On the Farthest Subconstituent of the q-Johnson Graph $J_q(n,k)$

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Abstract. *The farthest subconstituent of the q-Johnson graph $J_q(n,k)$ is well-known to be isomorphic to the bilinear forms graph $M_k(q)$ in the case of $n = 2k$. This fact is generalized for $n \geq 2k$.*

Keywords: q-Johnson graph, farthest subconstituent.

1 Introduction

Let $\mathbb{F}_q$ be the finite field of q-elements and $V = \mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$ consisting of row vectors :

$$V = \{\mathbf{v} = (v_1, v_2, ..., v_n) \mid v_i \in \mathbb{F}_q, \quad (1 \leq i \leq n)\}.$$

Let X denote the set $\binom{V}{k}_q$ of k-dimensional subspace of V and define a symmetric relation $\sim$ on X by

$$U_1 \sim U_2 \iff dim(U_1 \cap U_2) = k - 1.$$

The graph $\Gamma = (X, \sim)$ is called the q-Johnson graph and denoted by $J_q(n,k)$. We may assume $k \leq \frac{n}{2}$, since $J_q(n,k)$ is isomorphic to $J_q(n, n-k)$. Thus the diameter of Γ is k:

$$k = max\{\partial(U_1, U_2) \mid U_1, U_2 \in X\},$$

where $\partial(U_1, U_2)$ is the distance between U_1 and U_2 in Γ, i.e., the length of shortest paths joining U_1 and U_2.

Note it holds that

$$\partial(U_1, U_2) = k - dim(U_1 \cap U_2).$$

The reader is referred to [1] for basic properties of $J_q(n,k)$.

Fix a base vertex $U_0 \in X$ and define the i-th subconstituent $\Gamma_i(U_0)$ by

$$\Gamma_i(U_0) = \{U \in X \mid \partial(U_0, U) = i\}.$$

We call $\Gamma_k(U_0)$ the farthest subconstituent.

In the case of $n = 2k$, it is well-known that there is a bijection ϕ from $\Gamma_k(U_0)$ to the set $M_k(q)$ of $k \times k$ matrices over $\mathbb{F}_q$ such that $U_1, U_2 \in \Gamma_k(U_0)$ are adjacent if and only if $\phi(U_1) - \phi(U_2)$ has rank 1. In other words, the farthest subconstituent $\Gamma_k(U_0)$ is isomorphic to the bilinear forms graph

$M_k(q)$. This paper aims to generalize this fact for $n \geq 2k$.

The general linear group $GL(n, q)$ acts on $X = \binom{V}{k}_q$ from the right naturally as a group of graph automorphisms of $J_q(n, k)$. This action is distance-transitive, namely, $GL(n, q)$ acts on X transitively and the stabilizer of U_0 in $GL(n, q)$ acts on each $\Gamma_i(U_0)$ transitively $(0 \leq i \leq k)$. From this point of view, $J_q(n, k)$ can be regarded as an association scheme (the q-Johnson scheme) rather than a graph, and in the case of $n = 2k$, the farthest subconstituent $\Gamma_k(U_0)$ is isomorphic to the bilinear forms scheme $M_k(q)$ as an association scheme. The papers [2], [3] treat $\Gamma_k(U_0)$ in the general case as an association scheme and determine the parameters. We note that the problem dealt with in this paper is different from the one in [2], [3].

2 The Standard Basis of a Subspace of $V = \mathbb{F}_q^n$

For a vector $\mathbf{v} = (v_1, v_2, ..., v_n) \in V = \mathbb{F}_q^n$, we denote the j-th entry v_j of $\mathbf{v}$ by $\mathbf{v}(j)$; $\mathbf{v}(j) = v_j$. Set

$$h(\mathbf{v}) = min\{j \mid \mathbf{v}(j) \neq 0, \quad 1 \leq j \leq n\}$$

and call $h(\mathbf{v})$ the <u>head</u> of $v \in V$.

Let U be a subspace of V. Then there exists a basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_t$ for U $(t = dim(U))$ such that

$$h(\mathbf{v}_1) < h(\mathbf{v}_2) < ... < h(\mathbf{v}_t). \tag{1}$$

We may assume that for $\nu = h(\mathbf{v}_j) \quad (1 \leq j \leq t)$,

$$\mathbf{v}_i(\nu) = \delta_{ij} \tag{2}$$

where δ_{ij} is the Kronecker delta, i.e., $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. A basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_t$ of U is called <u>standard</u> if it satisfies (1), (2). It is easy to see that a standard basis exists uniquely for each subspace of V.

For the standard basis $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_t$ of U, we set

$$supp(U) = \{h(\mathbf{v}_1), h(\mathbf{v}_2), ..., h(\mathbf{v}_t)\}$$

and call it the <u>support</u> of U.

Let $M_{k\times n}(q)$ denote the set of $k \times n$ matrices over $\mathbb{F}_q$. For a matrix $A \in M_{k\times n}(q)$, we denote the i-th row of A by A_i and the (i, j)-entry of A by $A_i(j)$. The subspace spanned by $A_1, A_2, ..., A_k$ is denoted by $row(A)$.

A matrix $E \in M_{k\times n}(q)$ is said to be in echelon form if $E_1, E_2, ..., E_t$ form the standard basis for $row(E)$ and $E_i = (0, ..., 0)$ holds for $t + 1 \leq i \leq k$. For a matrix E in echelon form, we set

$$supp(E) = \{h(E_1), ..., h(E_t)\},$$

where $t = rank(E)$, and call it the support of E. Obviously, $supp(E)$ coincides with $supp(row(E))$.

Let $M^t_{k\times n}(q)$ denote the subset of $M_{k\times n}(q)$ consisting matrices E such that E is in echelon form with $|supp(E)| = t$. Let $\binom{V}{t}_q$ denote the set of t-dimensional subspace of $V = \mathbb{F}_q^n$. The following lemma is an elementary fact of linear algebra.

Lemma 1.

(i) The following mapping is a bijection :

$$M^t_{k\times n}(q) \longrightarrow \binom{V}{t}_q \quad (E \mapsto row(E))$$

(ii) Let $E, F \in M_{k\times n}(q)$ be in echelon form. If $row(E) \supset row(F)$, then $supp(E) \supset supp(F)$.

Set $Y = M^k_{k\times n}(q)$. Then Y is bijectively mapped onto $X = \binom{V}{k}_q$ by sending $E \in Y$ to $row(E) \in X$.

3 The Farthest Subconstituent of $J_q(n,k)$

We keep the notation of the previous sections. So $X = \binom{V}{k}_q$, $Y = M^k_{k\times n}(q)$, and there is a natural bijection between X and Y. When considering the farthest subconstituent $\Gamma_k(U_0)$, we may choose the base vertex U_0 arbitrarily without loss of generality, since $GL(n,q)$ acts on X transitively as a group of graph automorphisms. We set

$$U_0 = \{\mathbf{v} \in V \mid \mathbf{v}(1) = \mathbf{v}(2) = ... = \mathbf{v}(n-k) = 0\}.$$

So $supp(U_0) = \{n-k+1, n-k+2, ..., n\}$ and the corresponding matrix in echelon form is $[O\ I]$, where O is the zero matrix of size $k \times n-k$ and I is the identity matrix of size k.

We observe that for $E \in Y$, $row(E)$ belongs to $\Gamma_k(U_0)$, i.e., $dim(U \cap U_0) = 0 \quad (U = row(E))$ if and only if

$$supp(E) \subseteq \{1, 2, ..., n-k\} \tag{3}$$

So the following proposition holds.

Proposition 1. *Set*

$$Y_k = \{E \in Y \mid E \textit{ satisfies } (3)\}.$$

Then the following mapping is a bijection :

$$Y_k \longrightarrow \Gamma_k(U_0) \quad (E \mapsto row(E)).$$

For $E, E' \in Y_k$, we set $U = row(E),\ U' = row(E')$ and ask when $U \sim U'$ holds in $\Gamma_k(U_0)$.

Suppose $U \sim U'$. Then $dim(U \cap U_0) = k-1$, so there exist $\alpha, \beta \notin supp(U \cap U')$ such that

$$supp(U) = supp(U \cap U') \cup \{\alpha\}, \tag{4}$$

$$supp(U') = supp(U \cap U') \cup \{\beta\}. \tag{5}$$

In view of $supp(U) = supp(E), supp(U') = supp(E')$, we set

$$supp(E) = \{i_1, i_2, ..., i_k\} \text{ with } \alpha = i_r, \tag{6}$$

$$supp(E') = \{i'_1, i'_2, ..., i'_k\} \text{ with } \beta = i'_s, \tag{7}$$

where $i_1 < i_2 < ... < i_k$, and $i'_1 < i'_2 < ... < i'_k$.

By symmetry, we may assume

$$\alpha \leq \beta. \tag{8}$$

Then we have

$$\begin{aligned} supp(U \cap U') &= \{i_1, ..., i_{r-1}, i_{r+1}, ..., i_k\} \\ &= \{i'_1, ..., i'_{s-1}, i'_{s+1}, ..., i'_k\} \end{aligned} \tag{9}$$

with $r \leq s$ and

$$i'_j = \begin{cases} i_j & \text{if } 1 \leq j \leq r-1, \quad s+1 \leq j \leq k, \\ i_{j+1} & \text{if } r \leq j \leq s-1. \end{cases} \tag{10}$$

Since $U = row(E)$, the standard basis of U is $E_1, E_2, ..., E_k$ and $h(E_j) = i_j$. So the standard basis of $supp(U \cap U')$ consists of

$$E_j + x_j E_r \quad (1 \leq j \leq r-1) \tag{11}$$

and

$$E_j \quad (r+1 \leq j \leq k) \tag{12}$$

for some $x_j \in \mathbb{F}_q \quad (1 \leq j \leq r-1)$.

Similiarly since $E'_1, E'_2, ..., E'_k$ form the standard basis of U' and $h(E'_j) = i'_j$, the standard basis of $supp(U \cap U')$ consists of

$$E'_j + x'_j E'_r \quad (1 \leq j \leq s-1) \tag{13}$$

and

$$E'_j \quad (s+1 \leq j \leq k) \tag{14}$$

for some $x'_j \in \mathbb{F}_q \quad (1 \leq j \leq s-1)$.

Since $U \cap U'$ has a unique standard basis, we have the following equations :

$$E_j + x_j E_r = E'_j + x'_j E'_s \quad (1 \leq j \leq r-1) \tag{15}$$

$$E_{j+1} = E'_j + x'_j E'_s \quad (r \leq j \leq s-1) \tag{16}$$

$$E_j = E'_j \quad (s+1 \leq j \leq k) \tag{17}$$

Conversely, if E, E' are distinct elements of Y_k and satisfy the condition (6), (7), (8) for their supports and the equations (15), (16), (17) for some $x_j \in \mathbb{F}_q \ (1 \leq j \leq r-1)$, $x'_j \in \mathbb{F}_q \ (1 \leq j \leq s-1)$, then $dim(U \cap U') = k-1$ holds, i.e., $U \sim U'$, where $U = row(E),\ U' = row(E')$.

We treat the case of $\alpha = \beta$ first. If $\alpha = \beta$, then $supp(E) = supp(E')$ and $r = s$. Since $i_j = i'_j \ (1 \le j \le k)$ and $\alpha = \beta = i_r$, we have $E_j(\alpha) = E'_j(\alpha) = 0 \quad (j \ne r)$ and $E_r(\alpha) = E'_r(\alpha) = 1$. So we have $x_j = x'_j (1 \le j \le r-1)$ from (15). The equation (15) becomes

$$E_j - E'_j = x_j(E'_r - E_r) \ \ (1 \le j \le r-1).$$

The equation (16) is empty. Thus we have the following theorem.

Theorem 1. *For $E, E' \in Y_k$, assume $E \ne E'$ and $supp(E) = supp(E')$. Set $U = row(E), U' = row(E')$. Then $U \sim U'$ in $\Gamma_k(U_0)$, i.e., $dim(U \cap U') = k-1$ if and only if $rank(E - E') = 1$.*

In the case of $n = 2k$, the assumption of $supp(E) = supp(E')$ always holds in the above theorem, since the support of every element of Y_k is $\{1, 2, ..., k\}$.

We now treat the case of $\alpha < \beta$. In this case, $r < s$ holds in (9). We want to solve the equations (15), (16) under the conditions (6), (7), (10).

Since $\alpha = i_r$, we have $E_j(\alpha) = 0 \quad (j \ne r)$ and $E_r(\alpha) = 1$. Since $h(E'_s) = i'_s = \beta > \alpha$, we have $E'_s(\alpha) = 0$. From (15), we get

$$x_j = E'_j(\alpha) \quad (1 \le j \le r-1) \tag{18}$$

Since $\beta = i'_s$, we have $E'_j(\beta) = 0 \ \ (j \ne s)$ and $E'_s(\beta) = 1$. From (15), we get

$$x'_j = E_j(\beta) + x_j E_r(\beta) \ \ (1 \le j \le r-1). \tag{19}$$

By (16) we get

$$x'_j = E_{j+1}(\beta) \ \ (r \le j \le s-1). \tag{20}$$

Thus we have the following conclusion.

Proposition 2. *Let $E, E' \in Y_k$ and set $U = row(E), U' = row(E')$. If $U \sim U'$ in $\Gamma_k(U_0)$, i.e., $dim(U \cap U') = k-1$, then either $supp(E) = supp(E')$ or $|supp(E) \cap supp(E')| = k-1$.*

Theorem 2. *Fix $E \in Y_k$ arbitrarily and set $U = row(E)$.*

(i.) Pick $E' \in Y_k$ that satisfies $|supp(E) \cap supp(E')| = k-1$ and set $U' = row(E')$. Define α, β, r, s by $supp(E) = (supp(E) \cap supp(E')) \cup \{\alpha\}$, $supp(E') = (supp(E) \cap supp(E')) \cup \{\beta\}$, $\alpha = h(E_r)$, $\beta = h(E'_s)$. Assume $U \sim U'$ in $\Gamma_k(U_0)$, i.e., $dim(U \cap U') = k-1$. If $\alpha < \beta$, then (15), (16), (17) hold for $x_j \quad (1 \le j \le r-1)$ in (18) and $x'_j \quad (1 \le j \le s-1)$ in (19), (20).

(ii.) Conversely, for arbitrarily chosen $r, s \in \mathbb{Z} \quad (1 \le r < s \le k)$ and $x_j \in \mathbb{F}_q \quad (1 \le j \le r-1)$, define $x'_j \quad (1 \le j \le s-1)$ by (19), (20). Set $\alpha = h(E_r)$. Choose an arbitrary vector $\mathbf{v} \in V = \mathbb{F}_q^n$ such that $h(E_s) < h(\mathbf{v}) < h(E_{s+1})$ and $\mathbf{v}(\beta) = 1 \quad (\beta = h(\mathbf{v}))$, $\mathbf{v}(\nu) = 0 \quad (\nu \in supp(E), \nu \ne \alpha = h(E_r))$. Let E' be a $k \times n$ matrix over $\mathbb{F}_q$ such that $E'_s = \mathbf{v}$ and $E'_j \quad (j \ne s)$ are given by (15), (16), (17). Then $E' \in Y_k$ and $U \sim U'$ in $\Gamma_k(U_0)$, where $U' = row(E')$.

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