A Particle Based Solver for the Three Dimensional Fluid Flow through an Elastic Porous Medium

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Abstract. *Fluid flow through elastic porous medium is an interesting and complex problem in fluid dynamics. Its complexity arises due to the necessity to take into account both the stress and the strain in the structure of the elastic solid as it interacts with the fluid. In our work, we simulate this phenomenon by using the Smoothed Particle Hydrodynamics (SPH) method. In the SPH method, the fluid and the elastic solid are represented using particles in the Lagrangian frame. We compare various formulations of the interaction between the fluid an the elastic solid on a few test cases, and we present the results of our simulations.*

Keywords: Smoothed Particle Hydrodynamics, Elastic solid, Fluid-structure interaction, Porous medium

1 Introduction

Fluid flow through an elastic porous medium is an interesting topic encountered in many research fields of science and engineering. It has many application such as filtration process, blood vessel, reservoir characterization, etc. Here we want to investigate the phenomenon with a simple model of the sponge.

We try to simulate the fluid pass through the porous of the solid body. The position of the fluid is on the top of the solid body. The fluid goes down because of the gravitation (body force). The deformation of the elastic solid (sponge) occurred due to action of the fluid. We consider the fluid-structure interaction to model the action of the fluid to the solid body, vice versa.

In addition, we simulate this phenomenon by using Smoothed Particle Hydrodynamics (SPH) method. SPH is a meshless Lagrangian particle method to obtain numerical solutions of the fluid dynamics equations. The method was introduced in [5] to solve an astrophysical problem. Recently, the SPH method was developed in many application in computational fluid dynamics. We give a brief introduction about the basic SPH method in this paper.

Moreover, we model the case with different parameter discretization and size of pore radius of the sponge. It will show us the robustness of our method to solve this problem.

2 Governing Equations

We consider the governing equations of the motion of the fluid and solid body in isothermal condition. It consists of continuity and momentum equation.

2.1 Continuity equation

We use the continuity equation in Lagrangian framework. It can be written as,

$$\frac{D\rho}{Dt} + \rho\left(\nabla \cdot \mathbf{v}\right) = 0 \tag{1}$$

where ρ is density, t is time, $\mathbf{v}$ is the velocity. Here $\frac{D}{Dt}$ is a material derivative. It is equal to $\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla)$. The last term in material derivative is called a convection derivative part. This equation guarantee the mass conservation.

2.2 Momentum equation

The governing equation of momentum equation or the equation of motion can be written as,

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla \cdot \sigma + \rho\mathbf{g} \tag{2}$$

where σ is the Cauchy stress tensor. It consists of pressure and deviatoric stress tensor. The last term in equation (2) is body force term. The body force that we use in this case is gravitational force.

2.2.1 Momentum equation of the fluid

Since we model the motion of the incompressible Newtonian fluid, the equation (2) can be shown as,

$$\rho_f\frac{D\mathbf{v}_f}{Dt} = -\nabla p_f + \eta\Delta\mathbf{v}_f + \rho_f\mathbf{g}_f \tag{3}$$

where p_f and η denote the pressure and the dynamic viscosity. Note $(.)_f$ represents the physical quantity of the fluid. The term ∇p_f is pressure gradient term and the $\eta\Delta\mathbf{v}_f$ is dissipation or viscosity term.

2.2.2 Momentum equation of the solid body

The motion of the solid body is described by the following equation,

$$\rho_f\frac{D\mathbf{v}_s}{Dt} = -\nabla p_s + \nabla \cdot \mathbf{S}_s + \rho_s\mathbf{g}_s \tag{4}$$

where $\mathbf{S}_s$ is deviatoric stress tensor. Note $(.)_s$ represents the physical quantity of the solid body. This equation is obtained by the stress tensor decomposition into its isotropic and deviatoric parts:

$$\sigma^{ij} = -p_s\delta^{ij} + S_s^{ij}$$

from the equation (2). In [3] Monaghan proposed the following equation to compute the rate change of deviatoric stress,

$$\frac{DS_s^{ij}}{Dt} = 2G_s\left(\dot{\epsilon}_s^{ij} - \frac{1}{3}\delta^{ij}\dot{\epsilon}_s^{ij}\right) + \sum_k S_s^{ik}\Omega_s^{jk} + \Omega_s^{ik}S_s^{kj} \tag{5}$$

where

$$\dot{\epsilon}_s^{ij} = \frac{1}{2}\left(\frac{\partial v_s^i}{\partial x_s^j} + \frac{\partial v_s^j}{\partial x_s^i}\right) \tag{6}$$

is the rate of deformation tensor

$$\Omega_s^{ij} = \frac{1}{2}\left(\frac{\partial v_s^i}{\partial x_s^j} - \frac{\partial v_s^j}{\partial x_s^i}\right) \tag{7}$$

is the spin tensor. Then, we use the equation of state to obtain pressure in the fluid and the solid body,

$$p = \frac{\rho_0 c^2}{\gamma}\left(\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right) \tag{8}$$

where ρ_0 is the reference density, c is the speed of sound, and γ is a constant parameter.

3 Numerical Methods

3.1 Basic of SPH method

SPH approximate the Dirac delta function in an integral representation of a function by a smoothing function $W^h(d)$,

$$f(\mathbf{r}) = \int_\Omega f(\mathbf{r}')W^h(d)d\Omega(\mathbf{r}') \tag{9}$$

for

$$d = \|\mathbf{r} - \mathbf{r}'\|$$

where f is the function of a position vector, h is the length of the smoothing function, and $d\Omega(\mathbf{r}')$ represents a finite control volume in three dimensional space.

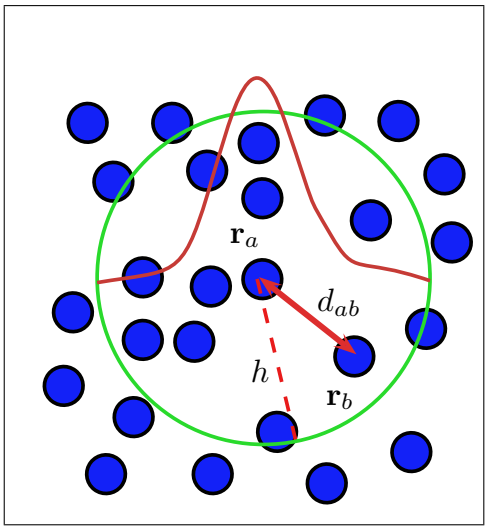


Figure 1: Two dimensional ilustration of a smoothing function over a support domain

By using SPH particle approximation, we substitute the finite control volume $d\Omega(\mathbf{r}')$ with the particle $\frac{m_b}{\rho_b}$, the equation (9) can be written as,

$$f(\mathbf{r}_a) = \sum_b \frac{m_b}{\rho_b} f(\mathbf{r}_b)W^h(d_{ab}) \tag{10}$$

The quantity of the smoothing function depend on the distance between the particles d_{ab}. m_b and ρ_b denote mass and density of each particle. The illustration can be shown in figure 1. More details about the SPH method can be found in [2] and [4].

3.2 Approximation of governing equations by SPH method

The SPH approximation of the continuity equation (1) for fluid and solid particles is,

$$\frac{D\rho_a}{Dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab} \tag{11}$$

By using this equation we can update the density of each particle. This equation shows that the time rate of density change of each particle is proportional to the relative velocities between the center particle a and its neighbour particles b.

According to the equation (3), the SPH approximation for the material derivative of the velocity of particle a is,

$$\frac{Dv_a^i}{Dt} = -\sum_{b\in\Omega_f} m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2}\right) \nabla_a W_{ab} + \frac{\eta}{\rho_a} \sum_{b\in\Omega_f} m_b \mathbf{v}_{ab} \nabla_a^2 W_{ab} + \mathbf{g} \tag{12}$$

where

$$W_{ab} = W^h(d_{ab}) \qquad \text{and} \qquad \mathbf{v}_{ab} = \mathbf{v}_a - \mathbf{v}_b$$

The summation of the equation (12) is just for the fluid particles Ω_f. This symmetric formulation satisfies the action-reaction principle between the center particle a and particle b. To compute the pressure gradient term we use smoothing function which has been proposed by [6],

$$W^h_{spiky}(d) = \frac{15}{\pi h^6} \begin{cases} (h-d)^3, & \text{if} \quad 0 \le d \le h \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

It is called spiky kernel. The gradient of this kernel will not vanish near the center. The kernel can generates the repulsive force if the particles get too close to each other when we compute the pressure gradient term[7].

To compute the viscosity term in equation (12) we use use this following smoothing function which has been introduced by [7],

$$W^h_{viscosity}(d) = \frac{15}{2\pi h^3} \begin{cases} -\frac{d^3}{2h^3} + \frac{d^2}{h^2} + \frac{h}{2d} - 1, & \text{if} \quad 0 \le d \le h \\ 0, & \text{otherwise} \end{cases} \tag{14}$$

The SPH approximation of the momentum equation (4) of the solid particle a, which yields

$$\frac{Dv_a^i}{Dt} = \sum_{b\in\Omega_s} m_b \sum_j \left(\frac{\sigma_a^{ij}}{\rho_a^2} + \frac{\sigma_b^{ij}}{\rho_b^2} + \Pi_{ab}\delta^{ij} + \left(\frac{R_a^{ij}}{\rho_a^2} + \frac{R_b^{ij}}{\rho_b^2}\right) f_{ab}^n\right) \frac{\partial W_{ab}}{\partial x_a^j} + \mathbf{g} \tag{15}$$

The summation of the equation (15) is just for the solid particles Ω_s. The term $\Pi_{ab}\delta^{ij}$ is a dissipation term which has been proposed by [1]. It is used to remove velocity oscillations and prevent unphysical penetration when the particles get too close each other in the SPH method. We compute the dissipation term by using the following equation,

$$\Pi_{ab} = \begin{cases} \frac{\alpha c_0 \mu_{ab}}{\rho_{ab}}, & \text{if} \quad \mu_{ab} < 0 \\ 0, & \text{otherwise} \end{cases} \tag{16}$$

$$\mu_{ab} = \frac{h(\mathbf{v}_a - \mathbf{v}_b) \cdot (\mathbf{r}_a - \mathbf{r}_b)}{d_{ab}^2 + (0.1h)^2}$$

In [3], Monaghan proposed artificial stress to removes tensile instability in the SPH method. Tensile instability is the clustering phenomenon that can be happened when the solid particles is stretched each other. The idea is by adding the repulsive force in terms of the kernel function and the stress tensor when the particles in tensile stress condition. We obtain the artificial stress by these following equations,

$$R_a^{ij} = -\varepsilon \begin{cases} \sigma_a^{ij}, & \text{if} \quad \sigma_a^{ij} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

and

$$f_{ab} = \frac{W^h(d_{ab})}{W^h(\Delta s)} \tag{18}$$

is the repulsive force which increases as the distance decreases and Δs is initial distance between solid particles.

We use the XSPH velocity correction from [4] to compute the strain tensor $\dot{\epsilon}$ and the spin tensor Ω^{ij} in RHS of (5) and (11). The equation is given by

$$\hat{\mathbf{v}}_a = \mathbf{v}_a + 0.5 \sum_{b \in \Omega_s} \frac{m_b}{\bar{\rho}_{ab}} \mathbf{v}_{ba} W_{ab} \tag{19}$$

When compute velocity correction in (19), we use a smoothing function from [3],

$$W_{poly6}^h(d) = \frac{315}{64\pi h^9} \begin{cases} (h^2 - d^2)^3, & \text{if} \quad 0 \leq d \leq h \\ 0, & \text{otherwise} \end{cases} \tag{20}$$

Then to obtain the velocity gradient of the particle in the equation (6) and (7) is given by

$$\frac{\partial \hat{v}_a^i}{\partial x_a^j} = -\sum_{b \in \Omega_s} \frac{m_b}{\rho_b} \hat{v}_{ab}^i \frac{\partial W_{ab}}{\partial x_a^j} \tag{21}$$

3.3 Fluid-structure interaction

To solve FSI problem, Antoci in [1] considered the shape of the solid body to compute the pressure gradient. The results satisfy the experimental data. But it can be a big problem if the method deals with a complex shape and deformation of the solid body.

However, we model the interaction between the fluid and the solid particles by using the SPH approximation of the continuity and the momentum equation of the fluid particles. The FSI problem is modeled by using these following rules:

- treat the solid particles as the fluid particles when we update the density of the fluid particles, vice versa

$$\frac{d\rho_a}{dt} = \sum_b m_b \mathbf{v}_{ab} \cdot \nabla_a W_{ab} \tag{22}$$

- treat the solid particles as the fluid particles when we compute the force exerted on the fluid particles due to the solid particles,

$$\mathbf{f}_{a(s \to f)} = -\sum_{b \in \Omega_s} m_b \rho_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} \right) \nabla_a W_{ab} + \eta \sum_{b \in \Omega_s} m_b \mathbf{v}_{ba} \nabla_a^2 W_{ab} \tag{23}$$

- add repulsive force which has been introduced by [8] to the equation (22),

$$\mathbf{f}_{a_{repulsive}} = K_f \sum_{b \in \Omega_s} \frac{m_a m_b}{\rho_a \rho_b} \nabla_a W_{ab} \tag{24}$$

- The force exerted on the solid particles due to the fluid particles is obtained by this rule,

$$\mathbf{f}_{a(f \to s)} = -\mathbf{f}_{a(s \to f)} \tag{25}$$

In [9], Amini et al. modified the repulsive force of (24) because he used the cubic spline kernel. The gradient of the kernel will vanish near the center. Since we use the kernel which has non zero value near the center, we do not have to modified the repulsive force in (24).

4 Implementations of the SPH method and discussions

The position of the fluid is on the top of the solid body (sponge). The initial distance between the bottom of the fluid particles and the top of the solid particles is 0.0045m. The thickness of the sponge is 0.004m. It is attached to right and left side of the wall. The gravitation (g=9.8m/s) is applied to all particles.

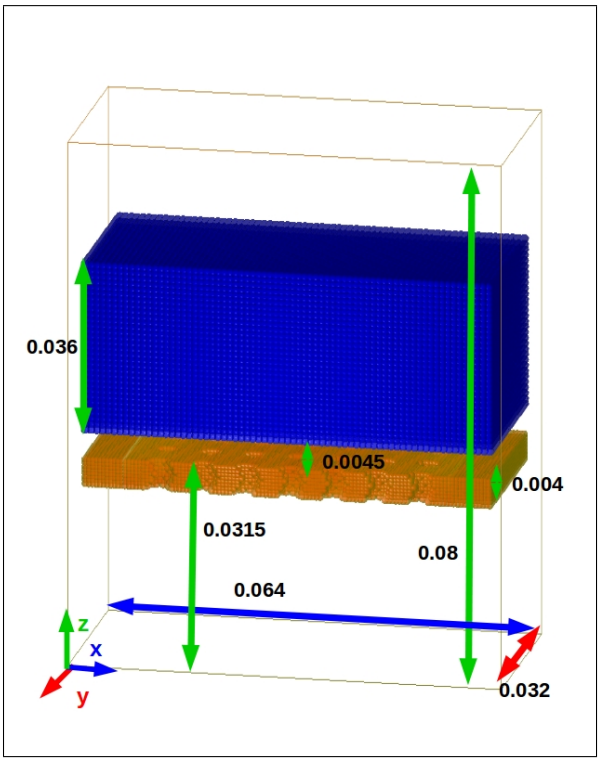


Figure 2: Geometry of the simulation

Water is used as the fluid with density $\rho_f = 1000\text{kg/m}^3$. The small sound speed of the water is chosen c_f=30m/s and dynamic viscosity of the water is η=3.5x10^{-3}kg/ms. The dimension of the water as shown in figure 2.

The sponge is modeled with density $\rho_s = 950\text{kg/m}^3$. The small sound speed of the sponge is chosen c_s=31.6m/s. Also the shear modulus G_s=2x10^{-6}N/m^2. The shape of the pore as shown in figure 3.

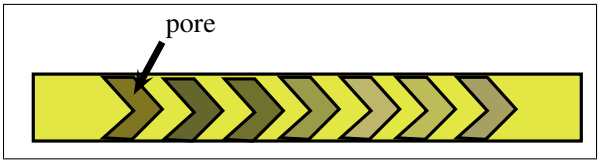


Figure 3: Side view of the sponge

There are two kind of the simulations in this paper. The first one, we try to simulate the case with different ratio of the pore radius r_{pore} to kernel length h. The top view of the sponge as shown in figure 4. The other one, we try to simulate the case with different kernel length but same size of the pore.

4.1 The case with different size of the pore radius

We use the ratio of the pore radius to the length kernel to set the different size of the sponge pore. There are four value for the ratio including 0.5, 1, 1.5 and 2. Each sponge has the regular size of the pore and identical number of the porous. The parameter of the sponge is setted equal for every simulation in this case. Then, we compare the average displacement of the solid particles each time interval between the sponges.

The length of the kernel function h=0.001, the distance between fluid particles d_{fluid}=8x10^{-4} and solid particles d_{solid}=5x10^{-4} are chosen. Also, the number particles for the fluid n_{fluid}=68231 and the solid n_{solid}: 43696 (r_{pore}=0.5h), 41533 (r_{pore}=h), 37978 (r_{pore}=1.5h), and 33160 (r_{pore}=2h). The time interval Δt=10^{-5}s is adopted.

We obtain the average displacement dis_{avg} of the solid particles of the sponge at certain time t by using the following equation,

$$dis_{avg} = \frac{1}{N}\sum_{i=1}^{N}(r_i - r_0)$$

where N is the number of solid particles, r_i is the position of the solid particle at time t and r_0 is the position of the solid particle at initial position. The results as shown in figure 5. The largest

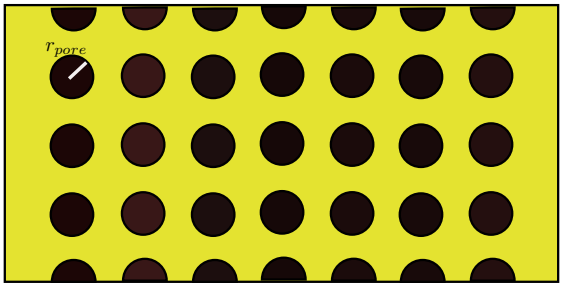


Figure 4: An illustration of the sponge from the top

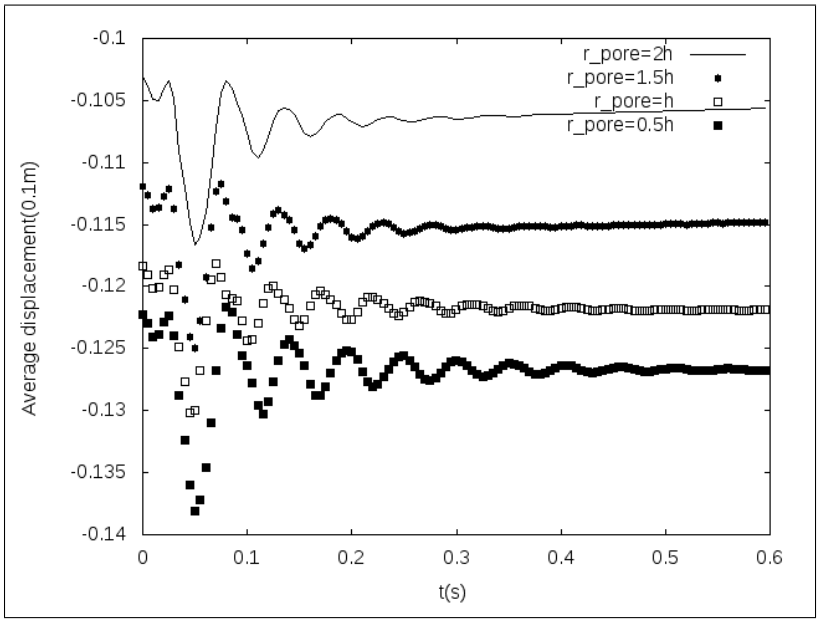


Figure 5: The displacement of the solid particles relative to its initial position for different size of pore radius

average displacement of the solid particles is the sponge with the size of pore radius equal to $0.5h$. The smallest average displacement of the solid particles is the sponge with the size of pore equal to $2h$. The different results is because the fluid which pass through the pore of the sponge r_{pore}=$2h$ is easier than the pore of the sponge which smaller than $2h$.

In addition, we can see the figure 6 as the explanation of the results. At the beginning t=0s the amount of the fluid is equal each case. At t=0.05s the solid particles is stretched because of the force from the fluid particles. The amount of the solid particles which passed through the solid particles for r_{pore}=$2h$ is greater than r_{pore}=$1.5h$ at t=0.075s. Moreover, at t=0.1s the strain of the sponge r_{pore}=$1.5h$ is greater than the sponge r_{pore}=$2h$.

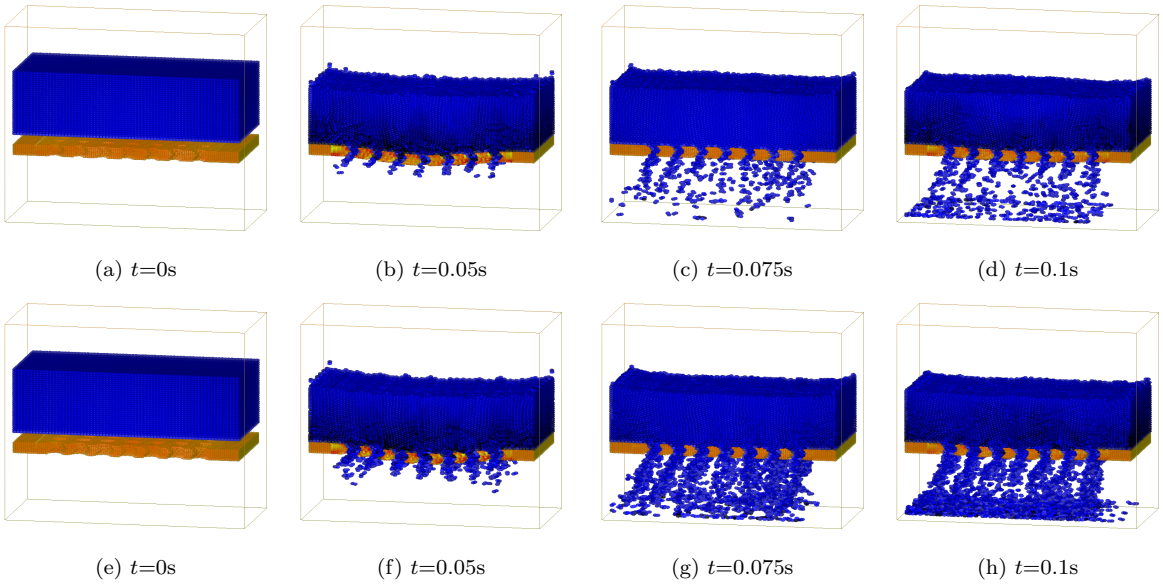
Hence, we conclude that the result for this case depends on the ratio of the kernel length to the radius of pore.

4.2 The case with different length of the kernel

In this case, we want to test the robustness of our method. A robustness requirement aims to ensure that the method does not depend on the discretization parameter h. The physical parameter that we use same as the first case with r_{pore}=0.0015m. The different with the first case is just we only change the kernel length. The value of h=0.0013, 0.0012, 0.0011, and 0.001m are adopted.

In Figure 7 shows the average displacement of the solid particles relative to its initial position. It can be seen the results at time 0 until 0.15 seconds is quietly similar. But at time 0.15 until 0.5 seconds the displacement of some sponges are different.

Our FSI method depends on the repulsive force parameter in (24). Here, we fix the value of $K_f = 10^8$. To get the best results we have to adjust the parameter K_f in terms of h.



(a) t=0s (b) t=0.05s (c) t=0.075s (d) t=0.1s

(e) t=0s (f) t=0.05s (g) t=0.075s (h) t=0.1s

Figure 6: The snapshots of the simulations for r_{pore}=1.5h (a-d) and r_{pore}=2h (e-h)

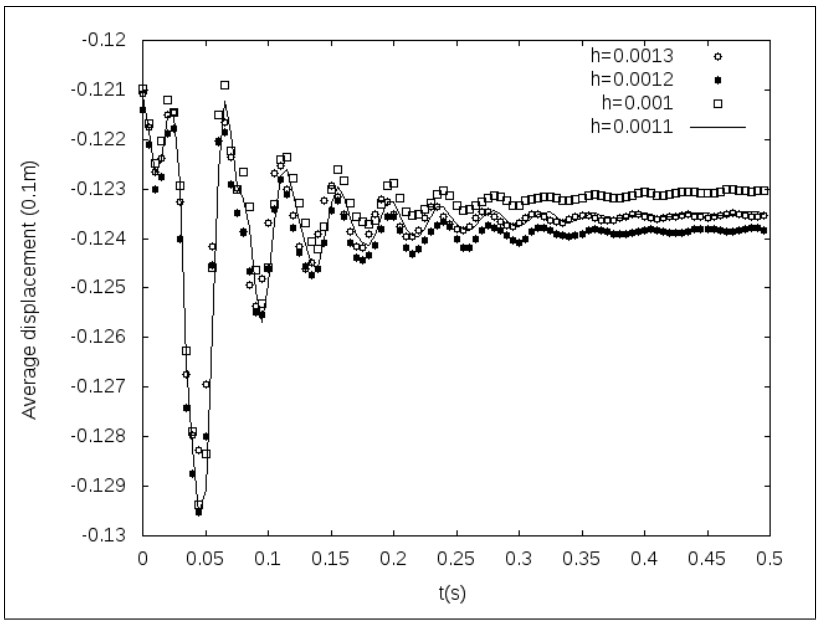


Figure 7: The displacement of the solid particles relative to its initial position for different length of h

5 Conclusions

We have presented SPH method for the fluid flow simulation through porous medium with various radius of pore. In addition our method to simulate fluid-structure interaction and deformation of the elastic solid (sponge) due to action of the fluid has been proposed. We should investigate the relation between K_f and h to achieve the robustness of our method. In future work, the fluid-structure interaction method can be extend to simulate the multi-phase flow through porous medium (e.g. oil-filled pores), solidification process, etc.

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