Numerical Construction of Energy-Theoretic Crack Propagation based on a Localized Francfort-Marigo Model

Armanda Ikhsan[a,b], Masato Kimura[b], Takeshi Takaishi[c], Maharani Ahsani Ummi[a,b]

[a]Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha 10, Bandung 40132 Indonesia, E-mail: armandaikhsan@gmail.com, maharaniahsaniummi@gmail.com
[b]Institute of Science and Engineering, Kanazawa University, Kakuma, Kanazawa 920-1192 Japan, E-mail: mkimura@se.kanazawa-u.ac.jp
[c]Faculty of Information Design, Hiroshima Kokusai Gakuin University, Nakano, Aki-ku, Hiroshima 739-0321 Japan, E-mail: t.takaishi@hkg.ac.jp

Abstract. *We simulate straight crack propagation using idea from the classical Griffith theory and Francfort-Marigo energy. According to the energy-theoretic model proposed by Francfort and Marigo, propagation of a straight crack is described by means of sum of elastic and surface energies. We modify the Francfort-Marigo model by replacing the global minimum by a local one in order to be consistent with the Griffith theory. We numerically construct some energy-theoretic propagation of straight cracks using finite element method and show some discontinuous growth behaviour of the crack.*

Keywords: crack propagation, elasticity problem, finite element method

1 Introduction

Crack propagation can cause serious problems. Therefore, to understand the crack propagation phenomenon is important. Griffith's theory [3] tells that a crack will propagate only if the elastic energy released during the crack growth is greater than or equal to the surface energy which is proportional to the area of the new crack surfaces.

Francfort and Marigo [2] proposed a crack propagation model based on the total energy which is a sum of elastic energy and surface energy. They extended the classical Griffith theory and proposed the model to describe crack propagation. This is one of the most naive models for crack propagation, but is not suitable for numerical simulation.

Although a number of crack propagation models and numerical algorithms have been proposed in engineering and physics, as far as the authors know, any mathematically closed model which can be numerically computable has not been established. Among the models, in particular, phase field model proposed in [4] is remarkable for its easy numerical treatment. It is described as a gradient flow of an approximate energy of the Francfort-Marigo energy by using the idea of Ambrosio-Tortorelli regularization [1]. Since it has a similar form of the reaction-diffusion equation, its numerical simulation is relatively easier than the other models. However, the accuracy of this model has not been studied well due to the lack of an established model.

As a theoretically reliable model, we investigate the energy-theoretic one and numerically construct some crack propagation. Which will be used as a reference solution to check the accuracy of other crack propagation models such as the phase field model.

The outline of this paper is as follows. In the second section, we will give a two dimensional setting of an antiplane displacement and mode III crack propagation with brief explanation about the classical Griffith theory and the Francfort-Marigo model and how we use it to make simulation. In the third section we will propose a localized Francfort-Marigo model and show some numerical examples. In the last section we will give conclusions and future work.

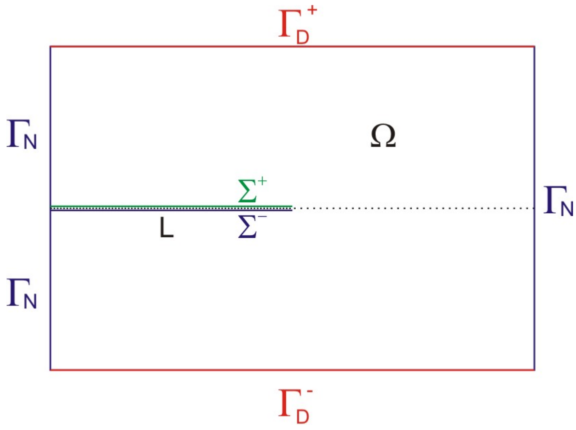

Figure 1: Domain

2 Classical Griffith Theory and the Francfort-Marigo Model

We consider a mode III crack propagation in a plate by the deformation perpendicular to the plate. The plate is supposed to be an isotropic elastic material with constant thickness. We suppose that $x = (x_1, x_2) \in \mathbb{R}^2$ is a Cartesian coordinate parallel to the plate, and that x_3 is a coordinate perpendicular to the plate. We assume that the deformation of the plate is limited to the x_3-direction. Therefore, we treat the problem as a two dimensional domain Ω. We assume a displacement field $\bar{u}$ has the following form :

$$\bar{u}(x_1, x_2, x_3) = (0, 0, u(x_1, x_2))^T$$

Let Ω be a bounded two dimensional domain, with a piecewise smooth boundary $\Gamma = \partial\Omega$. We suppose that Γ_D is a non empty open portion of Γ and set $\Gamma_N := \Gamma \setminus \Gamma_D$. A crack in Ω is denoted by $\Sigma \subset \Omega$ and its upper and lower sides are denoted by Σ^+ and Σ^- respectively. We suppose Σ is a straight crack of length L with only one tip in Ω as shown in Figure 1.

According to [4], the antiplane displacement u satisfies the following equations :

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \Sigma \\ u = g & \text{on } \Gamma_D \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \Gamma_N \cup \Sigma^{\pm} \end{cases} \tag{1}$$

We denote the solution u to (1) by $\hat{u}(L, g) \in H^1(\Omega \setminus \Sigma)$. We define a corresponding bilinear form and an elastic energy :

$$a(u, v; \Omega \setminus \Sigma) := \frac{\mu}{2} \int_{\Omega \setminus \Sigma} \nabla u \cdot \nabla v \, dx,$$
$$E(L, g) := \min_{u|_{\Gamma_D = g}} a(u, u; \Omega \setminus \Sigma) = a\left(\hat{u}(L, g), \hat{u}(L, g); \Omega \setminus \Sigma\right),$$

where $\mu > 0$ is a rigidity constant.

We suppose that the Dirichlet boundary condition depends on time t, that is $u = g(t) = g(\cdot, t)$ on Γ_D. The classical Griffith's criterion for the crack propagation is derived from a balance of the

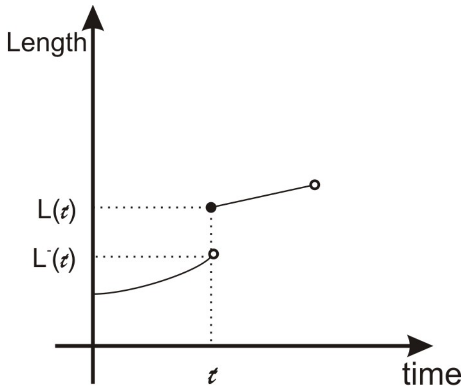

Figure 2: A discontinuous crack propagation

released energy and the created crack length. According to the theory, so-called energy release late G is defined as

$$G := -\frac{\partial E}{\partial L}(L, g(t)) \geq 0.$$

The crack Σ can propagate only if

$$G \geq \gamma. \tag{2}$$

Where the constant $\gamma > 0$ is called fracture toughness. This is called Griffith's criterion. Francfort and Marigo introduced the following total energy in (1):

$$\mathcal{E}(L, g) := E(L, g) + \gamma L \qquad (L := |\Sigma|),$$

which is a sum of the elastic energy $E(L, g)$ and a surface energy γL. The condition (2) is also equivalent to

$$\frac{\partial \mathcal{E}}{\partial L}(L, g(t)) \leq 0. \tag{3}$$

They proposed the following simple crack propagation model as an extension of the Griffith theory. They considered that $L(t) = |\Sigma(t)|$ is given by

$$\begin{cases} L(t) := \underset{L^-(t) \leq L \leq L_\infty}{\arg\min} \ \mathcal{E}(L, g(t)) \\ L^-(t) := \sup_{s<t} L(s), \end{cases} \tag{4}$$

where L_0 is the width of Ω. We call (4) Francfort-Marigo model in this paper. We remark that $L(t)$ is a non-decreasing function, but can be discontinuous as shown in Figure 2.

We suppose that the boundary condition $g(t)$ on Γ_D is given in the following form :

$$g(t) = tg_0(x) \ (x \in \Gamma_D),$$

where $x = (x_1, x_2)$ denotes the space variable. Since $\hat{u}(L, tg_0)$ is given as $\hat{u}(L, tg_0) = t\hat{u}(L, g_0)$, we get

$$E(L, tg_0) = a(t\hat{u}(L, g_0), t\hat{u}(L, g_0), \Omega \setminus \Sigma) = t^2 a(\hat{u}(L, g_0), \hat{u}(L, g_0), \Omega \setminus \Sigma) = t^2 E(L, g_0).$$

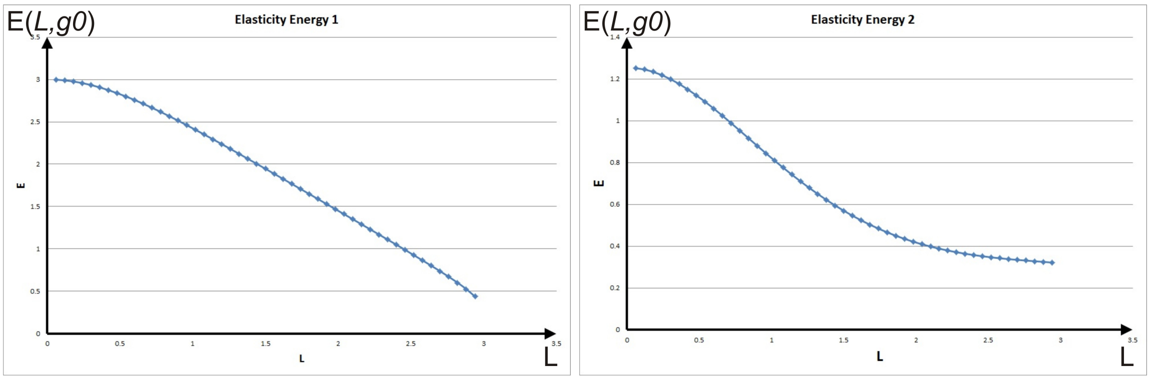

Figure 3: Graphs of elastic energy $E(L, g_0)$ with $g_0 = 1$ (left) and $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$ (right)

Therefore the total energy becomes

$$\mathcal{E}(L, t\ g_0) = t^2 E(L, g_0) + \gamma L.$$

In this paper, for simplicity, we assume $\Omega = (0, L_\infty) \times (-l, l)$ and $\Sigma(t) = (0, L(t)] \times \{0\}$ as shown in Figure 1 and set $\mu = 2$. Under the symmetry condition $g_0(x_1, -l) = -g_0(x_1, l)$ on Γ_D, for the initial condition $g_0 = 1$ and $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$, we get graphs of $E(L, g_0)$ as seen in Figure 3. These graphs are computed by using FreeFem++ software [5].

From Figure 3, we can see the difference of the profiles of $E(L, g_0)$ between $g_0 = 1$ and $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$. On the left figure, the graph is concave. But on the right one, the graph has two parts, concave and convex. These shapes essentially affect the type of crack propagation as will be seen later.

3 Localized Francfort-Marigo model

From the Griffith theory, we know that a crack cannot propagate if (3) is not reached. But, as we will see later, the Francfort-Marigo model (4) allows a crack to have a jump even if Griffith's criterion is not satisfied. So, we need to modify (4) to make it consistent to the classical Griffith's criterion.

In order to replace the global minimum in (4), we define a notion of a nearest local minimum. We assume $f \in C^1([a, b])$, we define a nearest local minimum of f in $[a, b]$ as follows :

$$\underset{a \le x \le b}{\text{arg loc-min}}\, f(x) := \begin{cases} a & (\text{ if } f'(a) \ge 0), \\ \sup\left\{x \in (a, b];\ f'(y) < 0\ \left({}^\forall y \in [a, x)\right)\right\} & (\text{ if } f'(a) < 0). \end{cases}$$

We propose the following modification of the Francfort-Marigo model.

Problem 3.1 (localized Francfort-Marigo model)**.** *Let L_∞ be the maximum length of the straight crack in Ω. For a given initial crack Σ_0 of length $L_0 \in (0, L_\infty)$, find $\Sigma(t)$ of length $L(t)$ for $t \ge 0$ such that*

$$L(t) = \underset{L^-(t) \le L \le L_\infty}{\text{arg loc-min}}\, \mathcal{E}(L, g(t))(t \ge 0),$$

where

$$L^-(t) := \begin{cases} L_0 & (t=0), \\ \sup\limits_{0\le s<t} L(s) & (t>0). \end{cases}$$

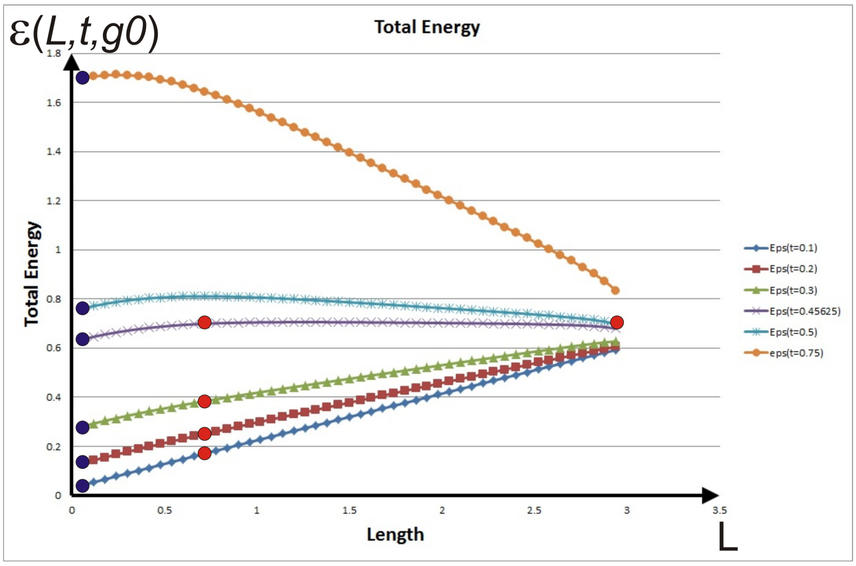

Figure 4: Graph of total energy $\mathcal{E}(L, t\ g_0)$ with $g_0 = 1$

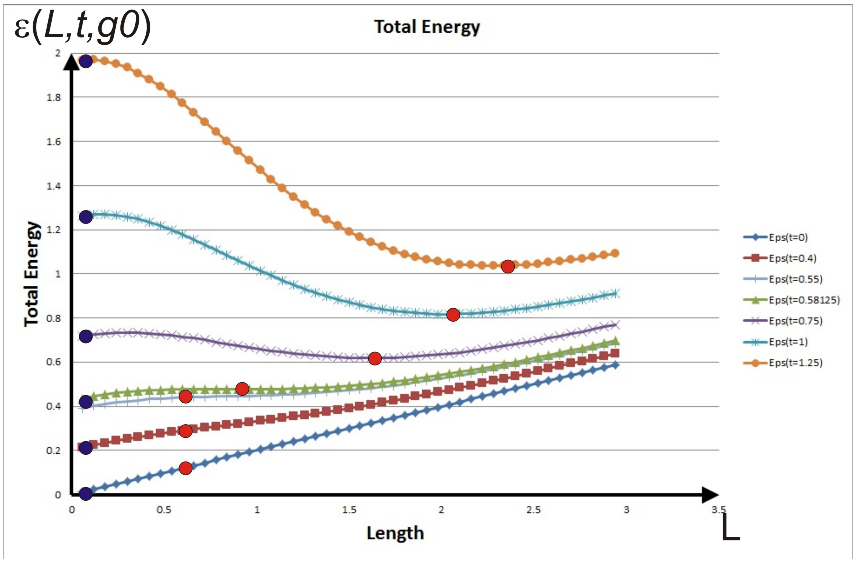

Figure 5: Graph of total energy $\mathcal{E}(L, t\ g_0)$ with $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$

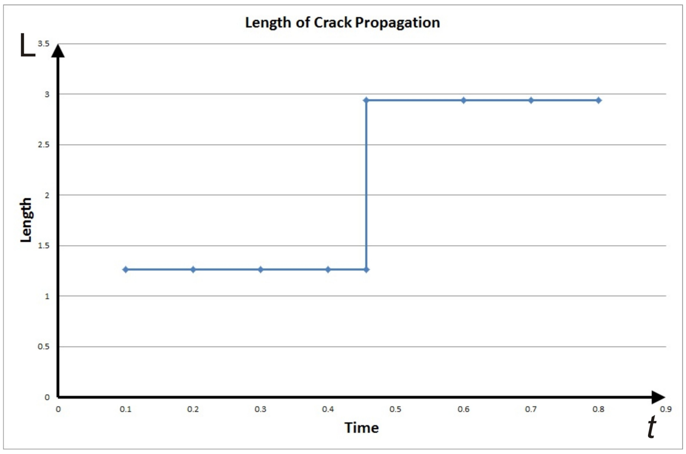

Figure 6: Length of crack propagation with $g_0 = 1$

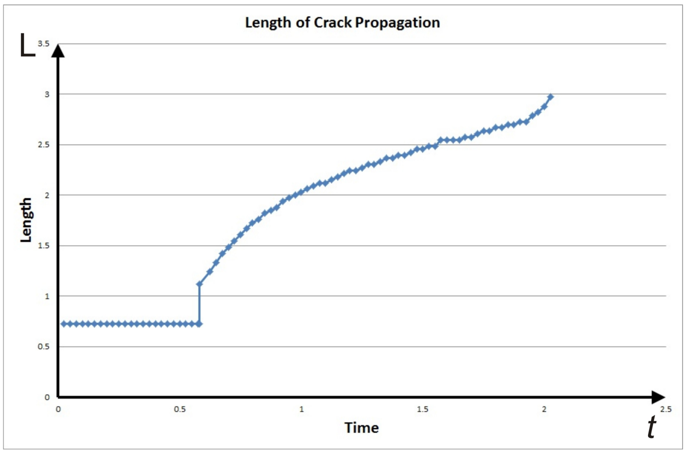

Figure 7: Length of crack propagation with $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$

We set $\gamma = 0.5$ then using relation $\mathcal{E}(L, t\ g_0) = t^2 E(L, g_0) + \gamma L$, we get graphs of the total energies as shown in Figures 4 and 5. In both cases in Figures 4 and 5, if the initial crack length L_0 is small enough as drawn by dots, the condition (4) is not reached until the end of these simulations. We remark that a solution of Problem 3.1 does not propagate too, but one of the original Francfort-Marigo model (4) does. But if we take L_0 near 0.6 as drawn by square, length suddenly jumps to the end, in one time for the case of $g_0 = 1$.

On the other hand, in the case of $g_0 = \cos^2(\frac{\pi x_1}{2L_\infty})$ there is a little jump and after that the crack starts propagating smoothly until the end of length. The obtained solutions to Problem 3.1 are shown in Figures 6 and 7. We remark that the difference of the behaviors of these solutions arises from the difference of the profiles of $E(L, g_0)$ as shown in Figure 3.

4 Conclusions and Future work

We studied energy-theoretic crack propagation models analytically and numerically in this paper. We computed the elastic energy $E(L, g_0)$ for each L by using FreeFem++. For the Francfort-Marigo model, we investigated the relations of the behaviour of a solution and the profile of the graph of $E(L, g_0)$. We pointed out that the solution can propagate with a jump even if Griffith's criterion is not satisfied. We proposed a localized Francfort-Marigo model to make it more consistent to the classical Griffith's criterion and numerically constructed some solutions. We observed that they exhibit several discontinuous behaviours but are consistent to Griffith's criterion.

The behaviour of the solutions of Problem 3.1 which were constructed numerically in this paper is expected to represent theoretical straight crack propagation in an ideal setting. They will be useful for checking the accuracy and reliability of other models, such as the phase field model. Some comparison results with Takaishi-Kimura model will be reported in our forthcoming paper.

Acknowledgment

One of the authors would like to thank MEXT scholarship of SGU for the financial support in his research.

References

[1] L. Ambrosio and V.M. Tortorelli (1992). On the approximation of free discontinuity problems. *Boll. Un. Mat. Ital.*, **7**, 6-B, 105 – 123.

[2] G.A. Francfort and J.-J. Marigo (1998). Revisiting brittle fracture as an energy minimization problem. *J. Mech. Phys. Solids*, **46**, 1319 – 1342.

[3] A.A. Griffith (1920). The phenomenon of rupture and flow in solids. *Phil. Trans. Royal Soc. London A*, **221**, 163 – 198.

[4] T. Takaishi and M. Kimura (2009). Phase field model for mode III crack growth in two dimensional elasticity. *Kybernetika*, **45**, no.4, 605 – 614.

[5] F. Hecht (2012). New development in FreeFem++. *J. Numer. Math.*, **20**, no.3-4, 251 – 265.