# Portfolio optimization with conditional Value-at-Risk under CEV model

**Sci. Rep. Kanazawa Univ.**
Vol. 66, pp. 17–27, 2023

**ARTICLES**

# Portfolio optimization with conditional Value-at-Risk under CEV model

Yuri IMAMURA[1,*] and Benyanee KOSAPONG[2,3]

[1] Faculty of Mathematics and Physics, Institute of Science and Engineering, Kanazawa University
Kanazawa, 920-1192, Japan
[2] Division of Mathematical and Physical Sciences, Graduate School of Natural Science and Technology, Kanazawa University
Kanazawa, 920-1192, Japan
[3] Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University
Bangkok, 10330, Thailand

(Received October 22, 2022 and accepted in revised form July 27, 2023)

**Abstract** Portfolio optimization is a well-known and beneficial procedure used by shareholders to select their portfolios. An investor must seek an equilibrium between risk and profit while making investment decisions. The fundamental concern is risk because the responsibility of the risk for each investor is different. A risk profile of each investor is characterized as a risk measure. In this paper, we focus on Conditional Value-at-Risk (CVaR). We numerically consider an optimal portfolio which minimizes CVaR under CEV model. Finally, the numerical results of CVaR and the optimal portfolio are discussed.

**Keywords.** Portfolio optimization, risk measure, Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), CEV model, numerical analysis.

## 1 Introduction

Portfolio optimization is an important process of selecting the optimal portfolio that provides the highest profit (portfolio returns) for each unit of risk accepted by the investors or minimizes risk for a given return. In 1952, Markowitz developed portfolio selection theory by using variance as a risk measure [11]. The variance, or equivalently standard deviation, is a famous tool for measuring a risk of portfolio. The axiom of coherent was introduced as properties of measures of risk [3, 4]. Artzner et al. [3] proposed the use of conditional Value-at-Risk (CVaR) (which is also called expected shortfall, average value at risk, tail value at risk, expected tail loss, superquantile, etc.) to alleviate the problems inherent in value-at-risk (VaR). VaR is a loss distribution percentile, and CVaR is the average of loss values which excess a specific percentage (e.g., 90%, 95%, and 99%) of the worst-case loss scenarios. Rockafellar and Uryasev [10] suggested another risk measure used CVaR to optimize portfolios. They showed that risk management with CVaR can be

* *Corresponding author* Email: imamuray@se.kanazawa-u.ac.jp

optimized with linear programming methods, whereas optimization with VaR is difficult numerically (see also [3, 4, 9, 12]). Rockafellar and Uryasev [10] applied CVaR for minimizing the risk of an investment portfolio in the stock market. They reformulated them as convex optimization problems, and a similar approach appears in Krokhmal, Palmquist and Uryasev [8]. Alexander et al. [2] applied the CVaR model to a portfolio optimization problem and numerically solved it by Monte Carlo simulation. In those previous results, under the assumption of the existence of a density function, the optimization problem is reduced to a linear programming problem. Since the density functions of some stock price models are not known, we aim to remove the density function assumption. Moreover, since option prices are lower than stock prices and options can make more profits than stocks, we consider a portfolio which consists of not only stocks but also options. For simplicity, the target derivatives of investment are restricted to one stock and its call and put options with the same strike. Under Black-Scholes economy (BS model), the analytical forms of call and put options are known since the price process is expressed by geometric Brownian motion. Then the distribution function of the loss function is obtained which is explained in Theorem 4.1. Since the behavior of volatility is not stable in the real market, we focus on solving the optimization problem under the constant elasticity of variance model (CEV model) which was proposed by [5]. Since the distribution of price process is not known, the motivation of this paper is numerically to show properties of the optimal portfolio under the CEV model.

This paper is organized as follows. In Section 2, we study the stock and option market. In Section 3, we explain definitions of VaR and CVaR and then we define the loss function of investment that is used to optimize the portfolio. In Section 4, we study optimization problem of portfolio with options under BS model and CEV model. Moreover, we show numerical results of the optimal portfolio using approximations of the stock prices and loss functions. In Section 5, we provide discussions and conclusions.

## 2 Option market

An option or a derivative is a contract that gives its owner a right to buy or sell assets at a fixed price on or before a given date. The option value is based on the value of the underlying asset. Call option and put option are the basic types. European call (or put) options allow the owner to buy (or sell) an asset at a specified price on a given maturity. For example, in the case of Japan, the Osaka exchange exists as an options market. Then, we trade call and put options with certain maturity times and strike prices for Nikkei 225 like stocks. The primary distinction between options and stocks is that stocks are shares of ownership in individual companies, while options are contracts with other investors. We can purchase a lot of options to make a profit, however it can make a loss at the same time. In this paper, we focus on a portfolio constructed by not only stock but also options.

## 3 VaR, CVaR and Optimal Portfolio

VaR is a statistic quantifying the magnitude of a possible bad situation or loss over a specific time. CVaR quantifies the amount of tail risk in an investment portfolio, and it is calculated by taking a weighted average of losses that exceed the VaR cutoff point. Figure 1 shows an example of histogram of loss values with a portfolio of investment.

We consider a portfolio of the investment strategy consisting of specific three claims; the asset, a call option and a put option, where the options are associated with the same asset and they have

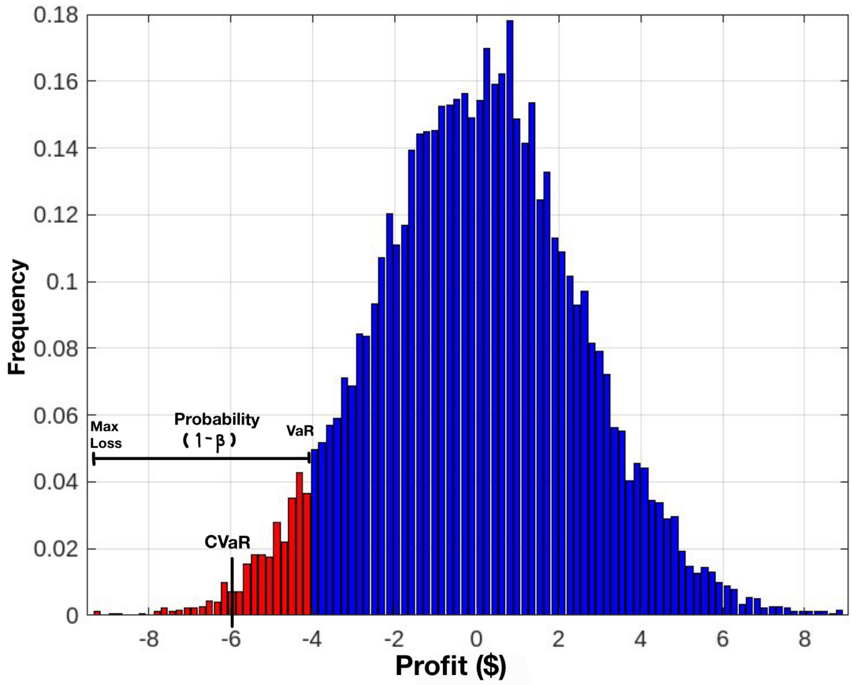



Figure 1: The graphical representation of VaR, CVaR and max loss (min profit)

the same strike $K$ and the same maturity $T$. Let $x_0$, $x_c$ and $x_p$ be portfolio weights investing in cash, a call option and a put option, respectively. We assume that the portfolio weights are restricted with $x_0+x_c+x_p=1$, and we denote by $\mathscr{A}(\subset \mathbb{R}^3)$ the set of all of the weights. We denote the initial prices of a call option and a put option by $C(K)$ and $P(K)$, respectively. The returns of each contract are determined as the ratio between the gain at the maturity time $T$ and the initial prices $S_0$, $C(K)$ and $P(K)$. We consider that the loss function of the portfolio with the weights $(x_0,x_c,x_p)\in\mathscr{A}$ is given by

$$\begin{aligned} &f(x_0,x_c,x_p;S_T) \\ &= -x_0\left[\frac{S_T-S_0}{S_0}\right]-x_c\left[\frac{(S_T-K)1_{\{S_T>K\}}-C(K)}{C(K)}\right]-x_p\left[\frac{(K-S_T)1_{\{K>S_T\}}-P(K)}{P(K)}\right]. \end{aligned} \tag{3.1}$$

We note that the loss function with a variable $S_T$ is bounded below by $-\dfrac{x_0K}{S_0}+1$ if $-x_0C(K)-x_cS_0\geq 0$ and $-x_0P(K)+x_pS_0\leq 0$, and the loss function with a variable $S_T$ is bounded above by $-\dfrac{x_0K}{S_0}+1$ if $-x_0C(K)-x_cS_0\leq 0$ and $-x_0P(K)+x_pS_0\geq 0$. In the other cases, the loss function is unbounded.

For each $(x_0,x_c,x_p)\in\mathscr{A}$ and $\alpha\in\mathbb{R}$, the probability that the loss does not exceed $\alpha$ is denoted by

$$\Psi(x_0,x_c,x_p;\alpha)=\mathbb{P}\left(f(x_0,x_c,x_p;S_T)\leq\alpha\right). \tag{3.2}$$

Then, for $(x_0,x_c,x_p)\in\mathscr{A}$ and confidence level $\beta$ in $(0,1)$, $\beta$-VaR $\alpha_\beta(x_0,x_c,x_p)$ and $\beta$-CVaR $\phi_\beta(x_0,x_c,x_p)$ are denoted by

$$\alpha_\beta(x_0,x_c,x_p)=\inf\{\,\alpha\in\mathbb{R}\mid\Psi(x_0,x_c,x_p;\alpha)\geq\beta\,\}, \tag{3.3}$$

and

$$\phi_\beta(x_0,x_c,x_p)=(1-\beta)^{-1}\mathbb{E}[f(x_0,x_c,x_p;S_T)1_{\{f(x_0,x_c,x_p;S_T)\geq\alpha_\beta(x_0,x_c,x_p)\}}]. \tag{3.4}$$

The aim of this paper is to find an investment portfolio weight $(x_0, x_c, x_p)$ which is the optimal value of the following minimization:

$$\begin{array}{ll} \min & \phi_\beta(x_0, x_c, x_p), \\ \text{subject to} & (x_0, x_c, x_p) \in \mathscr{A}. \end{array}$$

# 4 Optimal Portfolio under CEV Model

We assume that the asset price process follows the CEV model which is described in [1, 13]. That is, the price process is given by a solution of the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t^\gamma dW_t, \; S_0 > 0, \tag{4.1}$$

where $\mu$ is a parameter characterising the drift, $\sigma > 0$ is a volatility, $\gamma > 0$ is the elasticity which controls the relationship between volatility and price, and $\{W_t\}_{t\geq 0}$ is the Brownian motion. Generally option prices, VaR and CVaR are numerically calculated since the distribution of $S_T$ is unknown except in the case of BS model ($\gamma = 1$), i.e. the elasticity is one. Here the price under BS model is lognormally distributed with constant volatility.

In Section 4.1, we assume that the price process follows BS model. We give the closed form of option prices and the probability that the loss does not exceed a level, i.e. the cumulative distribution function of the portfolio loss value. Since the distribution of geometric Brownian motion is given explicitly, the optimal strategy which attains the minimal CVaR is determined. In Section 4.2, we focus on CEV model. CVaR is numerically calculated; (i) the Euler-Maruyama method provides the approximation of the discrete-time price process and (ii) Monte Carlo method gives the approximation of the probabilities.

## 4.1 Black-Scholes Model

In this section, we assume that the asset price process is given by the BS model, that is $\gamma = 1$ in (4.1). Here, the asset price process is offered by the solution of the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \; S_0 > 0. \tag{4.2}$$

By Itô's formula, SDE (4.2) is explicitly solved which is given by the following geometric Brownian motion:

$$S_t = S_0 \exp\left\{ (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t \right\}. \tag{4.3}$$

Since $W_T$ follows normal distribution with mean 0 and variance $T$, the prices of call option $C(K)$ and put option $P(K)$ are determined as follows (for example, see [6]);

$$\begin{aligned} C(K) &= S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2), \\ P(K) &= K e^{-rT} \Phi(-d_2) - S_0 \Phi(-d_1), \end{aligned} \tag{4.4}$$

where $r$ is the interest rate of the riskless security and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}} \left[ \log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right) T \right], \\ d_2 &= \frac{1}{\sigma\sqrt{T}} \left[ \log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right) T \right]. \end{aligned}$$

Here $S_0$ is the initial price, and $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ is the cumulative distribution function of standard normal distribution.

Then we get a closed form of the distribution function of the loss function.

**Theorem 4.1** *Under the BS model, for* $(x_0, x_c, x_p) \in \mathscr{A}$ *and* $\alpha \in \mathbb{R}$,

$$\Psi(x_0, x_c, x_p; \alpha) = \begin{cases} A^- - B + C & \text{if } -x_0C(K) - x_cS_0 \geq 0, -x_0P(K) + x_pS_0 \geq 0, \\ 1 - A^+ + C & \text{if } -x_0C(K) - x_cS_0 \leq 0, -x_0P(K) + x_pS_0 \geq 0, \\ A^- - C & \text{if } -x_0C(K) - x_cS_0 \geq 0, -x_0P(K) + x_pS_0 \leq 0, \\ 1 - A^+ + B - C & \text{if } -x_0C(K) - x_cS_0 \leq 0, -x_0P(K) + x_pS_0 \leq 0, \end{cases}$$

*where*

$$A^{\pm} = \Phi\left( \pm \frac{\log\left( \left( \frac{(\alpha-1)C(K) - x_cK}{-x_0C(K) - x_cS_0} \right) \vee \frac{K}{S_0} \right) - \left( \mu - \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}} \right),$$

$$B = \Phi\left( \frac{\log\left( \frac{K}{S_0} \right) - \left( \mu - \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}} \right),$$

$$C = \Phi\left( \frac{\log\left( \left( \frac{(\alpha-1)P(K) - x_pK}{-x_0P(K) + x_pS_0} \right) \wedge \frac{K}{S_0} \right) - \left( \mu - \frac{\sigma^2}{2} \right) T}{\sigma\sqrt{T}} \right).$$

*Here notations are defined by* $\log(z) := -\infty$ *when* $z \leq 0$, $a \vee b = \max\{a, b\}$ *and* $a \wedge b = \min\{a, b\}$.

## 4.2 Numerical Examples

We assume that the price process follows CEV model. In this section, we show some numerical examples for CEV model. Here we fix parameters as $\mu = 0$, $\sigma = 0.1206$, $T = \frac{1}{12}$, $S_0 = 295.42$ and $K = 300$. These values of parameters are from analyzing real data of S&P 500 mini index in Kosapong [7].

The prices of call option and put option are numerically obtained. The discrete-time price process is generated by Euler-Maruyama approximation; $S_0^{(n)} = S_0$ and for $1 \leq k \leq n$,

$$S_{\frac{k}{n}T}^{(n)} = S_{\frac{k-1}{n}T}^{(n)} + \mu S_{\frac{k-1}{n}T}^{(n)} \frac{T}{n} + \sigma \left\{ S_{\frac{k-1}{n}T}^{(n)} \right\}^{\gamma} Z_k,$$

where $\{Z_k\}_{k=1}^{n}$ is a sequence of independent normally distributed random variables with mean 0 and variance $T/n$. By Monte Carlo method, the arithmetic means for scenarios of the payoff functions give approximated prices of options. Here the number of time discretization of the path of the price process by Euler-Maruyama method is $10^5$ and the number of Monte Carlo simulation is $10^7$.

In the case $\gamma = 1.0$, we can explicitly calculate the prices of the call option and the put option by using (4.4), and the results are summarized in Table 1. We notice that Table 1 and the prices by

Table 1: The Call/Put options prices under Black-Scholes model

| Interest rate | Call price ($C(K)$) | Put price ($P(K)$) |
|---|---|---|
| 0.0 | 2.2418 | 6.8218 |
| 0.1 | 3.1563 | 5.2467 |

Euler-Maruyama method, in Table 2 at $\gamma = 1.0$, are close enough. Therefore, we can understand that an approximation error of Euler-Maruyama method is negligible.

By Euler-Maruyama method, we can generate a lot of scenarios of price process $S_T^{(n)}$, and then for each weight $(x_0, x_c, x_p) \in \mathscr{A}$, we obtain the loss functions $f(x_0, x_c, x_p; S_T^{(n)})$ defined by (3.1). By Euler-Maruyama method and Monte Carlo method, we can simulate the distribution of the loss function $f(x_0, x_c, x_p; S_T^{(n)})$. Therefore we obtain VaR as the percentile point defined by (3.3), and CVaR given by (3.4). In our simulation with fixed confidence level $\beta(= 0.90, 0.95, 0.99)$, VaR is calculated by using "prctile" function in MATLAB. We note that CVaR is a function of weight $(x_0, x_c, x_p)$. Hence by using optimization tool "fmincon" in MATLAB, we obtain the optimal weight which leads to the minimal CVaR. The code of optimization portfolio and the step of optimization are represented in Listing 1.

Listing 1: MATLAB code of portfolio optimization algorithm

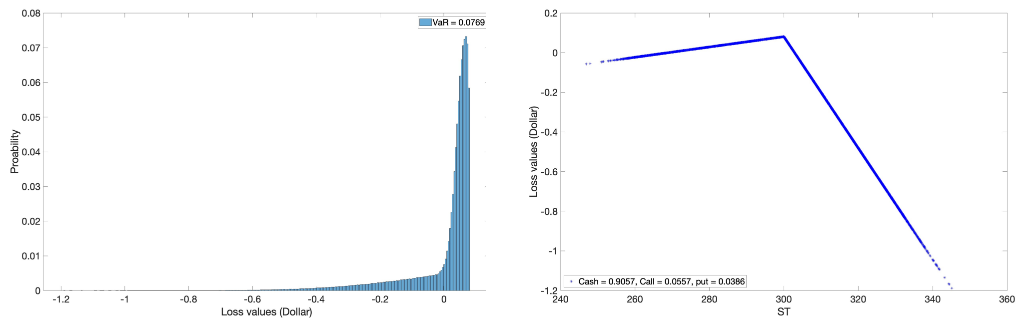
```
loss = @(x_0,x_c,x_p) -((x_0/(x_0+x_c+x_p))*returnMatrix(:,1)
     + (x_c/(x_0+x_c+x_p))*returnMatrix(:,2)
     + (x_p/(x_0+x_c+x_p))*returnMatrix(:,3));
VaR= @(x_0,x_c,x_p) prctile(loss(x_0,x_c,x_p),beta);
MeanLoss_tail = @(x_0,x_c,x_p) max(loss(x_0,x_c,x_p)
              -VaR(x_0,x_c,x_p),0);
CVaR = @(x_0,x_c,x_p) VaR(x_0,x_c,x_p)
     + (1/(1-beta/100))*mean(MeanLoss_tail(x_0,x_c,x_p));
options = optimset('Display','iter','PlotFcns',@optimplotfval
   );
[x_opt,fval,exitflag,output] = fmincon(@(x)CVaR(x(1),x(2),x
   (3)),x,'constraint',options);
```

Table 2: Options, optimal weights and corresponding VaRs and CVaRs

| $\gamma$ | interest rate | Call price | Put price | Confidence level | VaR | CVaR | optimal Cash | optimal Call | optimal Put |
|---|---|---|---|---|---|---|---|---|---|
| 0.75 | 0.0 | 0.0328 | 4.6180 | 90% | 0.0050 | 0.0051 | 0.9796 | $-7.7903 \times 10^{-5}$ | 0.0205 |
| | | | | 95% | 0.0087 | 0.0098 | 0.9737 | 0.0006 | 0.0257 |
| | | | | 99% | 0.0140 | 0.0164 | 0.9841 | $-4.5934 \times 10^{-4}$ | 0.0164 |
| | 0.1 | 0.2757 | 2.3945 | 90% | -0.0076 | -0.0075 | 0.9920 | $-7.6944 \times 10^{-4}$ | 0.0087 |
| | | | | 95% | -0.0072 | -0.0068 | 0.9856 | $-7.3612 \times 10^{-4}$ | 0.0152 |
| | | | | 99% | $-3.6817 \times 10^{-4}$ | $-2.8478 \times 10^{-4}$ | 0.9849 | $-8.3655 \times 10^{-4}$ | 0.0142 |
| 1.0 | 0.0 | 2.2232 | 6.8240 | 90% | 0.0737 | 0.0770 | 0.9845 | -0.0077 | 0.0232 |
| | | | | 95% | 0.0769 | 0.0786 | 0.9057 | 0.0557 | 0.0386 |
| | | | | 99% | 0.0796 | 0.0799 | 0.9531 | -0.0072 | 0.0541 |
| | 0.1 | 3.1477 | 5.2822 | 90% | -0.0072 | -0.0068 | 0.9931 | -0.0109 | 0.0178 |
| | | | | 95% | -0.0033 | $-7.9341 \times 10^{-5}$ | 0.9950 | -0.0126 | 0.0176 |
| | | | | 99% | 0.0119 | 0.0120 | 0.9729 | -0.0089 | 0.0360 |
| 1.25 | 0.0 | 14.9139 | 19.5677 | 90% | $2.4567 \times 10^{-4}$ | $2.4576 \times 10^{-4}$ | 0.9845 | -0.0497 | 0.0652 |
| | | | | 95% | 0.0071 | 0.0076 | 0.9768 | -0.0437 | 0.0669 |
| | | | | 99% | 0.0668 | 0.0695 | 0.9137 | -0.0224 | 0.1087 |
| | 0.1 | 16.1000 | 18.2876 | 90% | $-3.3081 \times 10^{-4}$ | $-1.2789 \times 10^{-5}$ | 0.9844 | -0.0490 | 0.0646 |
| | | | | 95% | 0.0013 | 0.0013 | 0.9834 | -0.0514 | 0.0681 |
| | | | | 99% | 0.0154 | 0.0176 | 0.9654 | -0.0375 | 0.0721 |
| 1.5 | 0.0 | 73.4303 | 78.1001 | 90% | $7.1485 \times 10^{-4}$ | $9.2344 \times 10^{-4}$ | 0.9847 | -0.2448 | 0.2601 |
| | | | | 95% | 0.0123 | 0.0128 | 0.9717 | -0.2448 | 0.2601 |
| | | | | 99% | 0.0319 | 0.0321 | 0.9528 | -0.2203 | 0.2674 |
| | 0.1 | 74.7960 | 77.0002 | 90% | -0.0068 | -0.0067 | 0.9913 | -0.2502 | 0.2588 |
| | | | | 95% | -0.0043 | -0.0034 | 0.9929 | -0.2519 | 0.2590 |
| | | | | 99% | 0.0479 | 0.0491 | 0.9415 | -0.1855 | 0.2440 |

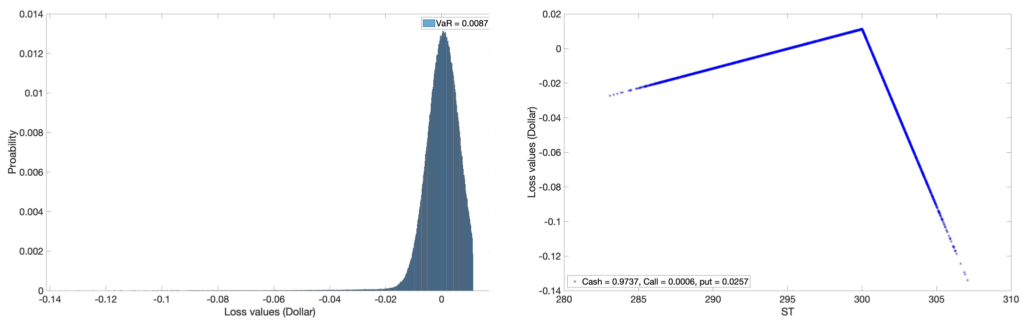
The simulation results are shown in Table 2, which demonstrates the Call and Put option prices, the optimal weights of cash, call option and put option, and corresponding VaR and CVaR for each $\gamma$ value in the CEV model. In Table 2, we observe that VaR and CVaR values are increasing for significance level. This fact is confirmed in the simulation results in Table 2. We also notice that for $\gamma = 1.00$, VaR and CVaR values are close. However, the CVaR values are always greater than VaR values. Those facts are mathematically correct by definitions of VaR and CVaR.

Figures 2, 3 and 4 show the histograms and graphs of loss functions of simulated prices for each $\gamma = 1.00, 0.75, 1.25$ under CEV model with no interest rate. Here the loss function is the value at the optimal weights. The optimal portfolios with 95% confidence level are given at the bottom of graphs 2(b), 3(b) and 4(b). The distributions of loss functions for $\gamma = 0.75$ and $\gamma = 1.25$ are numerically given, unlike the case $\gamma = 1.0$. The optimal solutions or the weights of investment that are represented in Figure 3 and Figure 4 are also unlike Figure 2. In these cases with $\gamma = 0.75$ and $\gamma = 1.0$, we notice that we need to sell both of call option and put option because the weights of investment are positive values. It opposes with the optimal solutions in case of $\gamma = 1.25$. We need to sell a put option and buy a call option since the weight of call option is a negative value.




(a) The histogram of loss values (b) The loss function

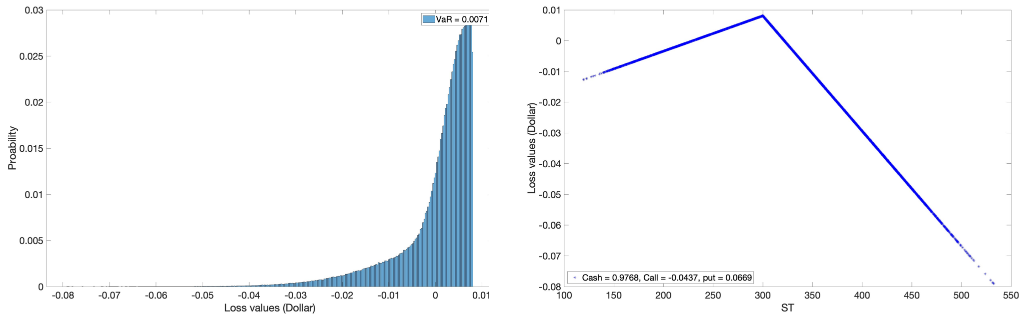
Figure 2: At 95% confidence level ($\gamma = 1.00$)




(a) The histogram of loss values (b) The loss function

Figure 3: At 95% confidence level ($\gamma = 0.75$)

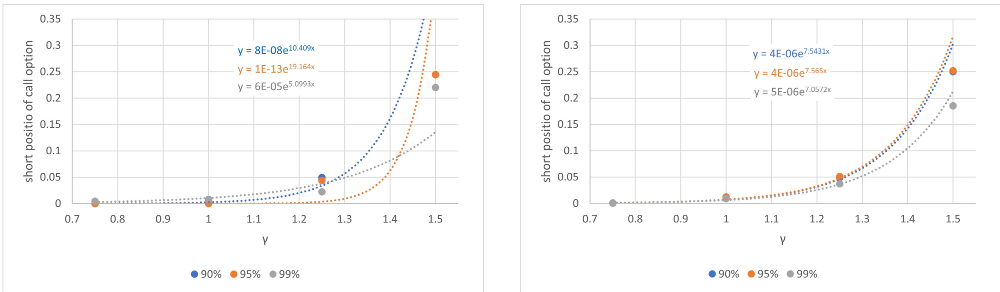
In addition, we consider the optimal portfolio with two interest rates, 0.0 and 0.1. The interest rates of return affect the simulated values of option prices. When we focus on the case $r = 0.0$, we confirm that VaR values are smaller than CVaR values and that they are positive values. However,




(a) The histogram of loss values (b) The loss function

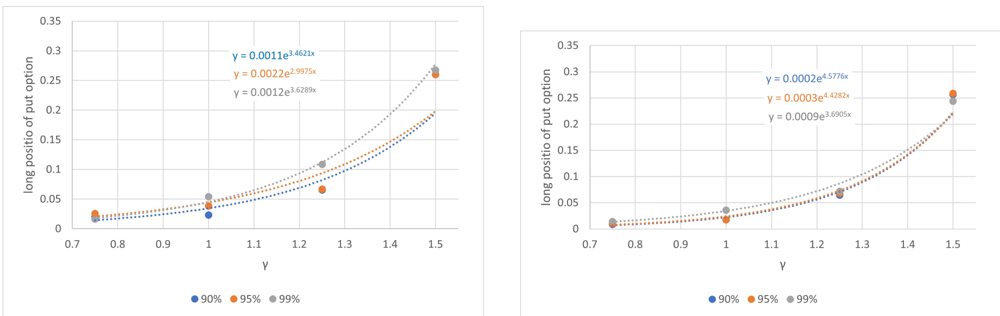
Figure 4: At 95% confidence level ($\gamma = 1.25$)

it is no trend of changes when we consider the VaR and the CVaR after changing the interest rates to 0.1. But CVaR values are slightly higher than VaR values. This is shown in Table 2.




(a) Interest rate 0.0 (b) Interest rate 0.1

Figure 5: The optimal short position of call option




(a) Interest rate 0.0 (b) Interest rate 0.1

Figure 6: The optimal long position of put option

Lastly, the $x$-axis and $y$-axis of Figure 5 and 6 are $\gamma$ values and the optimal solutions that we should invest in call and put options from Table 2. We find that there is a strong relationship between $\gamma$ and selling call options, or between $\gamma$ and buying put options. As it might be seen, the different interest rate of return makes the distinct graphs. The relative of interest rate return 0.1 is more than the relative of another one. Figure 5 looks like an exponential curve, and the proportion for selling call options significantly increases when $\gamma$ value rises up. Nevertheless, the slopes of $\gamma$ values and the proportion for buying put options are slightly increased in Figure 6.

## 5 Discussions and Conclusions

This paper presents an investigation into the optimization of an options portfolio with the objective of minimizing CVaR value. The study employs the CEV model and utilizes numerical methods, specifically the Euler-Maruyama method and Monte Carlo simulations, to calculate the loss function, VaR and CVaR. Subsequently, a numerical approach is employed to derive the optimal portfolio strategy. The results, as shown in Table 2, confirm that all CVaR values exceed their respective VaR values, with an upward trend observed as the confidence levels increase. Consequently, the proportion of cash decreases, while the proportions of call and put options increase correspondingly. It is noteworthy that in the absence of an interest rate (i.e., when the interest rate is zero), all VaR and CVaR values are positive, indicating consistent losses from these investments. Hence, the interest rate emerges as a parameter that significantly influences VaR and CVaR values. The findings suggest that CVaR values are contingent upon the confidence level and the interest rate of return. Moreover, based on the data presented in Table 2, Figure 5, and Figure 6, it is evident that a higher value of $\gamma$ correlates with a reduction in the number of call options (indicating increased selling activity) and an increase in the quantity of put options (reflecting augmented buying activity). This connection assumes significance from a financial standpoint as a higher $\gamma$ signifies greater volatility within the CEV model of the market, prompting investors to rely on options as a risk mitigation strategy. Consequently, the numerical analysis provides evidence that skillful construction of options portfolios can assist investors in managing volatility-related risks in the market. Moreover, we observe that CVaR values and the composition of the minimization portfolio depend not only on the confidence level and interest rate but also on the parameter $\gamma$.

In future research, we aim to extend our portfolio optimization approach to other markets such as Dow Jones and Nikkei 225, giving their status as large datasets that attract global investment interest. Additionally, we aspire to enhance our model to optimize portfolios in scenarios involving multiple maturity times, while considering various types of assets, including call and put options with different strike prices.

## Acknowledgements

The authors are grateful to the referee for the careful reading of the paper and for the comments and detailed suggestions which helped us improve considerably the manuscript. The first author was supported by JSPS KAKENHI Grant Number 20K03731. The second author is grateful for the Development and Promotion of Science and Technology Talents Project (DPST) scholarship, which enabled opportunity for this project.

## References

[1] Abramov, V., Klebaner, F. and Liptser, R. (2011). *The Euler-Maruyama approximations for the CEV model*. Discrete and Continuous Dynamical System-B, 16(1), 1–14.

[2] Alexander, S., Coleman, T. F., and Li, Y. (2006). *Minimizing CVaR and VaR for a Portfolio of Derivatives*. Journal of Banking and Finance, 30, 583–605.

[3] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1997). *Thinking Coherently*. Risk, 10, 68–71.

[4] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). *Coherent measures of risk*. Mathematical Finance, 9(3), 203–228.

[5] Cox, J. (1974). *Notes on Option Pricing I: Constant Elasticity of Diffusions*. Unpublished draft, Stanford University.

[6] Hull, J. C. (2012). *Options, futures, and other derivatives*. 8th ed. New Jersey: Pearson.

[7] Kosapong, B., Boonserm, P., and Rakwongwan, U. (2022). *Options portfolio optimization of exotic options written on Mini S&P500 Index in an illiquid market with Conditional Value-at-Risk (CVaR)*. Thai Journal of Mathematics, 169–183.

[8] Krokhmal, P., Palmquist, J., and Uryasev, S. (2002). *Portfolio optimization with conditional Value-at-Risk objective and constraints*. Journal of Risk, 4, 43–68.

[9] Rockafellar, R. T. (1970). *Convex analysis*. Princeton University Press.

[10] Rockafellar, R. T., and Uryasev, S. (2000). *Optimization of conditional value-at-risk*. Journal of risk, 2, 21–42.

[11] Rubinstein, M. (2002). *Markowitz's "portfolio selection": A fifty-year retrospective*. The Journal of Finance, 57(3), 1041–1045.

[12] Sarykalin, S., Serraino, G., and Uryasev, S. (2008). *Value-at-risk vs. conditional value-at-risk in risk management and optimization*. State-of-the-Art Decision-Making Tools in the Information-Intensive Age, Informs, 270-294.

[13] Schroder, M. (1989). *Computing the constant elasticity of variance option pricing formula*. The Journal of Finance, 44(1), 211–219.