e{}^aw^e=0,\\
&u_au^a=v_av^a=1-\lambda^2,\\
&w_aw^a=1,\\
&u_av^a=u_aw^a=v_aw^a=0.
\end{aligned}
\tag{12}
$$

If we put

$$\psi_b{}^a=\phi_b{}^a+\frac{1}{1-\lambda}(u_bv^a-v_bu^a), \tag{13}$$

where $\lambda\neq 1$, then we have

$$
\begin{aligned}
&\psi_b{}^c\psi_e{}^a=-\delta_b^a+w_bw^a,\\
&\psi_b{}^aw^b=0,\quad \psi_b{}^aw_a=0.
\end{aligned}
\tag{14}
$$

Putting $\tilde{g}_{ba}\psi_c{}^a=\psi_{cb}$, we obtain

$$\psi_{cb}=-\psi_{bc}. \tag{15}$$

Thus, we have following, from (14) and (15),

THEOREM 3.1. *The hypersurface which is tangent to u, v of a manifold with (f, g, u, v, λ)-structure admits an almost contact metric structure $(\psi, w, \tilde{g})$ if $\lambda\neq 1$ in $i(V)$.*

In a manifold M with normal (f,g,u,v,λ)-structure satisfying (4), we have (Yano and Okumura [5])

$$\nabla_j \nabla_i \lambda = -(1+\tau^2)\lambda g_{ji},$$

where τ is a constant.

Transvecting the equation above with $B_b{}^j B_a{}^i$ and using the equation of Gauss:

$$\nabla_b B_a{}^i = h_{ba} N^i,$$

where h_{ba} is a second fundamental tensor, as $u_i N^i = 0$ in $i(V)$, we have

$$\nabla_b \nabla_a \lambda = -(1+\tau^2)\lambda \tilde{g}_{ba} + h_{ba} N^j \nabla_j \lambda.$$

In a hypersurface $i(V)$ which is tangent to u^h and v^h, by virture of (7), (11), $N^j \nabla_j \lambda$ vanishes.

Thus, λ being not identically zero, by Obata's Theorem [1], the complete hypersurface $i(V)$ is isometric with a sphere. Consequentely, we have following

THEOREM 3.2. *A complete hypersurface which is tangent to u and v of a manifold M with normal (f, g, u, v, λ)-structure satisfying* (4) *is isometric with a sphere, if the function $\lambda(1-\lambda^2)$ is an almost everywhere non-zero function in M.*

4. Invariant hypersurface of a manifold with (f,g,u,v,λ)-structure

We assume that the hypersurface $i(V)$ is invariant, that is, the tangent hyperplane of $i(V)$ is invariant by linear transformation $f_i{}^h$. Then we have

$$f_i{}^h B_b{}^i = \phi_b{}^a B_a{}^h, \tag{16}$$

that is, we have in (8)

$$w_b = 0. \tag{17}$$

Hence, (10) becomes

$$\begin{gathered} \phi_b{}^c \phi_c{}^a = -\delta_b^a + u_b u^a + v_b v^a, \\ \alpha u^a + \beta v^a = 0, \\ \phi_c{}^a u^c = 0, \quad \phi_c{}^a v^c = 0, \\ \alpha^2 + \beta^2 = 1, \\ u_a u^a = 1-\alpha^2, \quad v_a v^a = 1+\beta^2, \quad u_a v^a = -\alpha\beta \end{gathered} \tag{18}$$

because of $\lambda = 0$ on $i(V)$.

If we put

$$V_\alpha = \{p \in i(V) \mid \alpha(p) \neq 0\},$$

$$V_\beta = \{p \in i(V) \mid \beta(p) \neq 0\},$$

then V_α and V_β are open in $i(V)$ and $V_\alpha \cup V_\beta = i(V)$.

Putting $\eta_b^{(\alpha)}=\frac{1}{\alpha}v_b$ in V_α and $\eta_b^{(\beta)}=-\frac{1}{\beta}u_b$ in V_β, we have $\eta_b^{(\beta)}=\eta_b^{(\alpha)}$ in $V_\alpha\cap V_\beta$. If we define a 1-form η by

$$\eta=\begin{cases}\eta_b^{(\alpha)}dy^b \text{ in } V_\alpha,\\ \eta_b^{(\beta)}dy^b \text{ in } V_\beta,\end{cases}$$

then η is well defined on $i(V)$ and we find

$$u_b=-\beta\eta_b,\ v_b=\alpha\eta_b. \tag{19}$$

Now, in a manifold M with (f,g,u,v,λ)-structure, we assume that

$$\begin{aligned} N_{(1)i}{}^h&=(\mathfrak{L}_u f)_i{}^h+v^h\nabla_i\lambda=0,\\ N_{(2)i}{}^h&=(\mathfrak{L}_v f)_i{}^h-u^h\nabla_i\lambda=0, \end{aligned}\tag{20}$$

that is

$$\begin{aligned} &u^j\nabla_j f_i{}^h-f_i{}^j\nabla_j u^h+f_j{}^h\nabla_i u^j+v^h\nabla_i\lambda=0,\\ &v^j\nabla_j f_i{}^h-f_i{}^j\nabla_j v^h+f_j{}^h\nabla_i v^j-u^h\nabla_i\lambda=0. \end{aligned}\tag{21}$$

Then transvecting (21) with $B_a{}^iB_b$ we have, by virtue of (19) and (17),

$$\begin{aligned} &(\nabla_b\phi_{ac})u^b+\alpha N^\cdot(\nabla_i f_i{}^h)B_a{}^iB_{\cdot h}-\phi_a{}^b\nabla_b u_c+\alpha\phi_a{}^bh_{bc}+\phi_{bc}(\nabla_a u^b)-\alpha\phi_{bc}h_a{}^b+v_c\nabla_a\lambda=0,\\ &(\nabla_b\phi_{ac})v^b+\beta N^\cdot(\nabla_i f_i{}^h)B_a{}^iB_{\cdot h}-\phi_a{}^b\nabla_b v_c+\beta\phi_a{}^bh_{bc}+\phi_{bc}(\nabla_a v^b)-\beta\phi_{bc}h_a{}^b-u_c\nabla_a\lambda=0. \end{aligned}\tag{22}$$

Subtracting the equation which is multiplied α to the second equation of (22) from the equation which is multiplied β to the first equation of (22), we have

$$(\nabla_b\phi_{ac})(\beta u^b-\alpha v^b)-\phi_a{}^b(\beta\nabla_b u_c-\alpha\nabla_b v_c)+\phi_{bc}(\beta\nabla_a u^b-\alpha\nabla_a v^b)+(\beta v_c+\alpha u_c)\nabla_a\lambda=0. \tag{23}$$

Substituting (19) into (23), we have

$$(\nabla_b\phi_{ac})\eta^b+\phi_{bc}\nabla_a\eta^b-\phi_a{}^b\nabla_b\eta_c=0,$$

because of $\alpha^2+\beta^2=1$ and $\alpha\nabla_a\alpha+\beta\nabla_a\beta=0$,
from which, we have

$$(\nabla_b\phi_a{}^c)\eta^b+\phi_b{}^c\nabla_a\eta^b-\phi_a{}^b\nabla_b\eta^c=0$$

or

$$N_a{}^c=(\mathfrak{L}_\eta\phi)_a{}^c=0.$$

On the other hand, we know (Yano and Okumura [6]) that an invariant hypersurface $i(V)$ of M with (f,g,u,v,λ)-structure admits an almost contact metric structure $(\phi,\ \eta,\ \tilde{g})$.

Thus, we conclude that the hypersurface $i(V)$ admits a K-contact metric structure (Sasaki and Hatakeyama [2]). Consequentely, we have following

THEOREM 4.1. *Let M be a manifold with (f, g, u, v, λ)-structure satisfying $N_{(x)i}{}^{h}=0$. Then the invariant hypersurface $i(V)$ of M admits a K-contact metric structure.*

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