Multiphase Mean Curvature Flow: Signed Distance Vector Approach

Rhudaina Z. Mohammad

Graduate School of Natural Science and Technology, Kanazawa University
Kakuma-machi, Kanazawa 920-1192 Japan
rhudaina@polaris.s.kanazawa-u.ac.jp

Abstract. *We introduce an algorithm for approximating mean curvature motion of interfaces separating multiple phases – a modification of the Merriman-Bence-Osher (MBO) thresholding method. We extend Esedoglu-Ruuth-Tsai's redistancing scheme to the multiphase case by adopting a variational method using reference vectors. We look at its efficiency in overcoming the well-known MBO time and grid restrictions on non-adaptive meshes.*

Keywords: Multiphase flow, mean curvature, vector-valued signed distance, variational method

1 Introduction

Mean curvature flow has a wide variety of applications, e.g. bubble simulation [14], grain growth [11, 5], image selective smoothing, edge detection, and so on in image processing (see, e.g. [1, 4]). Such motions can be quite challenging to approximate especially in the multiphase case, where junctions can occur. To consider such applications, a number of numerical methods, such as the front tracking method [3] explicitly approximate mean curvature motions. Unfortunately, such methods cannot handle complicated topological changes and are quite impractical to implement.

In 1994, Merriman, Bence, and Osher (MBO) [10] introduced an implicit scheme for realizing interfacial motions by mean curvature. In this scheme, the interface is defined as a boundary of some compact set which evolves through time by alternating two steps: Diffusion and Thresholding. This method has been extensively shown to converge to motion by mean curvature in the two-phase case [7, 2, 8]. Moreover, the MBO method can also be extented to multiphase case where the characteristic function of each phase region is diffused and sharpened separately [13, 10].

Despite its implicit nature allowing it to naturally handle complicated topological changes, it is also well-known that MBO algorithm suffers from time and grid restrictions on nonadaptive meshes [10]. Several modifications to the MBO method have been proposed to address this issue. Esedoglu, et.al. [6], for instance, replaced the thresholding step by a redistancing scheme. Unlike characteristic functions, signed distance functions provide subgrid accuracies on a uniform mesh which allows one to accurately locate the interface. Svadlenka, et.al. [14], on the other hand, retained the thresholding step but instead, keeps a record of the interface geometry. Another important feature of this modification is the variational approach to approximating multiphase mean curvature flow using reference vectors.

In this paper, we introduce a multiphase MBO-variant algorithm by adopting the variational method and constructing a signed distance vector-valued function as an extension of the redistancing scheme to the multiphase case. The idea is to address MBO restrictions on a multiphase case using signed distances instead of keeping information.

2 Construction of Signed Distance Vector

In this section, we construct the signed distance vector definition and formally state the algorithm for realizing multiphase mean curvature flow. We also show that interfaces evolving via such method, thus, indeed move by mean curvature.

2.1 Signed Distance Vector

Consider a domain $\Omega \subseteq R^N$, divided into k phases $P_1, P_2, \ldots, P_k$. Select a point $x \in \Omega$. Without loss of generality, say $x \in P_i$ with P_j as the nearest phase to x. Denote the interface between phase P_i and P_j by $\gamma_{ij} = \gamma_{ji}$. As in [14], we set up the reference vectors $\mathbf{p}_i$ corresponding to each phase P_i as vectors of dimension $k-1$ pointing from the centroid of a standard k-simplex to its vertices. Denote θ as the angle between any pair of reference vectors (depends only on the number of phases).

Let $d_1 : \mathbb{R}^N \to \mathbb{R}^+ \cup \{0\}$ be the distance of point x to the nearest interface, defined by:

$$d_1(x) := \min_{i \neq n} \operatorname{dist}(x, \gamma_{in}) = \operatorname{dist}(x, \gamma_{ij}),$$

and $s : \mathbb{R}^N \to \mathbb{R}^+ \cup \{0\}$, be the distance to the nearest junction point

$$s(x) := \min \{\operatorname{dist}(x, z) : z \in T_{ij}\},$$

where T_{ij} is the set of all junction points on interface γ_{ij}.

Definition 3.1. *We define the* ***signed distance vector*** $\boldsymbol{\delta} : \mathbb{R}^N \to \mathbb{R}^{k-1}$, *as follows:*

$$\boldsymbol{\delta}(x) := s(x)\boldsymbol{rot}_{ji}\left(\frac{\mathbf{p}_i + \mathbf{p}_j}{|\mathbf{p}_i + \mathbf{p}_j|}, \frac{d_1(x)}{s(x)}\frac{\theta}{2}\right).$$

Here, $\boldsymbol{rot}_{ji}(\mathbf{v}, \alpha)$ *means rotating vector* $\mathbf{v}$ *by angle* α *in the direction of rotation from reference vector* $\mathbf{p}_j$ *to reference vector* $\mathbf{p}_i$.

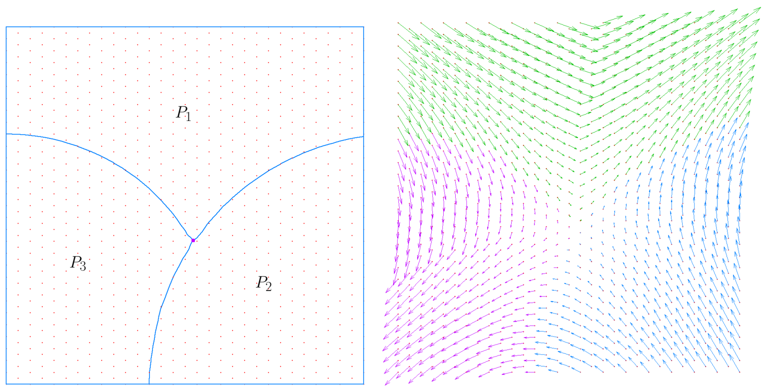


Figure 1: Three-phase initial condition (left) and its corresponding signed distance vector field (right).

From this construction, we see that $\boldsymbol{\delta}(x) = \mathbf{0}$ when x is a junction point; and

$$\boldsymbol{\delta}(x) = s(x)\frac{\mathbf{p}_i + \mathbf{p}_j}{|\mathbf{p}_i + \mathbf{p}_j|}$$

when point x lies on interface γ_{ij}.

Proposition 3.2. *If $x \in P_i$ with γ_{ij} as its nearest interface, then,*

$$\boldsymbol{\delta}(x) = s(x)\left[\cos\alpha(x)\frac{\mathbf{p}_i + \mathbf{p}_j}{|\mathbf{p}_i + \mathbf{p}_j|} + \sin\alpha(x)\frac{\mathbf{p}_i - \mathbf{p}_j}{|\mathbf{p}_i - \mathbf{p}_j|}\right],$$

where $\alpha(x) := \dfrac{d_1(x)}{s(x)}\dfrac{\theta}{2}$.

Proof. Let $\boldsymbol{\delta}(x) = s(x)\mathbf{r}$, where $\mathbf{r} := \mathbf{rot}_{ji}(\mathbf{v}, \alpha)$. Write $\mathbf{r} = a\mathbf{v} + b\mathbf{p}_i$ for some $a, b \in \mathbb{R}$. Since

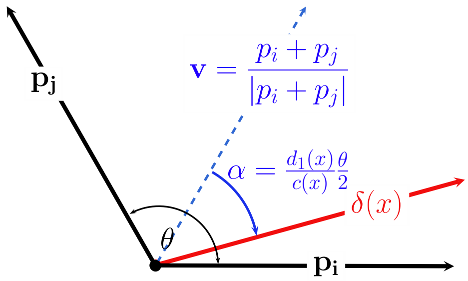


Figure 2: Construction of Signed Distance Vector

$\mathbf{p}_i \cdot \mathbf{p}_j = \cos\theta$, then taking the dot product $\mathbf{r}\cdot\mathbf{v}$ gives $\cos\alpha = a + b\cos\frac{\theta}{2}$. Hence, $a = \cos\alpha - b\cos\frac{\theta}{2}$. On the other hand, taking the dot product $\mathbf{r}\cdot\mathbf{p}_i$ gives

$$\cos\left(\tfrac{\theta}{2} - \alpha\right) = a\cos\tfrac{\theta}{2} + b = \left(\cos\alpha - b\cos\tfrac{\theta}{2}\right)\cos\tfrac{\theta}{2} + b,$$

which gives $b = \dfrac{\sin\alpha}{\sin\frac{\theta}{2}}$. Thus, we have

$$\begin{aligned}
\mathbf{r} &= \left(\cos\alpha - b\cos\tfrac{\theta}{2}\right)\frac{\mathbf{p}_i + \mathbf{p}_j}{|\mathbf{p}_i + \mathbf{p}_j|} + b\mathbf{p}_i \\
&= \left(\frac{\cos\alpha}{|\mathbf{p}_i + \mathbf{p}_j|} + \frac{b}{2}\right)\mathbf{p}_i + \left(\frac{\cos\alpha}{|\mathbf{p}_i + \mathbf{p}_j|} - \frac{b}{2}\right)\mathbf{p}_j \\
&= \cos\alpha\frac{\mathbf{p}_i + \mathbf{p}_j}{|\mathbf{p}_i + \mathbf{p}_j|} + \sin\alpha\frac{\mathbf{p}_i - \mathbf{p}_j}{|\mathbf{p}_i - \mathbf{p}_j|}.
\end{aligned}$$

□

We remark that for any point $x \in \mathbb{R}^N$ having $\gamma_{ij}(i, j \in \{1, 2, \ldots, k\})$ as its nearest interface, we have,

$$\boldsymbol{\delta}(x)\cdot(\mathbf{p}_i - \mathbf{p}_j) = \begin{cases} \sqrt{\frac{2(k-2)}{k-1}}\,s(x)\sin\alpha(x), & x \in P_i \\ -\sqrt{\frac{2(k-2)}{k-1}}\,s(x)\sin\alpha(x), & x \in P_j. \end{cases} \tag{1}$$

Meanwhile, if $x \in P_m \subset \mathbb{R}^N$ with $\gamma_{mn}(m, n \in \{1, 2, \ldots, k\})$ as its nearest interface, then we can find a constant $C > 0$ such that

$$|\boldsymbol{\delta}(x) \cdot (\mathbf{p}_i - \mathbf{p}_j)| \leq C|s(x)|. \tag{2}$$

In particular, $C = \sqrt{\frac{2k}{k-1}}$.

2.2 The Algorithm

Let $\Gamma := \bigcup\{\gamma_{ij} : i, j = 1, 2, \ldots, k\}$ be the interface to be evolved by mean curvature. Given a time step $\Delta t > 0$, we generate a sequence of time discrete evolutions $\{\Gamma_n\}$ at time $t = n\Delta t$. We obtain Γ_{n+1} from Γ_n, as follows:

1. For all $x \in \Omega$, set $\mathbf{u}_0 := \boldsymbol{\delta}(x)$ with respect to Γ_n.

2. Solve the vector-valued heat equation with initial condition $\mathbf{u}_0$:

$$\begin{cases} \mathbf{u}_t(t,x) & = & \Delta \mathbf{u}(t,x) & \text{on } (0, \Delta t] \times \Omega, \\ \dfrac{\partial \mathbf{u}}{\partial \boldsymbol{\eta}}(t,x) & = & 0 & \text{on } (0, \Delta t] \times \partial\Omega, \\ \mathbf{u}(0,x) & = & \mathbf{u}_0(x) & \text{in } \Omega. \end{cases} \tag{3}$$

3. For all $x \in \Omega$, identify the reference vector $\mathbf{p}_i$ which is closest to the solution $\mathbf{u}(\Delta t, x)$:

$$\mathbf{p}_i \cdot \mathbf{u}(\Delta t, x) = \max_{j=1,2,\ldots,k} \mathbf{p}_j \cdot \mathbf{u}(\Delta t, x)$$

   This redistribution of reference vectors determines the approximate new phase regions after time Δt, which in turn, defines the new interface Γ_{n+1}.

2.3 Velocity of the Interface

Next, we estimate the "normal velocity" of the interface evolving according to the signed distance vector scheme.

Theorem 3.3. *The velocity of the interface γ at any point $x \in \gamma \subset \mathbb{R}^N$ is*

$$v = -\kappa + O(\sqrt{t}) \quad \text{as time } t \to 0,$$

where κ is $(N-1)$-times the mean curvature of γ at x with respect to the outer normal $\boldsymbol{\eta}$.

Proof. For simplicity, consider $N = 2$. Fix a point $x \in \mathbb{R}^2$ on the interface. Without loss of generality, assume $x \in \gamma_{ij}$. Now, rotate and translate the coordinate system so that $x = 0$ in the new coordinate system and its outer normal $\boldsymbol{\eta}$ lies in the positive x_2 direction.

Fix $\epsilon > 0$, small enough so that $Q_\epsilon := [-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$ contains only two phases: P_i and P_j. Further, assume that there exists a smooth function f whose graph $(x_1, f(x_1))$ describes the interface γ_{ij} in Q_ϵ. Denote $H(x_1, f(x_1))$ as the curvature of interface γ_{ij} at $(x, f(x))$. Thus,

$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 0 \\ f''(0) &= -H(0) := -\kappa \end{aligned}$$

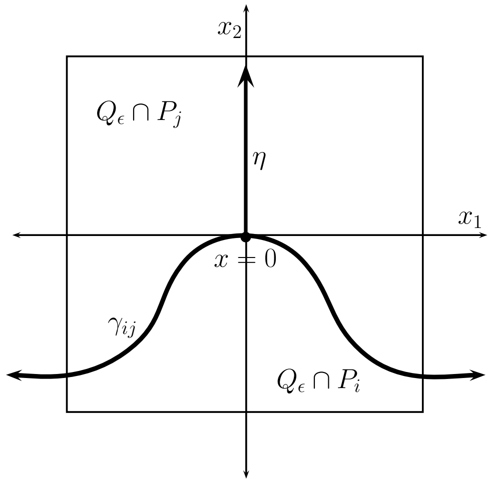


Figure 3: Setting up interface γ_{ij} in the new coordinate system.

Then, the normal velocity v of interface γ_{ij} can be found from the relation

$$\mathbf{u}(t, 0, vt) \cdot (\mathbf{p}_i - \mathbf{p}_j) = 0,$$

where $\mathbf{u}$ solves the vector-type heat equation:

$$\begin{cases} \mathbf{u}_t(t, x) &= \Delta \mathbf{u}(t, x) \quad \text{in } (0, \infty) \times \mathbb{R}^N, \\ \mathbf{u}(0, x) &= \boldsymbol{\delta}(x) \quad\quad\; \text{on } \{t = 0\} \times \mathbb{R}^N. \end{cases}$$

Hence,

$$0 \;=\; \frac{1}{4\pi t} \int_{Q_\epsilon} + \int_{\mathbb{R}^2 \setminus Q_\epsilon} \boldsymbol{\delta}(x) \cdot (\mathbf{p}_i - \mathbf{p}_j) e^{-\frac{|x - (0, vt)|^2}{4t}} dx := I + II$$

By equation (2) and the fact that $|s(x)| \leq |x| + M$ for some $M > 0$, we get

$$\begin{aligned} |II| &\leq \frac{1}{4\pi t} \int_{\mathbb{R}^2 \setminus Q_\epsilon} |\boldsymbol{\delta}(x) \cdot (\mathbf{p}_i - \mathbf{p}_j)| e^{-\frac{|x - (0, vt)|^2}{4t}} dx \\ &\leq \frac{C}{4\pi t} \int_{\mathbb{R}} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} + \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} \int_{\mathbb{R}} (|x_1| + |x_2| + M)\, e^{-\frac{x_1^2 + (x_2 - vt)^2}{4t}} dx_2 dx_1 \end{aligned} \tag{4}$$

Note that for $n = 1, 2, \ldots$, we have the following estimates

$$\left| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} x_1^n e^{-\frac{x_1^2}{4t}} dx_1 \right| \leq \frac{2}{\sqrt{4\pi t}} \int_\epsilon^\infty |x_1|^n \, e^{-\frac{x_1^2}{4t}} dx_1 \leq C\sqrt{t} e^{-\frac{\alpha}{t}} \tag{5}$$

for some $\alpha > 0$, and

$$\left| \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R} \setminus (-\epsilon, \epsilon)} x_2^n e^{-\frac{(x_2 - vt)^2}{4t}} dx_2 \right| \leq \frac{2}{\sqrt{4\pi t}} \int_\epsilon^\infty |x_2|^n \, e^{-\frac{(x_2 - vt)^2}{4t}} dx_2 \leq C\sqrt{t} e^{-\frac{\alpha}{t}}, \tag{6}$$

for some $\alpha > 0$. Using these estimates in equation (4) gives

$$|II| \leq C\left[2e^{-\frac{\alpha}{t}}\left(\sqrt{t}+t\right)+Me^{-\frac{\alpha}{t}}\left(\int_{\mathbb{R}} e^{-\frac{x_1^2}{4t}}dx_1+\int_{\mathbb{R}} e^{-\frac{(x_2-vt)^2}{4t}}dx_2\right)\right] \leq Ce^{-\frac{\alpha}{t}}, \qquad (7)$$

for some $\alpha > 0$.

On the other hand, by equation (1), it follows that for some $C > 0$,

$$\begin{aligned} I &:= \frac{1}{4\pi t}\int_{Q_\epsilon\cap P_i}+\int_{Q_\epsilon\cap P_j}\boldsymbol{\delta}(x)\cdot(\mathbf{p}_i-\mathbf{p}_j)e^{-\frac{|x-(0,vt)|^2}{4t}}dx \\ &= \frac{C}{4\pi t}\int_{Q_\epsilon} s(x)\sin\left(\frac{d(x)}{s(x)}\frac{\theta}{2}\right)e^{-\frac{|x-(0,vt)|^2}{4t}}dx \end{aligned}$$

where $d : \mathbb{R}^2 \to \mathbb{R}$ is the scalar signed distance to $Q_\epsilon \cap P_i$. Using the Taylor expansion of sine up to the third order, we have

$$I = \tfrac{1}{2}C\theta\frac{1}{4\pi t}\int_{Q_\epsilon} s(x)\left[\frac{d(x)}{s(x)}+O\left(\frac{d^3(x)}{s^3(x)}\right)\right]e^{-\frac{|x-(0,vt)|^2}{4t}}dx =: \tfrac{1}{2}C\theta\left[I_1+I_2\right]$$

Now, by the Taylor expansion of the scalar signed distance (cf. [6]), we get

$$I_1 = \frac{1}{4\pi t}\int_{Q_\epsilon}\left[\left(x_2+\tfrac{1}{2}\kappa x_1^2\right)+O\left(|x|^3\right)\right]e^{-\frac{|x-(0,vt)|^2}{4t}}dx =: I_1^1+I_1^2.$$

Claim 1: $I_1^1 = (v+\kappa)\,t + O(e^{-\frac{\alpha}{t}}\sqrt{t})$, as $t \to 0$.

Indeed,

$$\begin{aligned} \frac{1}{4\pi t}\int_{\mathbb{R}^2}\left(x_2+\tfrac{1}{2}\kappa x_1^2\right)e^{-\frac{|x-(0,vt)|^2}{4t}}dx &= \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}} x_2 e^{-\frac{(x_2-vt)^2}{4t}}dx+\frac{\kappa}{2\sqrt{\pi t}}\int_0^\infty x_1^2 e^{-\frac{x_1^2}{4t}}dx \\ &= \frac{1}{2\sqrt{\pi t}}\int_{\mathbb{R}}(x_2+vt)\,e^{-\frac{x_2^2}{4t}}dx+\kappa t = vt+\kappa t \end{aligned}$$

Moreover, by equation (5) and (6), we have

$$\begin{aligned} \left|\frac{1}{4\pi t}\int_{\mathbb{R}^2\setminus Q_\epsilon} x_2 e^{-\frac{|x-(0,vt)|^2}{4t}}dx\right| &\leq \frac{C}{4\pi t}\left|\int_{\mathbb{R}}\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}+\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\int_{\mathbb{R}} x_2 e^{-\frac{x_1^2+(x_2-vt)^2}{4t}}dx_2dx_1\right| \\ &\leq Ce^{-\frac{\alpha}{t}}\left[\sqrt{t}+\int_0^\infty\left(2\sqrt{t}\,|x_2|+vt\right)e^{-x_2^2}dx_2\right] \\ &\leq C\left(\sqrt{t}+t\right)e^{-\frac{\alpha}{t}}, \quad \text{for some } \alpha > 0 \end{aligned}$$

and,

$$\begin{aligned} \left|\frac{1}{4\pi t}\int_{\mathbb{R}^2\setminus Q_\epsilon}\tfrac{1}{2}\kappa x_1^2 e^{-\frac{|x-(0,vt)|^2}{4t}}dx\right| &\leq \frac{C}{4\pi t}\left|\int_{\mathbb{R}}\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}+\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\int_{\mathbb{R}} x_1^2 e^{-\frac{x_1^2+(x_2-vt)^2}{4t}}dx_2dx_1\right| \\ &\leq Ce^{-\frac{\alpha}{t}}\left[t\int_0^\infty x_1^2 e^{-x_1^2}dx_1+\sqrt{t}\int_{\mathbb{R}} e^{-x_2^2}dx_2\right] \\ &\leq C\left(t+\sqrt{t}\right)e^{-\frac{\alpha}{t}}, \quad \text{for some } \alpha > 0 \end{aligned}$$

which proves the claim.
Claim 2: $I_1^2 = O\left(t\sqrt{t}\right)$, as $t \to 0$.
Indeed,

$$\begin{aligned}
\left|I_1^2\right| &\leq \frac{C}{4\pi t}\int_{Q_\epsilon} \left(|x_1|+|x_2|\right)^3 e^{-\frac{x_1^2+(x_2-vt)^2}{4t}}dx \\
&\leq \frac{C}{4\pi t}\int_{Q_\epsilon} \left(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3\right) e^{-\frac{x_1^2}{4t}}e^{-\frac{(x_2-vt)^2}{4t}}dx_2dx_1 \\
&\leq Ct\sqrt{t}\int_0^{\frac{\epsilon}{2\sqrt{t}}}\int_{\frac{-\epsilon-vt}{2\sqrt{t}}}^{\frac{\epsilon-vt}{2\sqrt{t}}} \left(x_1^3+x_1^2\left|x_2+\sqrt{t}\right|+x_1\left|x_2+\sqrt{t}\right|^2+\left|x_2+\sqrt{t}\right|^3\right)e^{-x_1^2}e^{-x_2^2}dx_2dx_1 \\
&\leq C\left[t\sqrt{t}+\sqrt{t}e^{-\frac{\alpha}{t}}\right],
\end{aligned}$$

for some $\alpha > 0$, which proves the claim.

For the second term I_2, it is enough to consider the first two terms of the Taylor expansion (cf. [6]) of the scalar signed distance. Moreover, we can take $\epsilon > 0$, small enough so that $\min\limits_{x\in Q_\epsilon} s(x) > 0$. Thus,

$$\begin{aligned}
|I_2| &\leq C\frac{1}{4\pi t}\int_{Q_\epsilon}|s(x)|\left|\frac{d^3(x)}{s^3(x)}\right|e^{-\frac{|x-(0,vt)|^2}{4t}}dx \\
&\leq C\max_{Q_\epsilon}\frac{1}{s(x)^2}\frac{1}{4\pi t}\int_{Q_\epsilon}\left|d^3(x)\right|e^{-\frac{|x-(0,vt)|^2}{4t}}dx \\
&\leq \frac{C}{4\pi t}\int_{Q_\epsilon}\left(|x_2|+|x_1|^2\right)^3 e^{-\frac{x_1^2+(x_2-vt)^2}{4t}}dx \\
&\leq Ct\sqrt{t}\int_0^{\frac{\epsilon}{2\sqrt{t}}}\int_{\frac{-\epsilon-vt}{2\sqrt{t}}}^{\frac{\epsilon-vt}{2\sqrt{t}}}\left(\left|x_2+\sqrt{t}\right|+\sqrt{t}x_1^2\right)^3 e^{-x_1^2}e^{-x_2^2}dx_2dx_1 \\
&\leq C\sqrt{t}\left[t+e^{-\frac{\alpha}{t}}\right], \qquad (8)
\end{aligned}$$

for some $\alpha > 0$. Finally, it follows from claims 1 and 2, and equations (7) and (8) that

$$\begin{aligned}
0 &= I + II = (v+\kappa)\,t + O\left(t\sqrt{t}+e^{-\frac{\alpha}{t}}\right), \quad \text{for some } \alpha > 0 \\
&= v+\kappa+O\left(\sqrt{t}\right), \quad \text{as } t \to 0
\end{aligned}$$

which gives the desired result. □

3 Numerical Computation

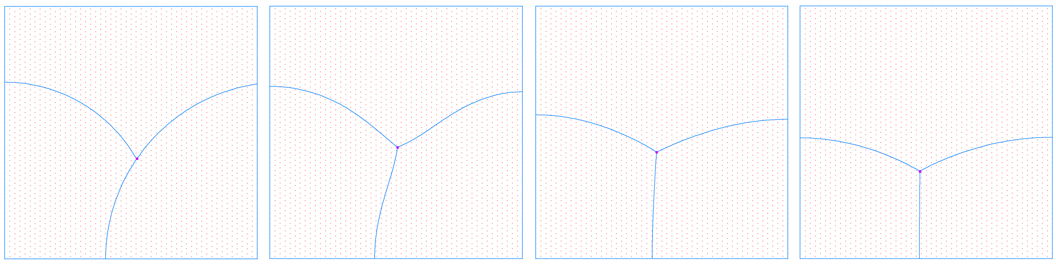
In this section, we present our numerical treatment of the method and some examples.

We adopt a variational approach, employing the idea of discrete Morse flow (DMF) [12, 9]. At each evolution step of the algorithm, we solve the vector-type heat equation (3) by discretizing time $\Delta t = h \times N$ and recursively minimizing its equivalent energy functional

$$\mathcal{F}_n^h(\mathbf{u}) = \int_\Omega \left(\frac{|\mathbf{u}-\mathbf{u}_{n-1}|}{2h}+\frac{|\nabla\mathbf{u}|^2}{2}\right)dx, \qquad (9)$$

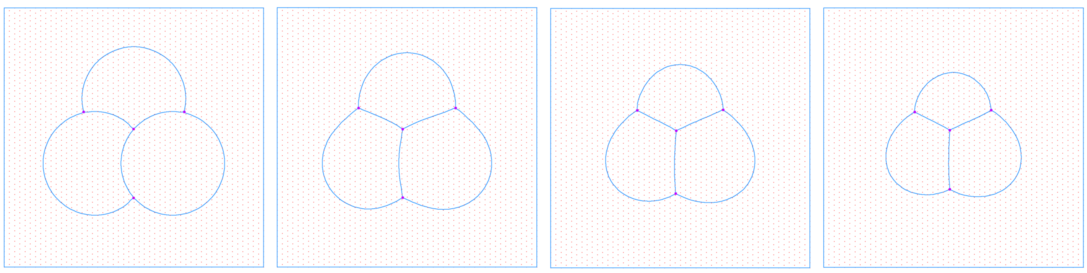
taking $\mathbf{u}_0$, as the initial condition.

Further, we approximate the functional values (9) via finite element method. The domain Ω is triangulated into a finite number of elements, over which the function is assumed to be linear and continuous. Finally, we search for the minimizers via steepest descent method and locate the interface using the final minimizer solution. In constructing the signed distance vector function, we evaluate distance to the interface using its exact distance to the piecewise linear interface. With this information, subgrid accuracy is achieved; thereby, preventing the algorithm from getting "stuck" and further its mean curvature evolution.



$t = 0.000$ $t = 0.005$ $t = 0.040$ $t = 0.150$

Figure 4: Three-phase Smooth Interface Problem



$t = 0.000$ $t = 0.010$ $t = 0.025$ $t = 0.035$

Figure 5: Shrinking Triple Bubble Problem

Examples of mean curvature evolution of a three-phase smooth interface and a four-phase triple bubble using the signed distance vector approach are shown in the figures 4 and 5, respectively. Here, the domain $\Omega := [0,1] \times [0,1]$ is triangulated into 5000 elements with 2576 nodes and the evolution time step $\Delta t = 0.0005$ is discretized into 30 DMF iterations. We also note that under these configurations, the MBO algorithm gets "stuck", while, the multiphase signed distance vector approach naturally alleviates the time and grid restriction (cf. [6]), without having to retriangulate elements based on the interfacial geometry.

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