Shallow Water - Navier-Stokes Coupling Method in Ocean Wave Simulation

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Abstract. *Modeling ocean wave propagation using particle method faces difficulty due to its high computational cost, especially when it is computed in large domains. By coupling a cheap model in the large deep sea region and a precise model near the shoreline, we can reduce this difficulty. There had been already proposed models coupling Navier-Stokes equations with a Boussinesq model. Herein we propose to couple a finite difference Shallow Water Equation (SWE) model for the deep sea with a Smoothed Particle Hydrodynamics Navier-Stokes Equation (NSE) model close to the shoreline where the waves break and overturn. The cheap SWE model represents the large computational domain whereas the NSE model is necessary to deal with the complex behavior of the free-surface at the shoreline. We present results of simulations of water waves for a flat bottom and an inclined bottom as a representation of coastal area.*

Keywords: Coupled Model, Ocean Wave Propagation, Navier-Stokes Equations, Shallow Water Equations, Smoothed Particle Hydrodynamics, Staggered Conservative Scheme

1 Introduction

The study of ocean waves is a very important field, including significant applications in coastal engineering, such as tsunami and coastal protection. Many models and numerical methods based on grid and particle approach were proposed to solve ocean wave problems in order to predict the phenomena and prevent devastation.

However, an accurate simulation of the issues still poses a problem, especially when it is computed in a large domain. Particle-based methods can describe the wave behavior, including complex aspects such as the free surface at the shoreline where the waves break and overturn. Nevertheless, these methods still require high computational resources. On the other hand, the computational cost of grid-based methods is cheaper than that of particle methods, but these methods cannot handle the above mentioned complex behaviors.

The computational cost problem can be reduced while keeping the quality of the result by coupling a cheap model for the deep sea with a precise model near the shoreline. The coupled model has been studied recently by several authors. Coupling smoothed particle hydrodynamics (SPH) with a 1-D Boussinesq-type wave model was proposed in [1] and [9].

In this research, we propose to couple a finite difference Shallow Water Equation (SWE) model with a 2-D Smoothed Particle Hydrodynamics Navier-Stokes Equation (NSE) model. The cheap SWE model represents the large computational domain whereas the NSE model is necessary to deal with the complex behavior of the free surface at the shoreline. We use a standard SPH formulation to solve the NSE model and staggered conservative scheme to solve the SWE model.

The purpose of our research is to reduce the computational cost of simulating ocean wave propagation from deep sea by coupling a finite difference shallow water model with a smoothed

particle hydrodynamics Navier-Stokes model. We compare the computational cost of the coupled model with computational cost of the pure Navier-Stokes model.

The outline of this paper is as follows. We introduce our governing equations in section 2. Discretization of the governing equations is explained in section 3. Section 4 is about the coupled model, including coupling strategies and algorithm. In the following section, we present numerical examples of flat and inclined bottom water wave simulations as well as a comparison between the pure SWE model with the coupled model. We summarize our research in the last section.

2 Governing Equations

2.1 Navier-Stokes Equations

The Navier-Stokes equations (NSE) are the basic governing equations for incompressible fluid flow over time t,

$$\begin{cases} \nabla \cdot \mathbf{u} = \mathbf{0} \\ \rho \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \mathbf{u} = \nabla p + \mu \nabla \cdot (\nabla \mathbf{u}) + \mathbf{f}, \end{cases} \tag{1}$$

where ρ is the density of the fluid, $\mathbf{u}$ is the velocity, p is the pressure, μ is the viscosity, and $\mathbf{f}$ is the total of external forces acting on the fluid.

We consider inviscid fluid ($\mu = 0$) and assume that the external force comes only from gravity. We solve the governing equations using smoothed particle hydrodynamics, where the fluid is treated as slightly compressible. In this case we can rewrite the equations (1) as follows

$$\begin{cases} \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \\ \frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p + \mathbf{g}, \end{cases} \tag{2}$$

where $\frac{D}{Dt}$ is the material derivative and $\mathbf{g}$ is the gravitational acceleration vector. The equations (2) are also known as Euler equations.

2.2 Shallow Water Equations

The shallow water equations (SWE) are model of fluid flows below a pressure surface (it can be a free surface but it is not necessary). The SWE can be derived from the NSE under the main assumption that the horizontal length scale is much greater than the vertical length scale [6]. As the sea becomes shallower, the vertical velocity of the fluid particles becomes more and more oval-shaped and at the shallow sea level the vertical velocity is assumed to be zero. Another assumption is that the horizontal velocity is homogeneous throughout the whole fluid depth.

We consider one dimensional shallow water equations with no frictional force as follows

$$\begin{cases} \eta_t + (hu)_x = 0 \\ u_t + u u_x + g \eta_x = 0, \end{cases} \tag{3}$$

where η is the fluid level above plane of reference, u is velocity of the fluid, $h = d + \eta$ is total depth of the fluid and d is the fluid depth below plane of reference.

The first equation in the equations (3) comes from mass conservation, whereas the second equation describes momentum conservation.

3 Numerical Method

3.1 Smoothed Particle Hydrodynamics

Smoothed particle hydrodynamics (SPH) is a particle method based on integral approximation using a kernel function that approximates the delta function

$$f(x) = \int_{\Omega} f(x')W\left(x - x'\right) dx', \tag{4}$$

where $x \epsilon \Omega, W$ is a smoothing kernel function with h as width of the kernel (see [2]). The method was proposed by Lucy, Gingold and Monaghan. At first the method was used for simulating astrophysical problems [7], [11]. Nowadays SPH is applied in many fields including wave simulation. The SPH method is convenient for this kind of wave problems since it does not require solving the free boundary.

The main idea of SPH is to discretize the fluid into a finite number of points that have some physical field quantities, e.g., mass-density, pressure, etc. [5]. The movement of the points (particles), depends on its governing equation, i.e., the equations (2). In the SPH method, the approximation of the governing equations is derived using particle approximation.

Unlike grid-based methods that compute spatial derivatives by taking the difference of values at neighboring grid points, the SPH method computes influence of all particles over a certain area depending on the support of the kernel through an integral approximation weighted by the kernel function.

The choice of the kernel function is important in the SPH method, similarly to the choice of schemes in finite difference method [5]. Depending on the situation, some kernel functions can be better than other ones. There are many types of kernel functions but the most common one is the beta cubic spline kernel

$$W\left(x - x', h\right) = \beta \begin{cases} \frac{3}{2} - q^2 + \frac{1}{2}q^3 & 0 \leq q < 1 \\ \frac{1}{6}(2-q)^3 & 1 \leq q < 2 \\ 0 & \text{otherwise} \end{cases}$$

where $q = \frac{|x-x'|}{h}$, $2h$ is the kernel radius (r_{kernel}). The value of β depends on the dimension of the problem, in 2-D case $\beta = \frac{15}{7\pi h^2}$.

To change the integral interpolation into particle approximation, we rewrite the integral (4) into discrete form as follows

$$\langle f(x_i) \rangle \quad = \quad \sum_{j=1}^{N} f(x_j) W_{ij} V_j,$$

where x_i represents the position of i-th particle, $V_j = \frac{m_j}{\rho_j}$ is the volume corresponding to the particle $j, W_{ij} = W(x_i - x_j, h)$, and $\rho_i = \sum_{j=1}^{N} m_j W_{ij}$. The summation is carried out for all neighboring particles in the kernel support.

The particle transformation of the NSE equations (2) results in the set of SPH equations as follows

$$\left\langle \frac{D\rho_i}{Dt} \right\rangle = \sum_{j=1}^{N} m_j \left(\mathbf{u}_i - \mathbf{u}_j\right) \cdot \nabla_i W_{ij}$$

$$\left\langle \frac{D\mathbf{u}_i}{Dt} \right\rangle = -\sum_{j=1}^{N} m_j \left(\frac{p_i + p_j}{\rho_i \rho_j} + \Pi_{ij} \right) \nabla_i W_{ij} + \mathbf{g},$$

where Π_{ij} is artificial viscosity, which is added in order to achieve numerical stability (see [2], [10]). We use artificial viscosity in [5] as follows

$$\Pi_{ij} = \begin{cases} \frac{-a\frac{1}{2}(c_i+c_j)\mu_{ij}+b\mu_{ij}^2}{\frac{1}{2}(\rho_i+\rho_j)} & \text{if } \mathbf{x}_{ij}\cdot\mathbf{u}_{ij}<0 \\ 0 & \text{otherwise} \end{cases}$$

where c_i and c_j are sound speeds of the particle, a and b are constants (typically $b = 0$), $\mathbf{u}_{ij} = (\mathbf{u}_i - \mathbf{u}_j)$, $x_{ij} = (x_i - x_j)$, and

$$\mu_{ij} = \frac{h\mathbf{u}_{ij}\cdot\mathbf{x}_{ij}}{\mathbf{x}_{ij}^2 + (\varepsilon h)^2}$$

with ε a small number, often taken as 0.1 [8].

To update the pressure of particles, we use equation of state that was proposed by Monaghan

$$p_i = B\left(\left(\frac{\rho_i}{\rho_0}\right)^{\gamma} - 1\right),$$

where γ is a constant, often taken as $\gamma = 7$. ρ_0 is the reference density. B is a parameter determining a restriction for the maximum change of density, which is often taken as the initial pressure [2].

3.2 Staggered Conservative Scheme

We use staggered discretizations for solving the equations (3). The scheme is often used in large-scale applications due to its advantages, e.g., its efficiency in combination with semi-implicit time integration (see [3]).

For the discretizations, we follow the ideas in [3] and [4]. In particular, we discrete the time interval $(0, T)$ into N_t time steps with length Δt. The spatial domain $\Omega := (0, L)$ is discretized into N_x cells with length Δx and partition points

$$x_{\frac{1}{2}} = 0, x_1, x_{\frac{3}{2}}, ..., x_{i-\frac{1}{2}}, x_i, x_{i+\frac{1}{2}}, ...x_{Nx+\frac{1}{2}} = L$$

Moreover, we define the depth η at full grid points and the velocity u at half grid points (see Fig.1). Therefore the total depth of the water h is also defined at the full grid.

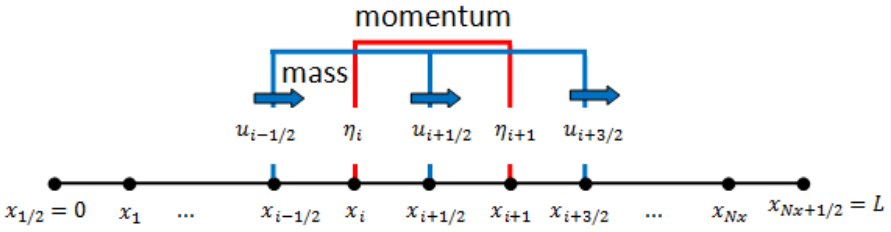

Figure 1: Staggered grid discretization scheme

Applying the scheme to the mass equation of the SWE, we obtain

$$\frac{\eta_i^{n+1}-\eta_i^n}{\Delta t}=-\left(\frac{{}^*h_{i+\frac{1}{2}}^n u_{i+\frac{1}{2}}^n - {}^* h_{i-\frac{1}{2}}^n u_{i-\frac{1}{2}}^n}{\Delta x}\right).$$

The total depth in the discretization of the mass equation appear at half grid but since we define them at full grid, we need to interpolate them. Here we denote by ${}^*h$ the missing points of h at half grid.

We approximate the missing points using first-order upwind scheme. Namely, when the wave comes from the left, we take information from the left side and when the wave comes from the right we take right-side values:

$${}^*h_{i+\frac{1}{2}}=\begin{cases} h_i & if\left(u_{i+\frac{1}{2}}^n \geq 0\right) \\ h_{i+1} & if\left(u_{i+\frac{1}{2}}^n < 0\right). \end{cases} \tag{5}$$

As for the momentum equation of the SWE, we rewrite the nonlinear part in the equations (3) as follows

$$u\frac{\partial u}{\partial x}=\frac{1}{h}\left(\frac{\partial\left(hu^2\right)}{\partial x}-u\frac{\partial\left(hu\right)}{\partial x}\right)=\frac{1}{h}\left(\frac{\partial\left(qu\right)}{\partial x}-u\frac{\partial q}{\partial x}\right).$$

Then, the discretization for the momentum equation is given as follows

$$\frac{u_{i+\frac{1}{2}}^{n+1}-u_{i+\frac{1}{2}}^n}{\Delta t}=-\frac{1}{\overline{h}_{i+\frac{1}{2}}}\left(\frac{\overline{q}_{i+1}u_{i+1}^*-\overline{q}_i u_i^*}{\Delta x}-u_{i+\frac{1}{2}}\frac{\overline{q}_{i+1}-\overline{q}_i}{\Delta x}\right)-g\left(\frac{\eta_{i+1}^{n+1}-\eta_i^{n+1}}{\Delta x}\right),$$

where

$$\overline{h}_{i+\frac{1}{2}}=\frac{h_i+h_{i+1}}{2},\quad \overline{q}_i=\frac{q_{i+\frac{1}{2}}+q_{i-\frac{1}{2}}}{2},\quad q_{i+\frac{1}{2}}={}^* h_{i+\frac{1}{2}}u_{i+\frac{1}{2}}$$

and ${}^*h$ satisfies the equation (5). u^* is the notation for missing points of u on the full grid which are approximated using first-order upwind scheme as follows

$$u_i^*=\begin{cases} u_{i-\frac{1}{2}} & if\left(\ \overline{q}_i \geq 0\right) \\ u_{i+\frac{1}{2}} & if\left(\overline{q}_i < 0\right). \end{cases}$$

Applying both discretizations of the mass and momentum equations, some oscillations will appear in the result. We add an artificial diffusion to smooth out the result:

$${}^*\eta_i^{n+1}=\left(1-\varepsilon\right)\eta_i+0.5\varepsilon\left(\eta_{i-1}+\eta_{i+1}\right) \tag{6}$$

with ε a smoothing parameter. The equation (6) is also known as first order Shapiro filter [6].

4 Coupled Model

4.1 Coupling Idea

The idea to couple the above two sets of governing equations is done by dividing the domain into two regions, SWE Region and NSE Region, with a buffer between these two regions [1], [9] (see Fig.2).

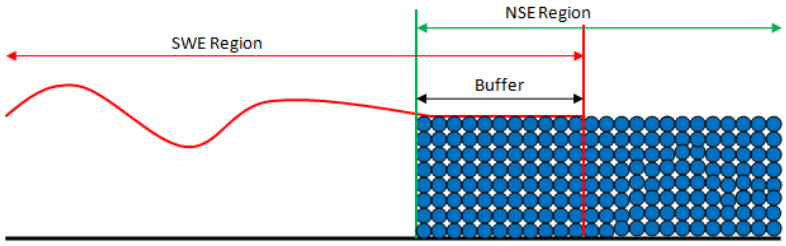

Figure 2: SWE - NSE coupling scheme

The buffer is a place for transferring information between the models. We still need to consider the way of selecting the information to be transferred between the models because of the fact that SWE model provides less information than NSE model.

Basically, we can set the whole computational domain as the SWE region but it will not be effective. Therefore, we choose the SWE region until near shoreline, so that the length of the region is larger than that of the NSE region to represent the large computational domain.

As for the NSE region, we set it up to start from close to the shoreline up to the end of the computational domain in order to deal with the complex behavior of the waves. The selection of the buffer length will be explained in the next section.

4.2 SWE to NSE Coupling

As we know, the SWE is simplified model so it has less information than the NSE model. We obtain homogeneous horizontal velocity for all depths at a given position, whereas the NSE model gives both vertical and horizontal velocity at each point of the domain.

To overcome the lack of information, we insert a wavemaker moving with a velocity given from the SWE model at the left boundary of the NSE region. The wavemaker is created from particles placed in a column as shown in Fig.3

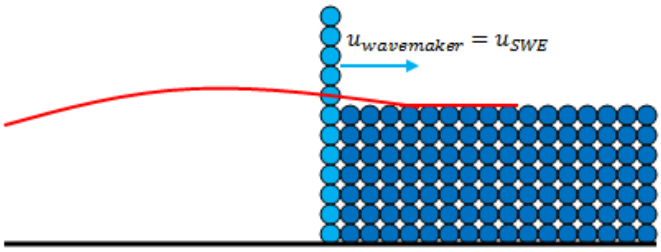

Figure 3: SWE to NSE coupling scheme

We impose the same velocity for every particle of the wavemaker since the SWE has homogeneous velocity. Moreover, the wavemaker moves only in the horizontal direction with the given velocity. In this way, the wavemaker will push the fluid particles and the influence of the SWE will propagate into the NSE through the wavemaker movement. By repeating the process, the influence will propagate over all of the NSE domain.

4.3 NSE to SWE Coupling

From the NSE model we obtain all necessary information for the SWE model. The difficulty in the feedback coupling is in reducing the information when transferring it to the SWE model. Ideally, we want to preserve mass and momentum. In this research, we do it by reconstructing wave-surface and averaging velocity over certain columns in the SPH result.

With the explicit staggered scheme, basically we need only data at one point for the feedback coupling. Hence, we prescribe the SWE boundary condition with the SPH result by taking the information at the right boundary of the SWE model as shown in the Fig.4

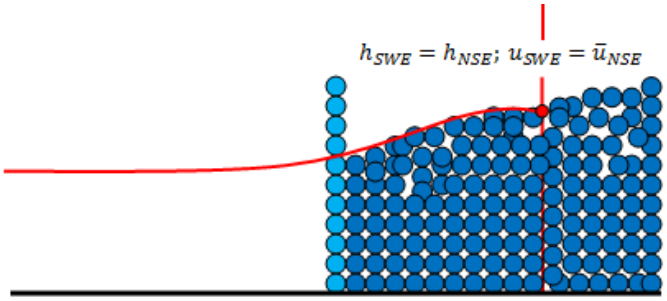

Figure 4: NSE to SWE coupling scheme where both of $\bar{u}_{\mathrm{NSE}}$ and h_{NSE} are computed by averaging particle horizontal velocity and by taking the maximum height of particles in suitable columns.

The choice of the suitable columns depends on parameters used in the SPH. When choosing the columns, we have to ensure that there are always particles in the columns. In our case, the smallest feasible width of the columns was $8\Delta x$, and we compute both $\bar{u}_{\mathrm{NSE}}$ and h_{NSE} over the columns centered on the boundary as follows

$$\begin{aligned} \bar{u}_{\mathrm{NSE}} &= \tfrac{1}{np} \sum_{k=1}^{np} u_1(k), \\ h_{\mathrm{NSE}} &= \max_{1 \le k \le np} y(k) \end{aligned}$$

where np is the number of particles in the column, u_1 is the horizontal velocity of the particles, and y is the vertical position of the particles.

The influence of the NSE model will propagate into the SWE region through the boundary condition. In the first iteration, the influence is at the boundary and the nearest grid from the boundary. By repeating the process, the influence will propagate over all of the SWE domain.

4.4 Coupling Algorithm

In the numerical computation, we have to make sure that the parameters of both models match before start the coupling simulation.

The time step for each model is chosen according to numerical stability requirements. For the SPH, when considering external force and viscous diffusion, the time step of SPH is chosen as follows (see [5])

$$\Delta t_{SPH} = \gamma \min_i \left(\Delta t_f, \Delta t_{cv} \right),$$

where $\Delta t_f = \min_i \left(\frac{h_i}{|\mathbf{f}_i|}\right)$, $\Delta t_{cv} = \min_i \left(\frac{h_i}{c_i + 0.6\left(ac_i + b\max_j \mu_{ij}\right)}\right)$, and $\gamma = 0.25$ or 0.4.
In [2], the time step is also chosen as the ratio between the smallest length of the kernel function and the maximum speed of numerical propagation

$$\Delta t_{SPH} = \min_i \left(\frac{h_i}{c}\right)$$

For the SWE staggered scheme, the CFL condition restricts the time step by the condition

$$\frac{\Delta t_{SWE}}{\Delta x}\left(\sqrt{g^* h^{n+1}_{i+\frac{1}{2}}} + \left|u^n_{i+\frac{1}{2}}\right|\right) \leq 1,$$

where ${}^*h$ satisfies the equation (5). In particular, these conditions usually mean that the SPH time step Δt_{SPH} has to be smaller than the time step for the finite difference method Δt_{SWE}. In our computations we put $\Delta t_{SPH} = 0.5\frac{h}{c}$ and select Δt_{SWE} as large as possible to save computational time.

The coupling algorithm is given as follows

Algorithm 1 Coupled the SWE - NSE algorithm

```
Input values of parameters and initial conditions for both models
for n = 1 ... N_max do
    t = nΔt_SWE
    Solve the SWE region
    Get velocity from SWE model at the SPH boundary
    for j = 1 ... N(N = Δt_SWE/Δt_SPH) do
        Update position of wavemaker
        Solve the NSE region
    Set boundary conditions for the SWE model as in section 4.3
```

5 Result and Discussion

As a numerical example, we simulated water waves on flat and inclined bottoms. The inclined bottom simulation represents coastal area. In this section, we also provide a validation of our model. We assume that the buffer length is large enough to prevent the wavemaker from going out of the buffer. In this simulation, we set the domain for the SWE region, buffer, and the NSE region as follows

$$\Omega_{SWE} := [0, 25], \Omega_{\text{buffer}} := [20, 25], \Omega_{NSE} := [20, 29].$$

Further, we set the constant of artificial viscosity $a = 0.03$, $g = 9.81$, $\Delta x = \frac{h_{\text{kernel}}}{0.9}$ and $\Delta t_{SPH} = \frac{1}{3}\Delta t_{SWE}$. These parameters are used for all cases of our numerical example.

5.1 Model Validation

First, we check the coupling idea by comparing the coupled SWE - SWE model with the pure SWE model. We divide the domain into two regions with a buffer for the coupled model and at

the buffer we transfer both depth and velocity between the models in the same way as in Algorithm 1. The results show that the coupled model and the pure model coincide, as expected.

We adopt our method to compare the coupled SWE - NSE model with pure SWE model. The results are shown in Fig.5.

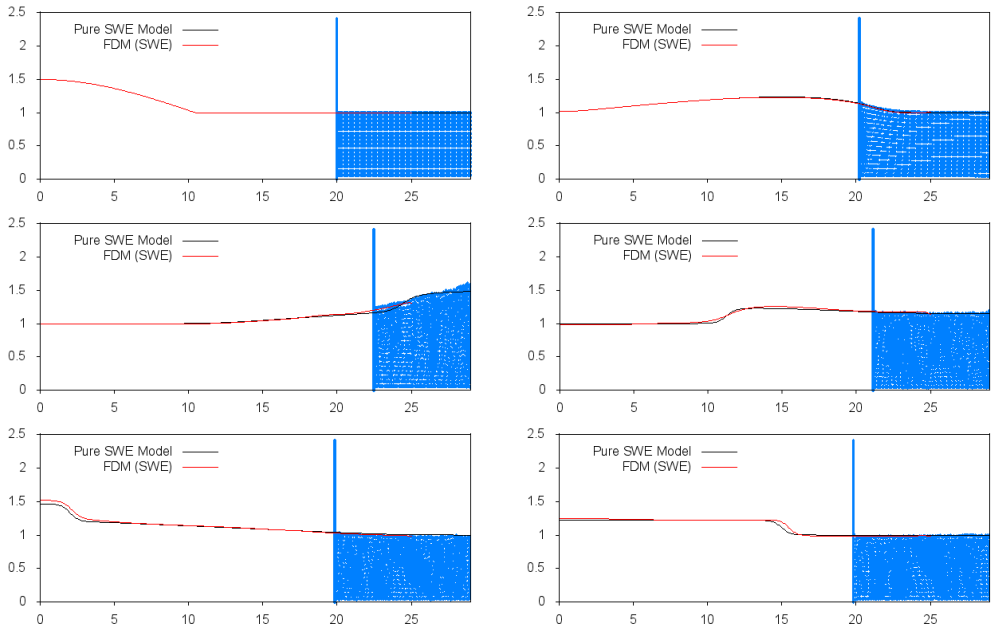

Figure 5: Comparison between the coupled model and the pure SWE model with $h_{\text{kernel}} = 0.0575$.

It is natural that result of the coupled model is different from the pure SWE model since the models are different. However, we can see that the influence of propagation among the models shifts smoothly and the waves resulting from the coupled model and the pure SWE model almost match within the buffer. Therefore, we may say that the result is good.

It is hard to say whether the coupling idea for the SWE - NSE model is correct or not, based only on the comparison method. In order to fully validate the model, it is necessary to compare its results with experimental data. In such a case, we need two-dimensional model for the SWE and three-dimensional model for the NSE.

5.2 Flat Bottom Water Waves Simulation

First, we set $h_{\text{kernel}} = 0.0270588$, obtaining the result shown in Fig.6.

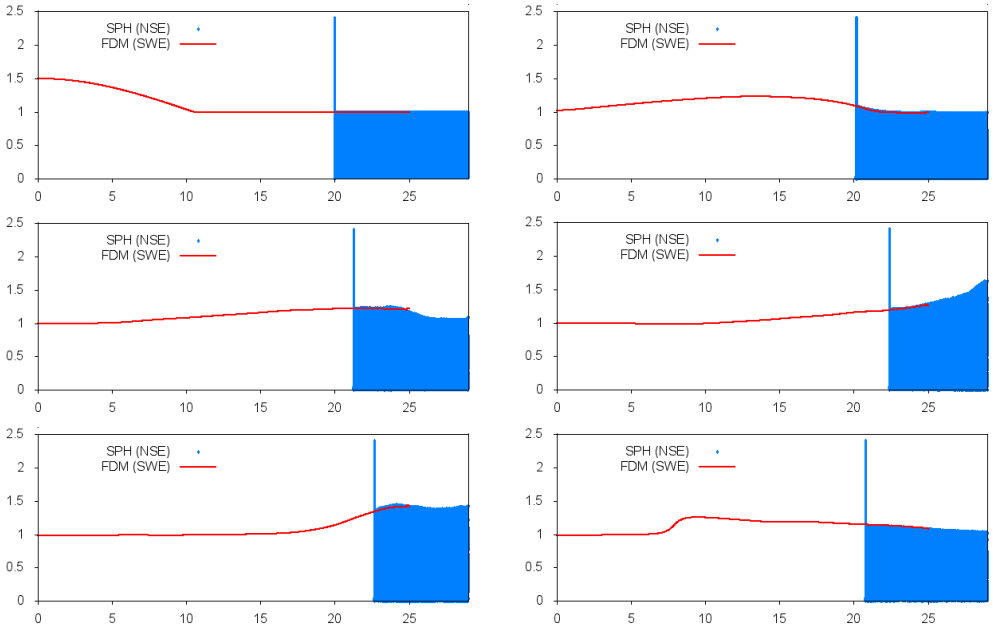

Figure 6: Wave propagation of coupled model at $t = 0, 3.49, 5.24, 6.98, 8.73, 12.23$.

Next, we increase the number of particles two times, which gives a radius of kernel function 0.714 times smaller than in the previous example. The result is shown in Fig.7.

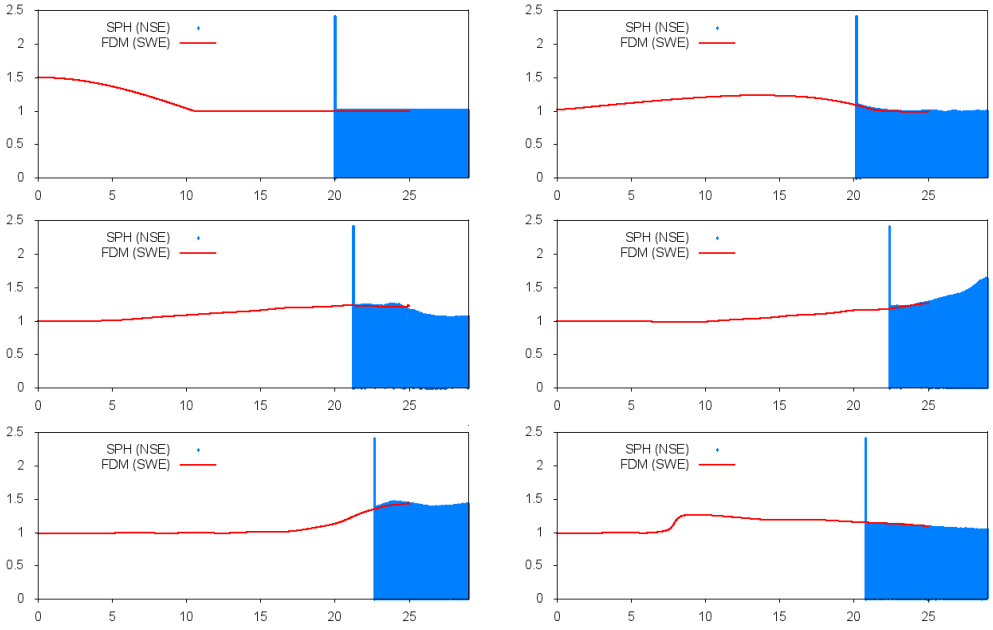

Figure 7: Wave propagation of coupled model with doubled number of particles (at the same time instants as above).

Decreasing the number of particles four times, the length of the kernel increases two times compared to the first case. The corresponding result is shown in Fig.8.

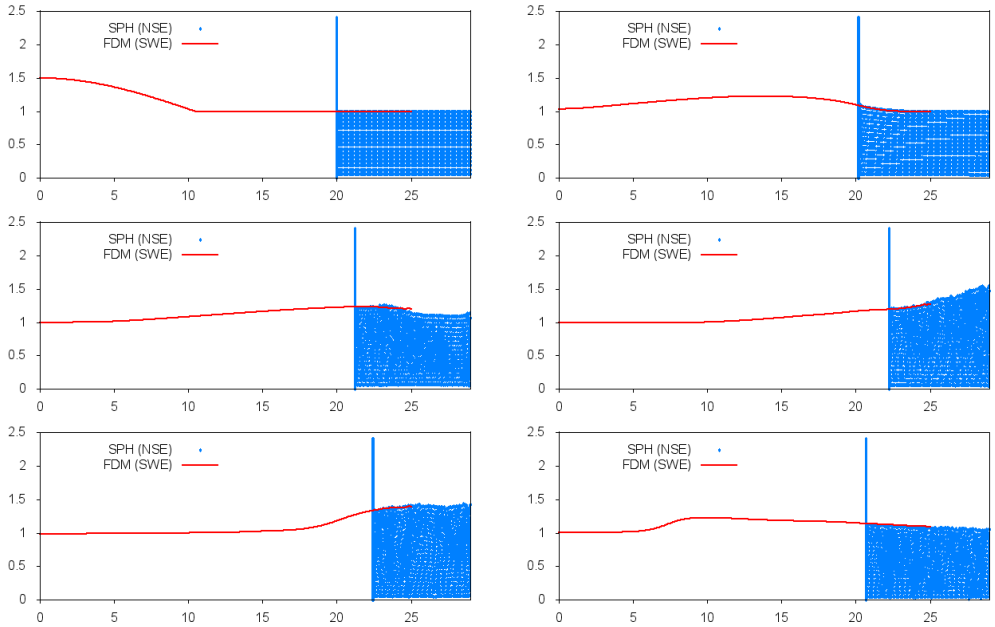

Figure 8: Wave propagation of coupled model with four times smaller number of particles (at the same time instants as above).

As we can see, the results in Fig.6, Fig.7, and Fig.8 do not essentially differ although the resolution of the SPH model was significantly changed. This confirms the correctness of the numerical results.

5.3 Inclined Bottom Water Waves Simulation

For the water wave simulation over inclined bottom, we use the same parameters as in the previous simulation with $h_{\text{kernel}} = 0.046$. The inclination starts from 26.05 with slope $\frac{1}{3}$. The results are shown in Fig.9.

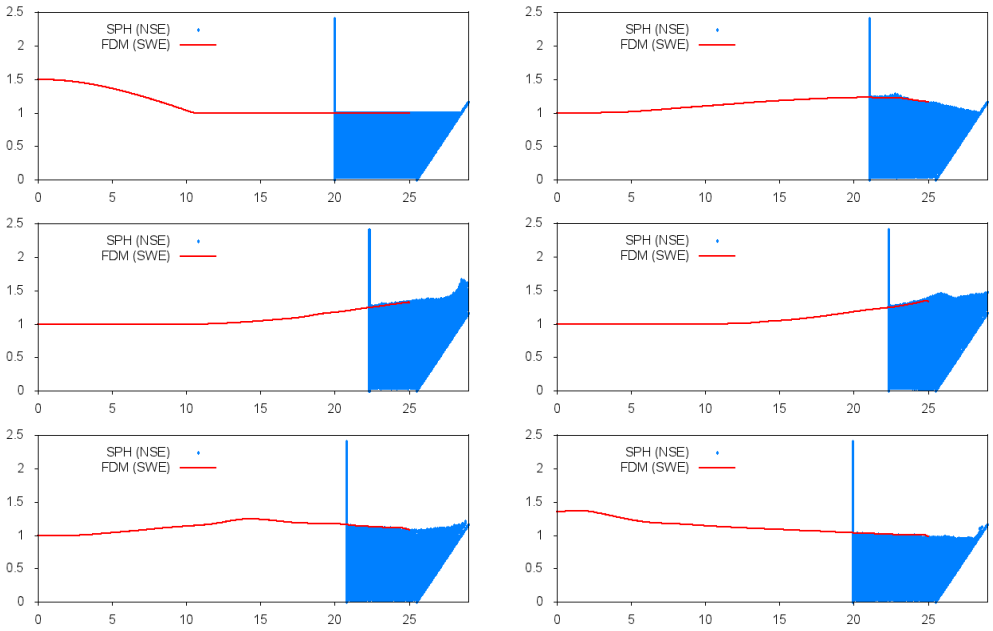

Figure 9: Wave propagation of coupled model at $t = 0, 5.05, 7.42, 8.91, 11.88, 14.85$.

6 Summary

We have presented coupled model between a finite difference Shallow Water Equation (SWE) model and a 2-D Smoothed Particle Hydrodynamics Navier-Stokes Equation (NSE) model. The coupling strategy for SWE to NSE is done by inserting a wavemaker that moves horizontally with velocity given from Shallow Water model. The NSE to SWE coupling is done by reconstructing the wave surface and by averaging velocity over certain columns in the SPH result. The computational speed for the case in Fig.8 was 2.37 times faster than for the pure NSE model, but it is expected to be much more effective for large SWE domains.

Coupling idea with inserting a wavemaker is intuitive. However, in real phenomena there is no such wavemaker that pushes water, therefore the authors would like to consider a more refined coupling idea mentioned in [1] that uses inlet/outlet boundary condition for the SPH method at the buffer for coupling from the SWE to the NSE. We keep this improvement as a future goal.

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