# On the Complex Charts of Four Variables

*by*

Yakichi SHIMOKAWA

## 1. Introduction

In this paper, the author tries to solve the nomographic charts of four complex variables which generalize the complex circular charts[1] and to illustrate the accuracy of this method, some simple examples are shown.

## 2. Complex chart matrix of the functional relation among four complex variables

We consider a square matrix of the fourth order of the form

$$M_4^c = \left\| \begin{array}{cccc} p_{11}\ (z_1) & p_{12}\ (z_2) & p_{13}\ (z_3) & p_{14}\ (z_4) \\ p_{21}\ (z_1) & p_{22}\ (z_2) & p_{23}\ (z_3) & p_{24}\ (z_4) \\ p_{31}\ (z_1) & p_{32}\ (z_2) & p_{33}\ (z_3) & p_{34}\ (z_4) \\ 1 & 1 & 1 & 1 \end{array} \right\| , \qquad (1)$$

where $z_k = x_k + iy_k$, $i^2 = -1$ $(k=1, 2, 3, 4)$ and $p_{1j}$, $p_{2j}$ and $p_{3j}$ are the analytic functions of $z_j$ $(j=1, 2, 3, 4)$ over an open domain R.

Now we consider the next two conditions :

1. Each column vector is not constant.
2. Vectors $(p_{1j}, p_{2j}, p_{3j}, 1)$ $(j=1, 2, 3, 4)$ are linearly dependent.

We call the matrix $M_4^c$, which satisfies the above two conditions, Massau's four variables complex chart matrix.

From the second condition, we have the relation

$$\det\ (M_4^c) = 0. \qquad (2)$$

Det $(M_4^c)$ is called Massau's complex chart determinant of the fourth order and above equation (2) is called a key equation or type equation for the four complex variables chart.

## 3. Method of nomographic solution of the key equation

We write the expression (2) as

$$\left| \begin{array}{cccc} f_1\ (z_1) & f_2\ (z_2) & f_3\ (z_3) & f_4\ (z_4) \\ g_1\ (z_1) & g_2\ (z_2) & g_3\ (z_3) & g_4\ (z_4) \\ h_1\ (z_1) & h_2\ (z_2) & h_3\ (z_3) & h_4\ (z_4) \\ 1 & 1 & 1 & 1 \end{array} \right| = 0. \qquad (3)$$

If we write the points which are represented by $f_j(z_j)$, $g_j(z_j)$ and $h_j(z_j)$ $(j=1,2,3,4)$ as $w_j$, $w_j{}^*$ and $w_j{}^{**}$ $(j=1, 2, 3, 4)$, respectively, in a Gaussian complex plane, we have the next relations:

$$\left.\begin{aligned} w_j&=f_j(z_j)=f_j(x_j+iy_j)=u_j(x_j, y_j)+iv_j(x_j, y_j) \quad ,\\ w_j^*&=g_j(z_j)=g_j(x_j+iy_j)=u_j^*(x_j, y_j)+iv_j^*(x_j, y_j) \quad ,\\ w_j^{**}&=h_j(z_j)=h_j(x_j+iy_j)=u_j^{**}(x_j, y_j)+iv_j^{**}(x_j, y_j) \quad . \end{aligned}\right\} \qquad (4)$$
$$(j=1, 2, 3, 4)$$

If a given functional relation of four complex variables

$$F(z_1, z_2, z_3, z_4)=0 \qquad (5)$$

be represented by the expression (3), we have three pairs of figures, namely, the first, the second and the third partial chart in which four families of curvilinear nets $w_j=f_j(z_j)$, $w_j^*=g_j(z_j)$ and $w_j^{**}=h_j(z_j)$ $(j=1, 2, 3, 4)$ are contained, respectively.

We consider the case where the values of $z_1$, $z_2$ and $z_3$ are given and the value of $z_4$ is unknown.

From the type equation

$$\begin{vmatrix} w_1 & w_2 & w_3 & w_4 \\ w_1^* & w_2^* & w_3^* & w_4^* \\ w_1^{**} & w_2^{**} & w_3^{**} & w_4^{**} \\ 1 & 1 & 1 & 1 \end{vmatrix} =0, \qquad (6)$$

if we have the relation

$$\begin{vmatrix} w_1 & w_2 & w_3 \\ w_1^* & w_2^* & w_3^* \\ 1 & 1 & 1 \end{vmatrix} \neq 0, \qquad (7)$$

we may find the complex number $a$, $b$ and $c$ satisfying the following relation

$$w_j^{**}=aw_j+bw_j^*+c \ . \ (j=1, 2, 3, 4) \qquad (8)$$

Relation (7) being not satisfied, we have the complex chart of three variables.[2)]
From the relation (8), we have the relation $w_j^{**}=aw_j+bw_j^*+c$ and $w_i^{**}=aw_i+bw_i^*+c$.
Eliminating $b$ in them, we have the relation

$$w_i^* w_j^{**}-w_i^{**}w_j^*=a(w_jw_i^*-w_iw_j^*)+c(w_i^*-w_j^*) \ . \ (i\neq j,\ i,\ j=1,2,3,4) \qquad (9)$$

If we put

$$w_i^* w_j-w_iw_j^*=\lambda_{ij}(w_i^*-w_j^*), \qquad (10)$$

we have the relation

$$w_i^* w_j^{**} - w_i^{**} w_j^* = (a\lambda_{ij}+c)\ (w_i^* - w_j^*) \tag{11}$$

from the relation (9).

Relation (10) and (11) are represented by next relations, respectively.

$$\begin{vmatrix} 0 & w_i^* & w_j^* \\ a\lambda_{ij}+c & w_i^{**} & w_j^{**} \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{12.1}$$

and

$$\begin{vmatrix} 0 & w_i^* & w_j^* \\ \lambda_{ij} & w_i & w_j \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{12.2}$$

$$(i \neq j,\ i,\ j=1,\ 2,\ 3,\ 4)$$

where $\lambda_{ij}=\lambda_{ji}$.

Similarly, eliminating $a$ in the relation (8), we have the next relations

$$\begin{vmatrix} 0 & w_i & w_j \\ b\mu_{ij}+c & w_i^{**} & w_j^{**} \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{12.3}$$

and

$$\begin{vmatrix} 0 & w_i & w_j \\ \mu_{ij} & w_i^* & w_j^* \\ 1 & 1 & 1 \end{vmatrix} = 0 \tag{12.4}$$

$$(i \neq j,\ i,\ j=1,\ 2,\ 3,\ 4)$$

where $\mu_{ij}=\mu_{ji}$.

According to the well-known theorem in the theory of functions of a complex variables, the above relations (12) show the next relations

$$\triangle\ O\ Q_iQ_j \backsim \triangle\ A_{ij}R_iR_j, \tag{13.1}$$

$$\triangle\ O\ Q_iQ_j \backsim \triangle\ B_{ij}P_iP_j, \tag{13.2}$$

$$\triangle\ O\ P_iP_j \backsim \triangle\ C_{ij}R_iR_j, \tag{13.3}$$

$$\triangle\ O\ P_iP_j \backsim \triangle\ D_{ij}Q_iQ_j, \tag{13.4}$$

$$(i \neq j,\ i,\ j=1,\ 2,\ 3,\ 4),$$

where vertices of triangles $P_j$, $Q_j$, $R_j$, $A_{ij}$, $B_{ij}$, $C_{ij}$ and $D_{ij}$ are represented by $w_j$, $w_j^*$, $w_j^{**}$, $a\lambda_{ij}+c$, $\lambda_{ij}$, $b\mu_{ij}+c$ and $\mu_{ij}$, respectively.

The point $A_{ij}$, $B_{ij}$, $C_{ij}$ and $D_{ij}$ being represented by $a\lambda_{ij}+c$, $\lambda_{ij}$, $b\mu_{ij}+c$ and $\mu_{ij}$, respectively, we have the next relations

$$\text{Hexagon } A_{12}A_{13}A_{14}A_{23}A_{24}A_{34} \backsim \text{Hexagon } B_{12}B_{13}B_{14}B_{23}B_{24}B_{34} \qquad (14.1)$$

and

$$\text{Hexagon } C_{12}C_{13}C_{14}C_{23}C_{24}C_{34} \backsim \text{Hexagon } D_{12}D_{13}D_{14}D_{23}D_{24}D_{34} \qquad (14.2)$$

( 1 ). If we know the value $w_4$ $(P_4)$ and $w_4^*$ $(Q_4)$, namely, $w_4$ and $w_4^*$ are constant, then, the value $w_j\,(P_j)$, $w_j^*\,(Q_j)$ and $w_j^{**}\,(R_j)$ $(j=1,\ 2,\ 3)$ being known, we know the point $B_{ij}$ and $D_{ij}$ $(i\neq j,\ i,\ j=1,\ 2,\ 3,\ 4)$ from the relation (13.2) and (13.4). From the relation (13.1) and (13.3), we know the point $A_{ij}$ and $C_{ij}$. $(i=j,\ i,\ j=1,2,3)$ Therefore, from the relation (14.1) and (14.2), the point $A_{i4}$ and $C_{i4}$ $(i=1,2,3)$ are known.

Using the relation (13.1) or (13.3), for example, $\triangle\ O\ Q_1Q_4 \backsim \triangle\ A_{14}R_1R_4$, we know the value $R_4$, namely, $w_4^{**}$ and reading the indices of curvilinear net passing through the point $R_4$, we know the required value $z_4$ $(=x_4+iy_4)$. (See Fig. 1)

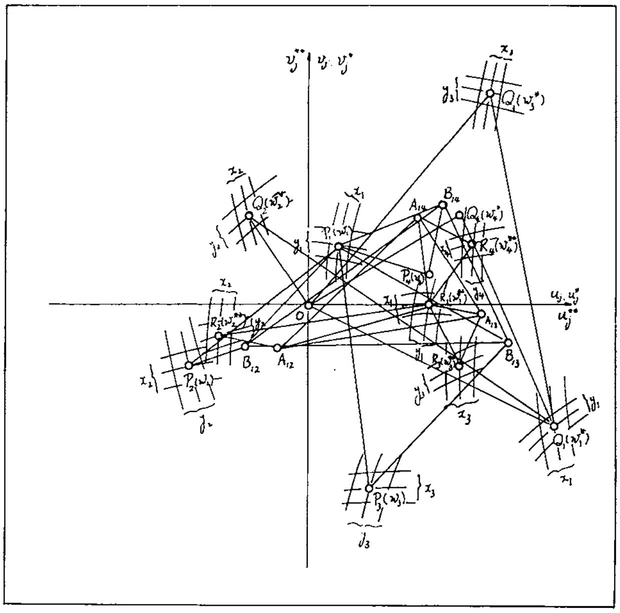

Fig. 1. First, second and third partial chart.

Step. 1. Find the point $A_{12}$, $A_{13}$, $B_{12}$, $B_{13}$ and $B_{14}$ from the relation $\triangle\ A_{12}R_1R_2 \backsim \triangle\ O\ Q_1Q_2 \backsim \triangle\ B_{12}P_1P_2$, $\triangle\ A_{13}R_1R_3 \backsim \triangle\ O\ Q_1Q_3 \backsim \triangle\ B_{13}P_1P_3$ and $\triangle\ B_{14}P_1P_4 \backsim \triangle\ O\ Q_1Q_4$.

2. Find the point $A_{14}$ from the relation $\triangle\ A_{14}A_{12}A_{13} \backsim \triangle\ B_{14}B_{12}B_{13}$.

3. Find the point $R_4$ from the relation $\triangle\ O\ Q_1Q_4 \backsim \triangle\ A_{14}\ R_1R_4$.

4. Read the indices $x_4$ and $y_4$ of curvilinear net passing through the point $R_4$.

( 2 ). If one of the values of $w_4$ and $w_4^*$, or both values are unknown, we can find the required value $z_4$ by the following method.

The value $w_j$ $(P_j)$, $w_j^*$ $(Q_j)$ and $w_j^{**}$ $(R_j)$ $(j=1,\ 2,\ 3)$ are known and using the relation (13), we can find the value $A_{ij}$, $B_{ij}$, $C_{ij}$ and $D_{ij}$. $(i\neq j,\ i,\ j=1,2,3)$

We search for the point $O'$ so that

$$\text{Quadrilateral } O\ B_{12}B_{13}B_{23} \backsim \text{Quadrilateral } O'\ A_{12}A_{13}A_{23} \tag{15.1}$$

or

$$\text{Quadrilateral } O\ D_{12}D_{13}D_{23} \backsim \text{Quadrilateral } O'\ C_{12}C_{13}C_{23}, \tag{15.2}$$

namely, for example, $\triangle\ O\ B_{12}B_{13} \backsim \triangle\ O'\ A_{12}A_{13}$ and the middle point $O''$ of $O\ O'$.

We search for the point $A'_{12}$ and $C'_{12}$ so thar $O'\ O'' \,/\!/\, A_{12}\ A'_{12}$ and $O'\ O'' \,/\!/\, C_{12}\ C'_{12}$, respectively.

We put the vector $O\ B_{12}$ of the first partial chart $w_j=f_j\ (z_j)\ (j=1, 2, 3, 4)$ drawn on the tracing paper and the vector $O\ D_{12}$ of the second partial chart $w_j^*=g_j\ (z_j)\ (j=1,2,3,4)$ drawn on another tracing paper, upon the vector $O''\ A'_{12}$ and $O''\ C'_{12}$ of the third partial chart $w_j^{**}=h_j\ (z_j)\ (j=1, 2, 3)$ $w_4^{***}=\frac{1}{2}\ w_4^{**}$ drawn on the stationary paper, respectively.

We seek for the point $P_4$, $Q_4$ and $R'_4$ which satisfy the relation

$$\frac{O''P_4}{O''P'_4}=\frac{O''B_{12}}{O''A'_{12}}=\frac{1}{|a|}\ ,\quad \frac{O''Q_4}{O''Q'_4}=\frac{O''D_{12}}{O''C'_{12}}=\frac{1}{|b|}\quad \text{and}\quad P'_4R'_4=R'_4Q'_4$$

and their indices $(x_4, y_4)$ are same and the value $x_4+iy_4=z_4$ is the required value. By the above process, we have $P'_4=aw_4+\frac{1}{2}c$, $Q'_4=bw_4^*+\frac{1}{2}c$ and $R'_4=\frac{1}{2}(P'_4+Q'_4)=\frac{1}{2}(aw_4+bw_4^*+c)$.

On the other hand, $R'_4=\frac{1}{2}\ w_4^{**}$ and therefore, $w_4^{**}=aw_4+bw_4^*+c$. If we use the mathematical instrument in this case, we can easily find the required value $z_4$. (See Fig. 2 and Fig. 3)

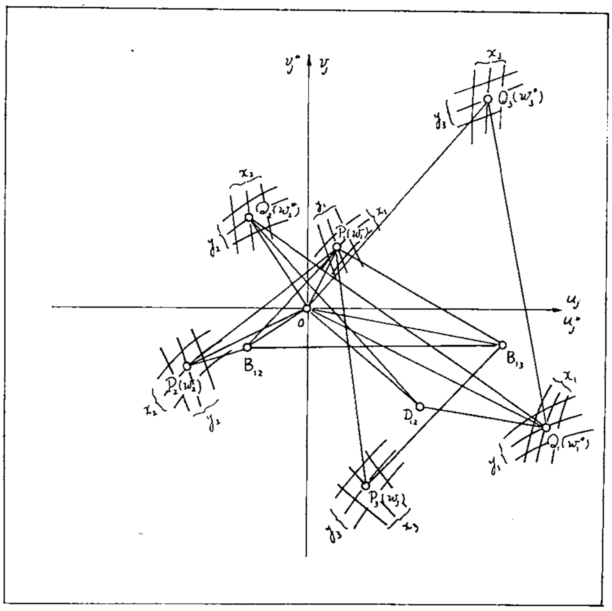

Fig. 2. 1. First and second partial chart.

Step. Find the point $B_{12}$, $B_{13}$ and $D_{12}$ from the relation
$\triangle\ B_{12}P_1P_2 \backsim \triangle\ O\ Q_1Q_2$, $\triangle\ B_{13}P_1P_3 \backsim \triangle\ O\ Q_1Q_3$ and $\triangle\ D_{12}Q_1Q_2 \backsim \triangle\ O\ P_1P_2$.

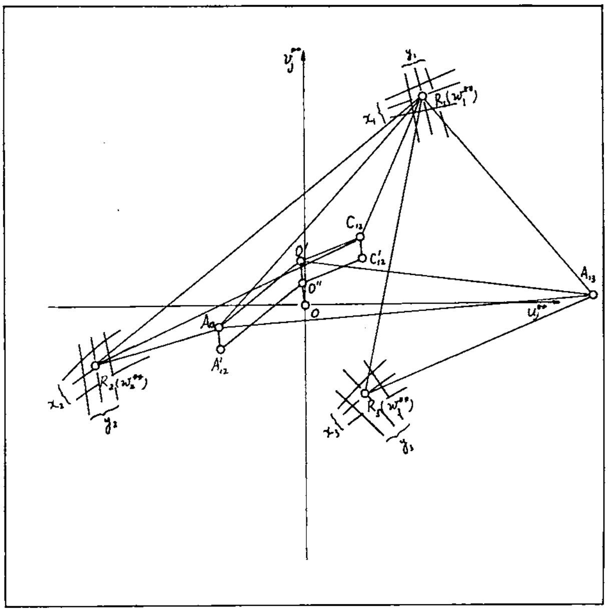

Fig. 2. 2. Third partial chart.

Step. 1. Find the point $A_{12}$, $A_{13}$ and $C_{12}$ from the relation $\triangle\ A_{12}R_1R_2 \backsim \triangle\ O\ Q_1Q_2$, $\triangle\ A_{13}R_1R_3 \backsim \triangle\ O\ Q_1Q_3$ and $\triangle\ C_{12}\ R_1R_2 \backsim \triangle\ O\ P_1P_2$.

2. Find the point $O'$ from the relation $\triangle\ O\ B_{12}B_{13} \backsim \triangle\ O'\ A_{12}A_{13}$.
3. Seek for the middle point $O''$ of $O\ O'$.
4. Find the point $A'_{12}$ and $C'_{12}$ so that $O'\ O''\ \underline{\underline{//}}\ A_{12}\ A'_{12}\ \underline{\underline{//}}\ C_{12}\ C'_{12}$.

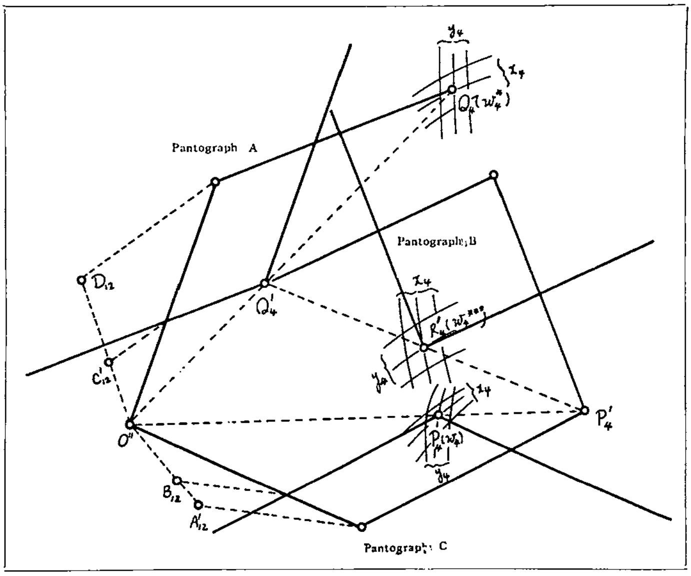


Fig. 3. Method of using the mathematical instruments (Pantograph A,B and C).

Step. 1. Put the vector $O\ B_{12}$ of the first partial chart and $O\ D_{12}$ of the second partial chart upon the vector $O''\ A'_{12}$ and $O''\ C'_{12}$ of the third partial chart, respectively.

2. Seek for the point $P_4$, $Q_4$ and $R'_4$ so that

$$\frac{O''\ P_4}{O''\ P'_4}=\frac{O''\ B_{12}}{O''\ A'_{12}}\ , \qquad \frac{O''\ Q_4}{O''\ Q'_4}=\frac{O''\ D_{12}}{O''\ C'_{12}} \text{ and } P'_4\ R'_4=R'_4\ Q'_4\ .$$

3. Read the indices $(x_4, y_4)$ of curvilinear nets passing through the point $P_4$, $Q_4$ or $R'_4$.

## 4. Affine transformation of the complex chart of four variables

Given the complex chart matrix $M_4^c$, we operate a matrix

$$A=\begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \ (\det(A) \neq 0), \tag{16}$$

where every element $a_{ij}$ is complex number, we have the next relation

$$A \cdot M_4^c=\begin{Vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} p_{11}(z_1) & p_{12}(z_2) & p_{13}(z_3) & p_{14}(z_4) \\ p_{21}(z_1) & p_{22}(z_2) & p_{23}(z_3) & p_{24}(z_4) \\ p_{31}(z_1) & p_{32}(z_2) & p_{33}(z_3) & p_{34}(z_4) \\ 1 & 1 & 1 & 1 \end{Vmatrix}$$

$$= \begin{Vmatrix} a_{11}p_{11}+a_{12}p_{21}+a_{13}p_{31}+a_{14} & a_{11}p_{12}+a_{12}p_{22}+a_{13}p_{32}+a_{14} & a_{11}p_{13}+a_{12}p_{23}+a_{13}p_{33}+a_{14} & a_{11}p_{14}+a_{12}p_{24}+a_{13}p_{34}+a_{14} \\ a_{21}p_{11}+a_{22}p_{21}+a_{23}p_{31}+a_{24} & a_{21}p_{12}+a_{22}p_{22}+a_{23}p_{32}+a_{24} & a_{21}p_{13}+a_{22}p_{23}+a_{23}p_{33}+a_{24} & a_{21}p_{14}+a_{22}p_{24}+a_{23}p_{34}+a_{24} \\ a_{31}p_{11}+a_{32}p_{21}+a_{33}p_{31}+a_{34} & a_{31}p_{12}+a_{32}p_{22}+a_{33}p_{32}+a_{34} & a_{31}p_{13}+a_{32}p_{23}+a_{33}p_{33}+a_{34} & a_{31}p_{14}+a_{32}p_{24}+a_{33}p_{34}+a_{34} \\ 1 & 1 & 1 & 1 \end{Vmatrix} \tag{17}$$

$$=\bar{M}_4^c \ .$$

We call the matrix $A$ a complex transformation matrix and $\bar{M}_4^c$ a transformed complex chart matrix.

When $\det(M_4^c)=0$, we have $\det(\bar{M}_4^c)=0$ from the relation (17) and this shows that, by an adequate affine transformation, we have another new complex chart which is convenient for use.

## 5. Example

1. We consider the next key equation

$$\begin{vmatrix} z_1 & \dfrac{20}{z_2} & \dfrac{4z_3}{1+i} & \dfrac{10}{z_4}+4-2i \\ \dfrac{20}{z_1+1-i} & z_2+2+i & \dfrac{20}{z_3}-2+3i & \dfrac{4z_4}{2+i} \\ \dfrac{4z_1}{1+2i} & \dfrac{20}{z_2}+1+i & z_3+3-i & 2z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0 \ .$$

When $z_1=1+2i$, $z_2=-4+2i$ and $z_3=2-i$ are given, we find $z_4=-1.34-0.29i$ and $z_4=3.75-1.08i$ by our method. On the other hand, we know the exact value $z_4=-1.336-0.290i$ and $z_4=3.753-1.086i$.

2. We consider the next key equation

$$\begin{vmatrix} z_1 & \dfrac{20}{z_2} & \dfrac{4z_3}{1+i} & 4+i \\ \dfrac{20}{z_1+1-i} & z_2+2+i & \dfrac{20}{z_3}-2+3i & 5+3i \\ \dfrac{4z_1}{1+3i} & \dfrac{20}{z_2}+1+i & z_3+3-i & z_4 \\ 1 & 1 & 1 & 1 \end{vmatrix}=0 .$$

When $z_1=1+2i$, $z_2=-4+2i$ and $z_3=2-i$ are given, we find $z_4=5.37+2.13i$ by our method.

We know the exact solution $z_4=5.367+2.142i$. The original charts of our examples have been drawn on the millimeter section papers $700\times1000$mm.

## 6. Conclusion

In this paper the author has found the method to obtain the solution of the nomographic chart of four complex variables. This method being graphical, we can not expect the higher accurate solution but we can find the first approximate value of the solution.

## References

1) Y. Shimokawa.: On the complex circular charts. The Proceedings of the 14th Japan National Congress for Applied Mechanics, 1964.

2) K. Morita and Y. Simokawa.: Nomographic Representation of the Functional Relations among Three Complex Variables. Z. A. M. M., 40 (1960), Ht. 7/8, S. 350-359.