Invexity and Weak Efficiency in Multiobjective Optimization Problems

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1. Introduction

In the last few decades, multiobjective optimization problems have been studied by many authors, and necessary and/or sufficient conditions for weak efficiency and/or efficiency have been developed. In most of these results, it seems that convexity mostly played an important part in sufficient conditions for optimality.

Recently, Hanson [2] introduced the concept of invexity in nonlinear programming, and proved that the Kuhn-Tucker necessary conditions for optimality are sufficient conditions and weak duality result holds under the assumption of invexity.

In this paper, we shall consider vector minimization problems, and give necessary and sufficient conditions for weak efficiency under the assumption of invexity. Moreover, we shall consider the Wolfe type dual problems and show that some duality relations hold.

To this aim, in section 2, we shall introduce some notations and definitions. In section 3, we shall consider unconstrained vector minimization problems. First, we shall extend the concept of invexity to vector-valued functions, and examine some properties of invexity, and then, we shall give necessary and sufficient conditions for weak efficiency under the assumption of invexity. In section 4, we shall consider inequality constrained vector minimization problems and give necessary and sufficient conditions for weak efficiency, by using results obtained in section 3. In section 5, we shall consider the Wolfe type dual problems to the problems given in section 4, and show that weak duality and strong duality results hold.

2. Preliminaries

In this section, we shall introduce some notations and definitions, which are used throughout the paper.

Let R^n be the n-dimensional Euclidean space. For any two vectors $x \equiv (x_1, x_2, \cdots, x_n)'$ and $y \equiv (y_1, y_2, \cdots, y_n)'$ in R^n, $\langle x, y \rangle \equiv \sum_{i=1}^{n} x_i y_i$ denotes the inner product, and x' denotes the transpose of the vector x. For any vectors x and y in R^n, we shall use the following notation:

$x \geqq y$ if and only if $x_i \geqq y_i$ for all $i=1, 2, \cdots, n$,

$x \geq y$ if and only if $x_i \geqq y_i$ for all $i=1, 2, \cdots, n$, and $x \neq y$,

$x > y$ if and only if $x_i > y_i$ for all $i=1, 2, \cdots, n$.

Let X be a non-empty subset of R^n and $f: R^n \to R^l$ be a vector-valued function from R^n to R^l given by $f(x) \equiv (f_1(x), f_2(x), \cdots, f_l(x))'$, where $f_i: R^n \to R$, $i=1, 2, \cdots, l$ are real-valued functions. Consider the following vector minimization problem:

$$
\text{(P)} \qquad \begin{cases} \text{minimize} \quad f(x), \\ \text{subject to} \quad x \in X. \end{cases}
$$

Definition 2.1. *A point $x^0 \in X$ is called weak efficient solution to the problem* (P) *if there exists no $x \in X$ such that $f(x) < f(x^0)$.*

In the case where we consider vector maximization problems, the "maximize $f(x)$" should be understood as "minimize $-f(x)$".

In the following, we assume f is differentiable on R^n and denote the gradient of f at $x^0 \in R^n$ by $\nabla f(x^0)$, where $\nabla f(x^0) \equiv (\nabla f_1(x^0), \nabla f_2(x^0), \cdots, \nabla f_l(x^0))'$ denotes the $l \times n$ matrix whose the jth row is vector $\nabla f_j(x^0)'$.

3. Unconstrained Case and Invexity

In this section, we shall consider unconstrained vector minimization problems, and give necessary and sufficient conditions for weak efficiency under the assumption of invexity.

Consider the following unconstrained vector minimization problem:

$$(\text{P1}) \qquad \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & x \in R^n. \end{cases}$$

Lemma 3.1. *If a point $x^0 \in R^n$ is a weak efficient solution to the problem* (P1), *then there exist a vector $\lambda \in R^l$ such that*

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) = 0, \tag{3.1}$$

$$\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_l)' \geq 0. \tag{3.2}$$

Proof. Let $x^0 \in R^n$ be a weak efficient solution to the problem (P1). Then the system

$$\langle \nabla f_i(x^0), d \rangle < 0, \quad i=1, 2, \cdots, l,$$

has no solution $d \in R^n$. Hence, by Gordon's theorem ([4], P.31), there exists a vector $\lambda \in R^l$ such that (3.1) and (3.2) hold. Q.E.D.

In order to derive sufficient conditions for weak efficiency, we introduce the concept of invexity for vector-valued functions.

Definition 3.1. *A differentiable vector-valued function $f: R^n \to R^l$ is said to be invex at $x^0 \in R^n$ if there exists a vector-valued function $\eta: R^n \times R^n \to R^n$ such that*

$$f(x) - f(x^0) \geqq \nabla f(x^0) \eta(x, x^0) \quad \textit{for all } x \in R^n.$$

It is said to be invex on R^n if f is invex at any $x^0 \in R^n$.

Remark 3.1. In case $l=1$, the above definition reduces to Hanson's one. In case $l \geqq 2$, however, even if f_i, $i=1, \cdots, l$, are invex at $x^0 \in R^n$, f is not invex at $x^0 \in R^n$.

Remark 3.2. If f_i, $i=1, \cdots, l$, are differentiable convex functions on R^n, then f is invex on R^n. In fact, it suffices to take $\eta(x, x^0) \equiv x - x^0$, but the converse does not hold in general.

Before giving sufficient conditions for weak efficiency, we shall give the following lemma.

Lemma 3.2. *Suppose that f is invex at $x^0 \in R^n$.*

(i) *If there exists a vector $\lambda \in R^l$ such that* (3.1) *and* (3.2) *hold, then $x^0 \in R^n$ is a global minimum point to the real-valued function $\langle \lambda, f(x) \rangle$ over R^n.*

(ii) *If $x^0 \in R^n$ is a global minimum point to a real-valued function $\langle \lambda, f(x) \rangle$ over R^n for some $\lambda \in R^l$ with $\lambda \geq 0$, then conditions* (3.1) *and* (3.2) *hold.*

Proof. (i). Suppose that there exists a vector $\lambda \in R^l$ such that (3.1) and (3.2) hold. Since f is invex at $x^0 \in R^n$, there exists a vector-valued function $\eta : R^n \times R^n \to R^n$ such that

$$f(x) - f(x^0) \geqq \nabla f(x^0)\eta(x, x^0) \qquad \text{for all} \quad x \in R^n.$$

From (3.1) and (3.2), it follows that

$$\langle \lambda, f(x) \rangle - \langle \lambda, f(x^0) \rangle \geqq 0 \qquad \text{for all} \quad x \in R^n,$$

which shows that $x^0 \in R^n$ is a global minimum point to $\langle \lambda, f(x) \rangle$ over R^n.

(ii) It will be proved easily. Q.E.D.

Now, we shall give necessary and sufficient conditions for weak efficiency.

Theorem 3.1. *Suppose that f is invex at $x^0 \in R^n$. Then $x^0 \in R^n$ is a weak efficient solution to the problem* (P1) *if and only if there exists a vector $\lambda \in R^l$ such that conditions* (3.1) *and* (3.2) *hold.*

Proof. It suffices to prove "if" part. Because "only if" part has been proved in Lemma 3.1. From Lemma 3.2.(i), we have

$$\langle \lambda, f(x) \rangle - \langle \lambda, f(x^0) \rangle \geqq 0 \qquad \text{for all} \quad x \in R^n. \tag{3.3}$$

Suppose that there exists an $\bar{x} \in R^n$ such that $f(\bar{x}) < f(x^0)$.
Since $\lambda \geq 0$, we have $\langle \lambda, f(\bar{x}) \rangle < \langle \lambda, f(x^0) \rangle$,

which contradicts (3.3). Q.E.D.

Theorem 3.2. *Suppose that f is invex at $x^0 \in R^n$. Then $x^0 \in R^n$ is a weak efficient solution to the problem* (P1) *if and only if there exists a vector $\lambda \in R^l$ with $\lambda \geq 0$ such that $x^0 \in R^n$ is a global minimum point to $\langle \lambda, f(x) \rangle$ over R^n.*

Proof. Necessity. Let $x^0 \in R^n$ be a weak efficient solution to the problem (P1). Then, by Lemma 3.1, there exists a vector $\lambda \in R^l$ such that (3.1) and (3.2) hold. Hence, from Lemma 3.2, it follows that $x^0 \in R^n$ is a global minimum point to $\langle \lambda, f(x) \rangle$ over R^n.

Sufficiency. Suppose to the contrary that there exists an $\bar{x} \in R^n$ such that $f(\bar{x}) < f(x^0)$. Since $\lambda \geq 0$, we have

$$\langle \lambda, f(\bar{x}) \rangle < \langle \lambda, f(x^0) \rangle,$$

which contradicts that $x^0 \in R^n$ is a global minimum point to $\langle \lambda, f(x) \rangle$ over R^n. Q.E.D.

Theorem 3.2 shows that unconstrained vector minimization problem (P1) is equivalent to a family of scalar minimization problems, and this equivalency is a well-known result in the convex case.

Remark 3.4. In the above theorem, note that we don't need invexity of f to prove the sufficiency.

In the rest of this section, we shall characterize invexity of vector-valued functions.

Theorem 3.3. *In order that f be invex at $x^0 \in R^n$, it is necessary and sufficient that for all $\lambda \in R^l$ satisfying* (3.1) *and* (3.2) *at $x^0 \in R^n$, $x^0 \in R^n$ is a global minimum point to the real-valued function $\langle \lambda, f(x) \rangle$ over R^n.*

Proof. It suffices to prove sufficiency. Necessity was proved in Lemma 3.2.(i). Let $\lambda \in R^l$ be any vector satisfying conditions (3.1) and (3.2), and $x^0 \in R^n$ be a global minimum point to the real-valued function $\langle \lambda, f(x) \rangle$ over R^n. Then, for each $x \in R^n$, the system

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) = 0,$$
$$\lambda \equiv (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_l)' \geq 0,$$
$$\sum_{i=1}^{l} \lambda_i (f_i(x) - f_i(x^0)) < 0,$$

has no solution $\lambda \in R^l$. Hence, by Gale's theorem ([4], P.33), there exists a vector $\eta(x,\ x^0) \in R^n$, possibly depending upon x^0 and x, such that

$$f(x) - f(x^0) \geqq \nabla f(x^0) \eta(x,\ x^0)$$

holds.

Next, we consider the case that there exists no $\lambda \in R^l$ such that (3.1) and (3.2) hold, that is, the system

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) = 0,$$
$$\lambda \equiv (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_l)' \geq 0,$$

has no solution $\lambda \in R^l$. Then by Gordon's theorem, there exists a vector $\eta(x^0) \in R^n$, possibly depending upon x^0, such that

$$\nabla f(x^0) \eta(x^0) < 0.$$

Hence, for each $x \in R^n$, there exists $t(x) > 0$, possibly depending upon x, such that

$$f(x) - f(x^0) \geqq \nabla f(x^0) t(x) \eta(x^0),$$

which shows that f is invex at $x^0 \in R^n$.　　Q.E.D.

From Theorem 3.3, we have the following corollaries.

Corollary 3.1. *If there exists no* $\lambda \in R^l$ *such that* (3.1) *and* (3.2) *hold at* $x^0 \in R^n$, *then* f *is invex on* $x^0 \in R^n$.

Corollary 3.2 f *is invex* at $x^0 \in R^n$ *if and only if for all* $\lambda \in R^l$ *satisfying* (3.1) *and* (3.2), $\langle \lambda,\ f(x) \rangle$ *is invex at* $x^0 \in R^n$.

4. Constrained Case

In this section, we shall consider inequality constrained vector minimization problems, and give necessary and sufficient coditions for weak efficiency, by using the results obtained in the preceding section.

Let $g_1, g_2, \cdots, g_m$, be real-valued functions defined on R^n, and define $g: R^n \to R^m$ by $g(x) \equiv (g_1(x), \cdots, g_m(x))'$. Consider the following inequality constrained vector minimization problem:

$$\text{(P2)} \qquad \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & x \in X, \end{cases}$$

where $X \equiv \{x \in R^n | g(x) \leqq 0\}$.

Let $x^0 \in R^n$ be a feasible solution to the problem (P2). We set $I(x^0) \equiv \{i \in \{1, \cdots, m\} | g_i(x^0) = 0\}$ and $g_I(x^0) \equiv (g_i(x^0))'$, $i \in I(x^0)$.

First, we shall give the Fritz-John type necessary conditions for weak efficiency to the problem (P2).

Lemma 4.1. *If $x^0 \in R^n$ is a weak efficient solution to the problem* (P2), *then there exist vectors $\lambda \in R^l$ and $\mu \in R^m$ such that*

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x^0) = 0, \tag{4.1}$$

$$\langle \mu, g(x^0) \rangle = 0, \tag{4.2}$$

$$(\lambda, \mu) \geq 0, \tag{4.3}$$

where $\lambda \equiv (\lambda_1, \lambda_2, \cdots, \lambda_l)'$, and $\mu \equiv (\mu_1, \mu_2, \cdots, \mu_m)'$.

Proof. Let $x^0 \in R^n$ be a weak efficient solution to the problem (P2). Then the system

$$\langle \nabla f_i(x^0), d \rangle < 0, \quad i = 1, 2, \cdots, l,$$
$$\langle \nabla g_j(x^0), d \rangle < 0, \quad j \in I\ (x^0)$$

has no solution $d \in R^n$. Hence, by Gordon's theorem, there exist $\lambda_i \in R$, $i = 1, \cdots, l$, and $\mu_j \in R$, $j \in I(x^0)$, not all zero such that

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x^0) + \sum_{j \in I(x^0)} \mu_j \nabla g_j(x_0) = 0,$$

$$\lambda_i \geqq 0,\ i = 1, \cdots, l,\ \mu_j \geqq 0,\ j \in I(x^0).$$

Since $g_j(x^0) = 0$ for $j \in I(x^0)$, we have $\mu_j g_j(x^0) = 0$. By setting $\mu_j = 0$ for $j \notin I(x^0)$, we have (4.1), (4.2) and (4.3). Q.E.D.

In order to derive the Kuhn-Tucker type necessary conditions, we need a constraint qualification. Now, we assume that Slater type constraint qualification holds, that is, there exists a vector $\bar{x} \in R^n$ such that $g_j(\bar{x})<0$, $j \in I(x^0)$ in the problem (P2).

Lemma 4.2. *Let $x^0 \in R^n$ be a feasible solution to the problem* (P2). *Suppose that there exists a vector $\bar{x} \in R^n$ such that $g_j(\bar{x})<0$, $j \in I(x^0)$, and g_I is invex at $x^0 \in R^n$. If $x^0 \in R^n$ is a weak efficient solution to the problem* (P2), *then there exist vectors $\lambda \in R^l$ and $\mu \in R^m$ such that*

$$\sum_{i=1}^{l}\lambda_i \nabla f_i(x^0)+\sum_{j=1}^{m}\mu_j \nabla g_j(x^0)=0, \tag{4.1}$$

$$\langle \mu,\ g(x^0)\rangle=0, \tag{4.2}$$

$$\lambda \geq 0, \text{ and } \mu \geqq 0. \tag{4.4}$$

Proof. By Lemma 4.1, it suffices to prove $\lambda \geq 0$. Suppose, to the contrary, that $\lambda=0$. Then from (4.1), we have

$$\sum_{j\in I(x^0)}\mu_j \nabla g_j(x^0)=0.$$

Since g_I is invex at $x^0 \in R^n$, there exists a vector-valued function $\eta: R^n \times R^n \to R^n$ such that

$$g_I(x)-g_I(x^0) \geqq \nabla g_I(x^0)\eta(x,\ x^0) \qquad \text{for all} \quad x \in R^n.$$

Since $\mu_j \geqq 0$, $j \in I(x^0)$ and $\mu_j=0$, $j \notin I(x^0)$, in view of (4.2), we have

$$\langle \mu,\ g(x)\rangle \geqq 0 \qquad \text{for all} \quad x \in R^n. \tag{4.5}$$

On other hand, since $g_j(\bar{x})<0$ and $\mu_j \geqq 0$, $j \in I(x^0)$, and $\mu_j>0$ for at least one $j \in I(x^0)$, we have $\langle \mu,\ g(\bar{x})\rangle<0$, which contradicts (4.5). Therefore, we have $\lambda \geq 0$. Q.E.D.

Remark 4.1. In Lemma 4.2, we may assume that Cottle's constraint qualification holds; that is, the system

$$\sum_{j\in I(x^0)}\mu_j \nabla g_j(x^0)=0,$$

$$\mu_j \geqq 0,\ j \in I(x^0), \text{ and } \mu_j>0 \text{ for at least one } j,$$

has no solution $\mu_j \in R$, $j \in I(x^0)$. In this case, from Corollary 3.1, it follows that g_I is invex at $x^0 \in R^n$, and Cottle's constraint qualification is equivalent to Slater's.

Let $h: R^n \to R^l \times R^m$ define by $h \equiv (f_1, \cdots, f_l, g_1, \cdots, g_m.)'$. Now, we shall give necessary and sufficient conditions for weak efficiency.

Theorem 4.1. *Let $x^0 \in R^n$ be a feasible solution to the problem* (P2). *Suppose that there exists a vector $\bar{x} \in R^n$ such that $g_j(\bar{x})<0$, $j \in I(x^0)$, and that h is invex at $x^0 \in R^n$.* Then *$x^0 \in R^n$ is a weak efficient solution to the problem* (P2) *if and only if there exist vectors $\lambda \in R^l$ and $\mu \in R^m$ such that* (4.1), (4.2), *and* (4.4) *hold.*

Proof. Necessity was proved in Lemma 4.2. Hence, we shall prove sufficiency. Suppose that (4.1), (4.2), and (4.4) hold. Since h is invex at $x^0 \in R^n$, from Lemma 3.2.(i), we have

$$\langle \lambda, f(x)\rangle + \langle \mu, g(x)\rangle \geqq \langle \lambda, f(x^0)\rangle + \langle \mu, g(x^0)\rangle \qquad \text{for all} \quad x \in R^n.$$

Since $\mu \geqq 0$ and $g(x) \leqq 0$ for all $x \in X$, in view of (4.2), we have

$$\langle \lambda, f(x)\rangle \geqq \langle \lambda, f(x^0)\rangle \qquad \text{for all} \quad x \in X,$$

which implies that $x^0 \in R^n$ is a weak efficient solution to the problem (P2). Q.E.D.

5. Duality

In this section, we shall consider the Wolfe type dual problem to the problem (P2), and show that weak duality and strong duality results hold between them. Consider the following vector maximization problem:

$$\text{(D)} \qquad \begin{cases} \underset{x,\,\lambda,\,\mu}{\text{maximize}} \quad f(x) + \langle \mu, g(x)\rangle e \\ \text{subject to} \end{cases}$$

$$\sum_{i=1}^{l} \lambda_i \nabla f_i(x) + \sum_{j=1}^{m} \mu_j \nabla g_j(x) = 0, \tag{5.1}$$

$$\lambda \geq 0, \ \sum_{i=1}^{l} \lambda_i = 1, \ \mu \geqq 0, \tag{5.2}$$

where $e \equiv (1, 1, \cdots, 1)'$, $e \in R^l$.

First, we shall prove weak duality result holds between the problem (P2) and the problem (D).

Theorem 5.1. *Let $x \in R^n$ be any feasible solution to the problem* (P2), *and* (y, λ, μ) *be any feasible solution to the problem* (D). *Suppose that* $h \equiv (f_1, \cdots, f_l, g_1, \cdots, g_m)'$ *is invex on* R^n. *Then, following inequality cannot hold*:

$$f(x) < f(y) + \langle \mu, g(y) \rangle e. \tag{5.3}$$

Proof. Since h is invex on R^n, in view of (5.1), from Lemma 3.2, we have

$$\langle \lambda, f(x) \rangle + \langle \mu, g(x) \rangle \geqq \langle \lambda, f(y) \rangle + \langle \mu, g(y) \rangle \qquad \text{for all} \quad x \in R^n.$$

From (5.2), we have

$$\langle \lambda, f(x) \rangle + \sum_{i=1}^{l} \lambda_i \langle \mu, g(x) \rangle \geqq \langle \lambda, f(y) \rangle + \sum_{i=1}^{l} \lambda_i \langle \mu, g(y) \rangle.$$

Hence, by using $\langle \mu, g(x) \rangle \leqq 0$, it follows that

$$\langle \lambda, f(x) \rangle \geqq \langle \lambda, f(y) + \langle \mu, g(y) \rangle e \rangle. \tag{5.4}$$

Since $\lambda \geq 0$, the following inequality cannot hold:

$$f(x) < f(y) + \langle \mu, g(y) \rangle e. \qquad \text{Q.E.D.}$$

Theorem 5.2. *Let $x^0 \in R^n$ be a weak efficient solution to the problem* (P2). *Suppose that the assumptions of Theorem* 4.1 *hold. Then there exist vectors $\lambda \in R^l$ and $\mu \in R^m$ such that* (x^0, λ, μ) *is a weak efficient solution to the problem* (D).

Proof. Let $x^0 \in R^n$ be a weak efficient solution to the problem (P2). Then, from Theorem 4.1, there exist vectors $\lambda \in R^l$ and $\mu \in R^m$ such that (5.1), (5.2), and (5.4) hold. Without any loss of generality, we may assume $\langle \lambda, e \rangle = 1$. Therefore, (x^0, λ, μ) is a feasible solution to the problem (D). It follows from Theorem 5.1 that for any other feasible solution $(y, \hat{\lambda}, \hat{\mu})$ to the problem (D),

$$f(x^0) < f(y) + \langle \hat{\mu}, g(y) \rangle e$$

does not hold. Since $\langle \mu, g(x^0) \rangle = 0$, it follows that

$$f(x^0) + \langle \mu, g(x^0) \rangle e < f(y) + \langle \hat{\mu}, g(y) \rangle e$$

cannot hold. This implies that (x^0, λ, μ) is a weak efficient solution to the problem (D). Q.E.D.

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