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Two-generator elliptic subgroups of $SL(2,C)$

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Abstract

We classify all two-generator, purely elliptic subgroup of $SL(2,C)$ by means of the order of generators and the crossing angle of their axes.

Key words

Möbius transformations, regular polyhedral groups, discrete groups, purely elliptic two-generator subgroup of $SL(2,C)$,

1 Introduction

We denote by H^3 the upper-half space and regard it as the 3-dimensional hyperbolic space. The special linear group $SL(2,C)$ acts in H^3 as a group of hyperbolic isometries. For an elliptic element $f \in SL(2,C)$, denote by A_f the geodesic joining two fixed points of f on $\hat{C} = \partial H^3$ and call it the axis of f. For two elliptic elements $f, g \in SL(2, C)$, let $\langle f, g \rangle$ be the group generated by f, g. It is well known that $\langle f, g \rangle$ is purely elliptic if and only if $A_f \cap A_g \neq \phi$ [1]. In this paper, we classify all two-generator, purely elliptic subgroups of $SL(2,C)$.

By $M(\hat{R}^3)$ we denote the group of Möbius transformations acting on $\hat{R}^3 = R^3 \cup \{\infty\}$. The special orthogonal group $SO(3)$ acts in the unit ball B^3 in R^3 as a rotation group. Let $\langle f, g \rangle$ be a purely elliptic subgroup of $SL(2,C)$. Then there exist $x_0 \in A_f \cap A_g$ and $h \in M(\hat{R}^3)$ so that $h(H^3) = B^3$, $h(x_0) = 0$ and $h\langle f, g \rangle h^{-1} \subset SO(3)$. Since $h(A_f)$ is the geodesic through the origin joining two fixed points of hfh^{-1}, $h(A_f)$ is also called the axis of hfh^{-1}. Thus, for the study of purely elliptic subgroups, it suffices to investigate subgroups of $SO(3)$. From now on, we assume $f, g \in SO(3)$. By $ord(f)$ we denote the order of f. For non-trivial elements f, g, $\langle f, g \rangle$ may be discrete or not. Suppose that $\langle f, g \rangle$ is discrete. Then the classification theory of elementary Kleinian groups yields that $\langle f, g \rangle$ can be classified into five kinds[3]: a cyclic group, a dihedral group, a tetrahedral group, a hexahedral group and a dodecahedral group[3]. The last three kinds are regular polyhedral groups. To describe the non-discrete case, we need some preparations.

For a subgroup G of $SO(3)$, we define

$$F_G = \{\, \xi \in S^2 = \partial B^3 \mid g(\xi) = \xi \ \textit{for some}\ g \in G - \{id\} \}.$$

For $k = 2, 3, \ldots, +\infty$, we also define a subset of F_G as

$$F_G^k = \{\, \xi \in F_G \mid g(\xi) = \xi\,,\ ord(g) = k \}.$$

To describe the action of non-discrete groups on S^2, we regard $cl(B^3)$ as the earth. Denote by N, S and E the North Pole, the South Pole and the equator, respectively. A non-discrete group G is said to belong to the class I and denoted by $G \in C_I$, if there exists $h \in SO(3)$ so that $F_{hGh^{-1}} = h(F_G) = \{N, S\}$. A group G belongs to the class II C_{II} if there exists $h \in SO(3)$ so that $cl(h(F_G)) = \{N, S\} \cup E$, $h(F_G) \cap E \subset f(F_G^2)$ and $h(F_G^k) \cap E = \phi$ for $k \geq 3$. A group G belongs to the class III C_{III} if $cl(F_G^\infty) = S^2$. We shall show in Theorem 1 that each non-discrete group in $SO(3)$ belongs to either C_I, C_{II} or C_{III}.

For non-trivial elements $f, g \in SO(3)$, denote by $\theta(f, g)$ the crossing angle formed by A_f and A_g at the origin. Note that $\theta(f, g)$ is contained in $[0, \pi/2]$. In terms of $ord(f)$, $ord(g)$ and $\theta(f, g)$, we classify

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all two-generator subgroups of $SO(3)$ into eight kinds.

2 Non-discrete groups

In the first place, we consider non-discrete groups. As to the classification of subgroups of $PSL(2C)$, the following result due to Sullivan is essential.

Proposition 1.[5] Let G be a subgroup of $PSL(2, C)$. Then, after replacing G by a subgroup of index 2 if necessary, there are three possibilities:

(1) G is discrete.

(2) G is solvable and conjugate to a subgroup of similarities.

(3) (a) G is dense in $PSL(2, C)$, or

(b) G is conjugate to a dense subgroup of $PSL(2, R)$, or

(c) G is conjugate to a dense subgroup of $SO(3)$.

Suppose that G is a purely elliptic, non-discrete subgroup of $SL(2, C)$. Then the proposition above yields that G is conjugate in $M(\hat{R}^3)$ to either (2) a solvable subgroup of similarities or one of its Z_2-extensions, or (3)(c) a dense subgroup of $SO(3)$. Assume that (2) holds. Taking a suitable conjugation in $SL(2, C)$, we may suppose that the fixed point set for G is $\{0, \infty\}$. Then G is isomorphic to a non-discrete subgroup of $SO(2)$ and hence belongs to C_I. Next we suppose that G has a normal subgroup G_0 of index 2, which is isomorphic to a non-discrete subgroup of $SO(2)$. Then, for $f_0 \in G-G_0$, we have a coset decomposition $G=G_0+f_0G_0$. The element f_0 is of order 2, so is $f_0 g$ for each $g \in G_0$. Hence $F_{f_0G_0}$ forms a dense subset of a great circle on S^2. Therefore we conclude $G \in C_{II}$. Finally we suppose that G satisfies (3)(c). Taking a suitable conjugation, we may assume that G is a subgroup of $SO(3)$ and acts in B^3. Since G is dense in $SO(3)$, so is F_G in S^2. To prove that G belongs to C_{III}, it suffices to show $F_G^{\infty} \neq \phi$. So we prove

Lemma 1. Let G be a non-discrete subgroup of $SO(3)$. Assume that G does not leave any great circle on S^2 invariant. Then G contains an element of infinite order.

Proof. We prove this lemma by contradiction. So we assume that every element in G is of finite order. Since G is non-discrete, G is an infinite group. On the other hand, Selberg's lemma shows that every finitely generated subgroup of G is finite. As a consequence, we find a finite subset X_0 of G such that the order of the group $\langle X_0 \rangle$ $|\langle X_0 \rangle|$ is finite and greater than 60. It follows that $\langle X_0 \rangle$ is conjugate in $M(\hat{R}^3)$ to an elementary Kleinian group. Therefore we deduce that $\langle X_0 \rangle$ is a cyclic, a dihedral or a regular polyhedral group. Note that the order of each regular polyhedral group is at most 60[6]. Hence the group $\langle X_0 \rangle$ is cyclic or dihedral. In any case, there exists a set V which consists of two antipodal points on S^2 so that V is left invariant by $\langle X_0 \rangle$. Hence we define a subgroup G_0 of G by $G_0=\{g \in G \mid g(V)=V\}$. Let Y be an arbitrary finite subset of G. As its order $|\langle X_0 \cup Y \rangle|$ is greater than 60, the group $\langle X_0 \cup Y \rangle$ is either cyclic or dihedral, too. Since V is left invariant by $\langle X_0 \rangle$, it is also $\langle X_0 \cup Y \rangle$ -invariant. It implies that $\langle X_0 \cup Y \rangle$ is a subgroup of G_0 and hence G_0 contains every finitely generated subgroup of G. It means $G=G_0$ and G leaves a great circle on S^2 invariant. It contradicts the assumption and our lemma is established.

We summarize the argument above as the following form.

Theorem 1. Let G be a non-discrete subgroup of $SO(3)$. Then G belongs to either C_I, C_{II} or C_{III}.

Let G be a group generated by $f, g \in SO(3)$. From now on, we add the condition $ord(f) \geq ord(g)$ in the notation $G=\langle f, g \rangle$. Obviously we have the following equivalences:

$G=\langle f, g \rangle \in C_I \Leftrightarrow \theta(f, g)=0$ and $ord(f)=\infty$.

$G=\langle f, g \rangle \in C_{II} \Leftrightarrow$ (1) $\theta(f, g)=\pi/2$ and $ord(f)=\infty$, $ord(g)=2$ or

(2) $\theta(f, g)/\pi$ is irrational and $ord(f)=ord(g)=2$.

As an immediate consequence of Theorem 1, we have the following which is closely related to Jørgensen's result[4].

Corollary 1. Let G be a purely elliptic subgroup of $SL(2, C)$. Assume that G does not leave any circle in C invariant. Then G is discrete if and only if each cyclic subgroup of G is discrete.

If the assumption of this corollary is not satisfied, we can construct a counter example as follows: Let G be a group consisting of all rotations of finite order which fix 0, ∞. Of course G is non-discrete, but every finitely generated subgroup is discrete.

3 Regular polyhedral groups

Each discrete subgroup of $SO(3)$ is a cyclic, a dihedral or a regular polyhedral group. So, to consider a discrete group $\langle f, g\rangle$, we may restrict f, g to elements in one of these groups. It suffices to investigate only regular polyhedral groups. First, we deal with the tetrahedral groups.

[I] **The tetrahedral group.** A tetrahedral group is generated by rotations $f_1,\dots, f_4$ of order 3, and g_1, g_2, g_3 of order 2. Each f_i is a rotation by $2\pi/3$ about the axis through a vertex and the center of the opposite face. Each g_i is a rotation by π about the axis joining two midpoints of opposite edges. For $\{f, g\}\subset\langle f_1, \dots, g_3\rangle$, only two cases can occur for $\langle f, g\rangle$ to be a tetrahedral group.

(1) $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287\cdots^\circ$ and $ord(f) = ord(g) = 3$.

(2) $\theta(f, g) = \arctan\sqrt{2} = 54.7376\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

In any other combination, $\langle f, g\rangle$ is a cyclic or a dihedral group.

[II] **The hexahedral group.** A hexahedral group is generated by three kinds of rotations: rotations f_1, f_2, f_3 of order 4, $g_1,\dots, g_4$ of order 3 and $h_1,\dots, h_6$ of order 2. Each f_i, g_i and h_i is a rotation by $\pi/2$, $2\pi/3$ and π about the axis through two centers of opposite faces, two opposite vertices and two midpoints of opposite edges, respectively. We find six combinations of $\{f, g\}\subset\langle f_1,\dots,h_6\rangle$ for $\langle f, g\rangle$ to be a regular polyhedral group. In (1),…, (4) $\langle f, g\rangle$ is a hexahedral group and a tetrahedral group in (5), (6).

(1) $\theta(f, g) = \pi/2$ and $ord(f) = ord(g) = 4$.

(2) $\theta(f, g) = \arctan\sqrt{2} = 54.7376\cdots^\circ$ and $ord(f) = 4$, $ord(g) = 3$.

(3) $\theta(f, g) = \pi/4$ and $ord(f) = 4$, $ord(g) = 2$.

(4) $\theta(f, g) = \arctan(1/\sqrt{2}) = 35.2643\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

(5) $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287\cdots^\circ$ and $ord(f) = ord(g) = 3$.

(6) $\theta(f, g) = \arctan\sqrt{2} = 54.7376\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

[III] **The dodecahedral group.** Generators of a dodecahedral group are rotations $f_1, \dots, f_6$ of order 5, $g_1, \dots, g_{10}$ of order 3 and $h_1, \dots, h_{15}$ of order 2. Each f_i, g_i and h_i is a rotation by $2\pi/5$, $2\pi/3$ and π about the axis through two centers of opposite faces, two opposite vertices and two midpoints of opposite edges, respectively. There are ten combinations for $\langle f, g\rangle$ to be a regular polyhedral group. Each group in (1), …, (8) corresponds to a dodecahedral group, and a tetrahedral group in (9), (10).

(1) $\theta(f, g) = \arctan 2 = 63.4349\cdots^\circ$ and $ord(f) = ord(g) = 5$.

(2) $\theta(f, g) = \arctan(3-\sqrt{5}) = 37.3773\cdots^\circ$ and $ord(f) = 5$, $ord(g) = 3$.

(3) $\theta(f, g) = \arctan(3+\sqrt{5}) = 79.1876\cdots^\circ$ and $ord(f) = 5$, $ord(g) = 3$.

(4) $\theta(f, g) = \arctan\{(\sqrt{5}-1)/2\} = 31.7174\cdots^\circ$ and $ord(f) = 5$, $ord(g) = 2$.

(5) $\theta(f, g) = \arctan\{(\sqrt{5}+1)/2\} = 58.2825\cdots^\circ$ and $ord(f) = 5$, $ord(g) = 2$.

(6) $\theta(f, g) = \arctan(2/\sqrt{5}) = 41.8103\cdots^\circ$ and $ord(f) = ord(g) = 3$.

(7) $\theta(f, g) = \arctan\{(3-\sqrt{5})/2\} = 20.9051\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

(8) $\theta(f, g) = \arctan\{(3+\sqrt{5})/2\} = 69.0948\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

(9) $\theta(f, g) = \arctan 2\sqrt{2} = 70.5287\cdots^\circ$ and $ord(f) = ord(g) = 3$.

(10) $\theta(f, g) = \arctan\sqrt{2} = 54.7356\cdots^\circ$ and $ord(f) = 3$, $ord(g) = 2$.

In any other case, $\langle f, g\rangle$ is a cyclic or a dihedral group.

By removing duplications, we have seventeen combinations for $\langle f, g\rangle$ to be a discrete group. In any other combination, $\langle f, g\rangle$ is not discrete.

4 The dodecahedral group

Cases of the tetrahedral and the hexahedral groups are quite simple, so we discuss only the dodecahedral groups. Let $P_1, \dots, P_{20}$ and $F_1, \dots, F_{12}$ be vertices and faces of the regular dodecahedron Δ, respectively (See Figures 1 and 2). For each face F_k, $k = 1,\dots,6$ of Δ, there exists the opposite

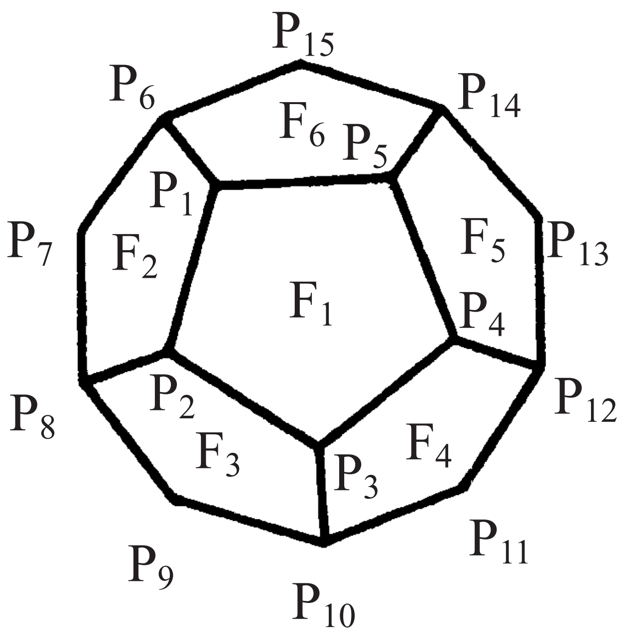

Figure 1

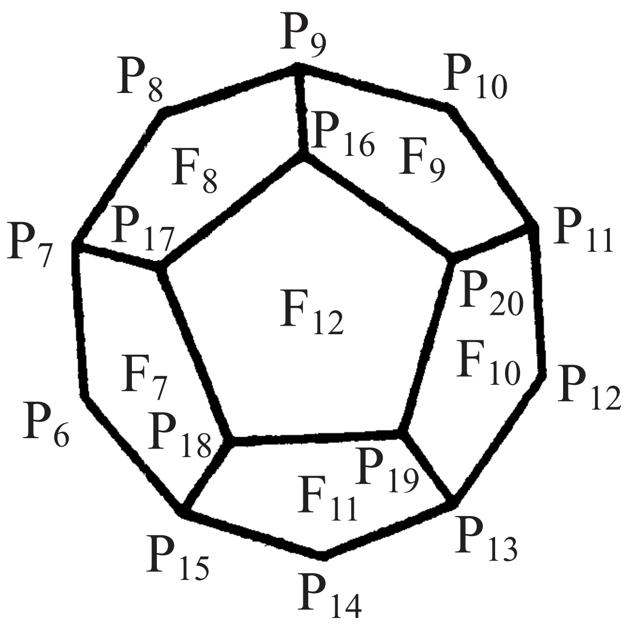

Figure 2

face F_l, $l=7$, … ,12. Denote by f_k the rotation by $2\pi/5$ about the line through the centers of F_k and F_l. This rotation f_k is oriented so as to interchange vertices of F_k clockwise. Similarly, for each P_k, $k=1,\ldots,10$ we can find the opposite vertex P_l, $l=11,\ldots,20$. Let g_k be the rotation by $2\pi/3$ about the line through P_k and P_l. So as to interchange three vertices around P_k clockwise, g_k is oriented. Let h_1 be the rotation by π about the line through the midpoints of edges P_1P_2 and $P_{20}P_{19}$. We denote it by h_1: (1,2) − (20,19). In this way, we can enumerate all rotations of order 2 as the following: h_2: (2,3) − (19,18), h_3: (3,4) − (18,17), h_4: (4,5) − (17,16), h_5: (5,1) − (16,20), h_6: (1,6) − (20,11), h_7: (2,8) − (19,13), h_8: (3,10) − (18,15), h_9: (4,12) − (17,7), h_{10}: (5,14) − (16,9), h_{11}: (6,7) − (11,12), h_{12}: (7,8) − (12,13), h_{13}: (8,9) − (13,14), h_{14}: (9,10) − (14,15), h_{15}: (10,11) − (15,6).

Now we denote by D_k, T and D the dihedral group of degree k, the tetrahedral and the dodecahedral group, respectively. The group D is generated by $S=\{f_1,\ldots,f_6, g_1,\ldots,g_{10}, h_1,\ldots,h_{15}\}$. We consider all two-generator subgroups of D. Since the order of each element in S is prime, it suffices to consider only groups that are generated by two elements in S. We may exclude cyclic subgroups of D.

Subgroups of D generated by two elements in S are classified into six types as the followings: (1) $\langle f_i, f_j\rangle$, (2) $\langle g_i, g_j\rangle$ (3) $\langle h_i, h_j\rangle$ (4) $\langle f_i, g_j\rangle$ (5) $\langle f_i, h_j\rangle$ and (6) $\langle g_i, h_j\rangle$. In any case, it suffices to show when $i=1$

(1) $\langle f_i, f_j\rangle$, $i, j=1,\ldots,6$ and $i\neq j$.

We may assume $j=2$. As $f_1f_2(P_k)=P_k$ ($k=1$, 20) and f_1f_2 transposes P_5, P_2 and P_6 cyclically, we deduce $f_1f_2=g_1$. Similarly, $f_2f_1(P_k)=P_k$ ($k=2$, 19) and f_2f_1 transposes P_1, P_3 and P_8 cyclically, so we have $f_2f_1=g_2$. In this way, we obtain $g_k\in\langle f_1, f_2\rangle$ for $k=1,\ldots,10$. On the other hand f_1g_2 interchanges P_1 and P_2, so we have $f_1g_2=h_1$. Therefore we conclude $h_k\in\langle f_1, f_2\rangle$ for $k=1,\ldots,15$ and hence $\langle f_1, f_2\rangle = D$. Let ξ be the dihedral angle of Δ and set $\vartheta=\theta(f_1, f_2)$. Since $\sin\xi = 2/\sqrt{5}$, $\xi > \pi/2$[2] and $\xi+\vartheta=\pi$, we have $\vartheta=\arctan 2$.

(2) $\langle g_i, g_j\rangle$ $i, j,=1,\ldots,10$ and $i\neq j$.

We discuss these groups by classifying them into two cases.

(i) The case where fixed vertices of g_1 and g_j belong to a common edge, i.e. $j=2$, 5, 6. Without loss of generalities, we may set $j=2$. As $g_2g_1(F_1)=F_1$ and $g_2g_1(P_1)=P_3$, it follows $g_2g_1=f_1^3$ and $f_1\in\langle g_1, g_2\rangle$. Noting $g_1f_1g_1^{-1}=f_2$, we conclude $\langle g_1, g_2\rangle \supset \langle f_1, f_2\rangle = D$. Denote by a the length of edges of Δ. We set $\vartheta=\theta(g_1, g_2)$. Since the radius of the circumscribed sphere of Δ is $\sqrt{3}(\sqrt{5}+1)a/4$[2], we obtain $\cos\vartheta=\sqrt{5}/3$ and so $\vartheta=\arctan(2/\sqrt{5})$.

(ii) The case where fixed vertices of g_1, g_j do not belong to any common edge, i.e. $j=3,4,7,8,9,10$. We may assume $j=3$ and denote $\vartheta=\theta(g_1, g_3)$. Since the length of the edge P_1P_3 is $(\sqrt{5}+1)a/2$, the second cosine formula yields $\vartheta=\arctan 2\sqrt{2}$. In general, two generators of order 3 whose axes cross at the angle $\arctan 2\sqrt{2}$ generate the tetrahedral group T. So we conclude $\langle g_1, g_3\rangle=T$.

(3) $\langle h_i, h_j\rangle$, $i, j=1,\ldots,15$ and $i\neq j$.

Since both generators are of order two, it follows that $\langle h_i, h_j\rangle$ is D_k for each $i, j, i\neq j$. We divide this case into three parts.

(i) $j=2, 5, 6, 7, 8, 10, 13, 15$. In this case we know $\vartheta=\theta(h_1, h_j)=2\pi/5$, $\langle h_i, h_j\rangle=D_5$.

(ii) $j=3, 4, 11, 12$. In this case we see $\vartheta=\pi/3$ and obtain $\langle h_1, h_j\rangle=D_3$.

(iii) If $j=9, 14$, we deduce $\vartheta=\pi/2$ and obtain $\langle h_1, h_j\rangle=D_2$.

(4) $\langle f_i, g_j\rangle$, $i=1,\ldots,6$ and $i=1,\ldots,10$.

Any g_j $(j=1, \ldots, 10)$ does not leave the axis of f_1 invariant. Thus, for each j, there exists k $(\neq 1)$ so that $g_j f_1 g_j^{-1}=f_k$. Therefore we conclude $\langle f_1, g_j\rangle\supset\langle f_1, f_k\rangle=D$. To calculate $\vartheta=\theta(f_1, g_j)$, we consider two possibilities:

(i) The case where F_1 contains a fixed vertex of g_j, that is $j=1,\ldots,5$.

(ii) The case where F_1 contains no fixed vertices of g_j, i.e. $j=6,\ldots,10$.

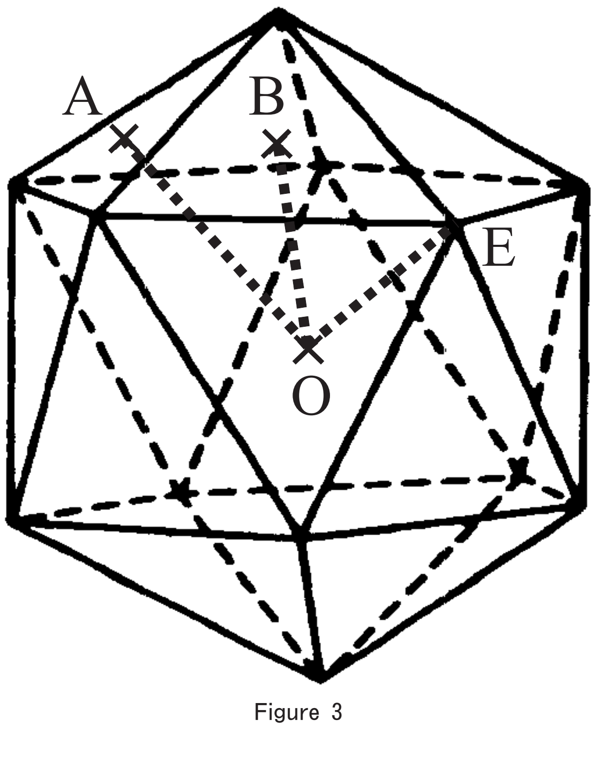

Figure 3

In both cases, we investigate the regular icosahedron Λ which is the dual polyhedron of Δ. We denote by E the vertex of Λ which corresponds to the center of the face F_1 of Δ. Centers of faces of Λ which correspond to vertices P_1, P_6 of Λ are denoted by B, A respectively (See Figure 3). Let O be the center of the circumscribed sphere of Λ (Δ). By setting $\angle BOA=\vartheta_1$, $\angle EOB=\vartheta_2$, we have $\theta(f_1, g_1)=\vartheta_2$ and $\theta(f_1, g_6)=\vartheta_1+\vartheta_2$. Let η be the dihedral angle of Λ. Since $\sin\eta=2/3$ and $\eta+\vartheta_1=\pi$, we see $\tan\vartheta_1=2/\sqrt{5}$. We denote by b the length of edges of Λ. It is well known that the radii of the inscribed and circumscribed spheres of Λ are $(3+\sqrt{5})b/4\sqrt{3}$ and $\sqrt{5+\sqrt{5}}b/2\sqrt{2}$, respectively[2)]. By observing the triangle ΔOBE, we have $\tan\vartheta_2=3-\sqrt{5}$ and so $\tan(\vartheta_1+\vartheta_2)=3+\sqrt{5}$. Therefore we obtain $\theta(f_1, g_1)=\arctan(3-\sqrt{5})$ and $\theta(f_1, g_6)=\arctan(3+\sqrt{5})$.

(5) $\langle f_i, h_j\rangle$, $i=1,\ldots,6$ and $j=1,\ldots,15$.

We classify these groups into three cases.

(i) The case where F_1 contains an invariant edge of h_j, i.e. $j=1,\ldots,5$.

Since $h_j f_1 h_j^{-1}=f_{1+j}$ for each j, it implies $\langle f_1, h_j\rangle\supset\langle f_1, f_{1+j}\rangle=D$. We set $\theta(f_1, h_1)=\vartheta$. As $2\vartheta+\xi=\pi$ with $\sin\xi=2/\sqrt{5}$, $\xi>\pi/2$, we see $\cos 2\vartheta=1/\sqrt{5}$. Therefore we conclude $\theta(f_1, h_1)=\arctan\{(\sqrt{5}-1)/2\}$.

(ii) The case where F_1 does not contain invariant edges of h_j and one of them contains a vertex of F_1, that is $j=6,\ldots,10$. We may assume $j=6$. Since $h_6 f_1 h_6^{-1}=f_4^{-1}$, it follows $\langle f_1, h_6\rangle\supset\langle f_1, f_4\rangle=D$. We have $2\theta(f_1,h_6)+\theta(f_1,h_4)=\pi$ and $\theta(f_1,h_4)=\arctan 2$ from the case (1). So we obtain $\tan 2\theta(f_1, h_6)=-2$ and hence $\theta(f_1, h_6)=\arctan\{(\sqrt{5}+1)/2\}$.

(iii) The case where the axes of f_1 and h_j cross at right angles, i.e. $j=11,\ldots,15$.

In this case we can easily see $\theta(f_1, h_j)=\pi/2$ and hence $\langle f_1, h_j\rangle=D_5$.

(6) $\langle g_i, h_j\rangle$, $i=1,\ldots,10$ and $j=1,\ldots,15$.

This case is divided into four parts.

(i) The case where P_1 belongs to an invariant edge E_j of h_j, i.e. $j=1, 5, 6$.

We deal only with $\langle g_1, h_1\rangle$. Noting $h_1g_1(F_1)=F_1$ and $h_1g_1(P_1)=P_2$, we have $h_1g_1=f_1^{-1}$ and hence $f_1\in\langle g_1, h_1\rangle$. Since $h_1 f_1 h_1^{-1}=f_2$, it implies $\langle g_1, h_1\rangle\supset\langle f_1, f_2\rangle=D$. We define $\theta(g_1, h_1)=\vartheta$. Denote the length of the edge P_1P_2 by a. Since r, the length of

OP_1, is the radius of the circumscribed sphere of Δ, $r=\sqrt{3}(\sqrt{5}+1)a/4$ and hence $\cos\vartheta=\sqrt{3}(\sqrt{5}+1)/6$. Therefore we obtain $\vartheta=\arctan\{(3-\sqrt{5})/2\}$.

(ii) The case where P_1 and E_j belong to a common face and P_1 is opposite to E_j in this face, that is $j=3, 12, 14$. We may suppose $j=3$. As $g_1^{-1}h_3(F_5)=F_5$, $g_1^{-1}h_3(P_4)=P_{14}$, it follows $g_1^{-1}h_3=f_5^2$ and $f_5\in\langle g_1, h_3\rangle$. Since $h_3f_5h_3^{-1}=f_3$, we know $\langle g_1, h_3\rangle\supset\langle f_3, f_5\rangle=D$. Denote $\theta(g_1, h_3)=\vartheta$. Then we can see $2\vartheta+\theta(g_1, g_6)=2\vartheta+\arctan(2/\sqrt{5})=\pi$. So we obtain $\vartheta=\arctan\{(3+\sqrt{5})/2\}$.

(iii) The case where P_1 adjoins an invariant edge E_j of h_j i. e. $j=2, 4, 7, 10, 11, 15$. We set $j=2$. As $h_2g_1(P_4)=P_4$, $h_2g_1(P_3)=P_{12}$, it implies $h_2g_1=g_4^{-1}$ and $\langle g_1, h_2\rangle=\langle g_1, g_4\rangle$. Obviously we have $\theta(g_1, g_4)=\theta(g_1, g_3)$. Therefore, from (2)(ii), we obtain $\langle g_1, h_2\rangle=\langle g_1, g_4\rangle=T$, a tetrahedral group, and hence $\theta(g_1, h_2)=\arctan 2\sqrt{2}$.

(iv) The case where P_1 and E_j do not belong to any common face, that is $j=8, 9, 13$. In this case we can easily obtain $\theta(g_1, h_j)=\pi/2$ and $\langle g_1, h_j\rangle=D_3$.

Thus we complete the proof of the dodecahedral case.

5. Two-generator groups

Let us state our main result.

Theorem 2. Let f, g be non-trivial elements in $SO(3)$. Then the following holds:

[I] **discrete case**

(1) If $\theta(f, g)=0$ and $ord(f)<\infty$, then $\langle f, g\rangle$ is a finite cyclic group.

(2) (a) If $\theta(f, g)=\pi/2$ and $ord(f)<\infty$, $ord(g)=2$, or

(b) If $\theta(f, g)/\pi$ is non-zero rational and $ord(f)=ord(g)=2$, then $\langle f, g\rangle$ is a dihedral group.

(3) (a) If $\theta(f, g)=\arctan 2\sqrt{2}$ and $ord(f)=ord(g)=3$, or

(b) $\theta(f, g)=\arctan\sqrt{2}$ and $ord(f)=3$, $ord(g)=2$, then $\langle f, g\rangle$ is a tetrahedral group.

(4) (a) If $\theta(f, g)=\pi/2$ and $ord(f)=ord(g)=4$, or

(b) If $\theta(f, g)=\arctan\sqrt{2}$ and $ord(f)=4$, $ord(g)=3$, or

(c) If $\theta(f, g)=\pi/4$ and $ord(f)=4$, $ord(g)=2$, or

(d) If $\theta(f, g)=\arctan(1/\sqrt{2})$ and $ord(f)=3$, $ord(g)=2$, then $\langle f, g\rangle$ is a hexahedral group.

(5) (a) If $\theta(f, g)=\arctan 2$ and $ord(f)=ord(g)=5$, or

(b) If $\theta(f, g)=\arctan(3-\sqrt{5})$ and $ord(f)=5$, $ord(g)=3$, or

(c) If $\theta(f, g)=\arctan(3+\sqrt{5})$ and $ord(f)=5$, $ord(g)=3$, or

(d) If $\theta(f, g)=\arctan\{(\sqrt{5}-1)/2\}$ and $ord(f)=5$, $ord(g)=2$, or

(e) If $\theta(f, g)=\arctan\{(\sqrt{5}+1)/2\}$ and $ord(f)=5$, $ord(g)=2$, or

(f) If $\theta(f, g)=\arctan(2/\sqrt{5})$ and $ord(f)=ord(g)=3$, or

(g) If $\theta(f, g)=\arctan\{(3-\sqrt{5})/2\}$ and $ord(f)=3$, $ord(g)=2$, or

(h) If $\theta(f, g)=\arctan\{(3+\sqrt{5})/2\}$ and $ord(f)=3$, $ord(g)=2$, then $\langle f, g\rangle$ is a dodecahedral group.

[II] **non-discrete case**

(1) If $\theta(f, g)=0$ and $ord(f)=\infty$, then $\langle f, g\rangle$ belongs to C_I.

(2) (a) If $\theta(f, g)=\pi/2$ and $ord(f)=\infty$, $ord(g)=2$, or

(b) If $\theta(f, g)/\pi$ is irrational and $ord(f)=ord(g)=2$, then $\langle f, g\rangle$ belongs to C_{II}.

(3) In any other than [I](1),…,[II](2), $\langle f, g\rangle$ belongs to C_{III}.

As a consequence of this theorem, we give a sufficient condition for $\langle f, g\rangle$ to be a group in C_{III}. Let f, g be non-trivial elements of $SO(3)$. If $\theta(f, g)\neq 0$, then $\langle f, g\rangle$ cannot only be a cyclic group but also a group in C_I. On the other hand, it is necessary that $\theta(f, g)=\pi/2$ or $ord(f)=ord(g)=2$ for $\langle f, g\rangle$ to be a dihedral group in C_{II}. Moreover, orders of elements in each regular polyhedral groups are at most 5[3),5)]. Hence we obtain

Corollary 2 Let f, g be non-trivial elements in $SO(3)$. If $\theta(f, g)$ is contained in $(0, \pi/2)$ and $ord(f)>5$, then $\langle f, g\rangle$ belongs to C_{III}.

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SL(2, *C*)の2元生成、純楕円的部分群

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要　　旨

SL(2, *C*)の純楕円的部分群は、離散群としては有限巡回群、2面体群、正4面体群、正6面体群、正12面体群からなり、非離散部分群は3つのクラス C_I、C_{II} 、C_{III} に分類される。本論分は*SL*(2, *C*)のすべての2元生成純楕円的部分群を、その生成元の位数および軸の交角により完全に分類した。