博士論文

A Probabilistic Approach to the Zero-Mass Limit Problem for a Spinless Particle in Relativistic Quantum Mechanics

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1. Introduction.

In this thesis, we want to make probabilistic approach to some limit problem in relativistic quantum mechanics.

Quantum mechanics is a theory to explain the phenomena in space-time $\mathbf{R}^d \times \mathbf{R} = \mathbf{R}^{d+1}$ which are caused by atoms, molecules and elementary particles. The measured results of the quantities of the quantum mechanical particle, for example, the total energy in other words *Hamiltonian*, the position and the momentum distribute in general probabilistically. This point is different from classical mechanics, which is deterministic theory. Such physical quantities are called *observables* and are expressed by self-adjoint operators in $L^2(\mathbf{R}^d)$. Especially, the quantum mechanical Hamiltonian, often written by H, is called *Schrödinger operator.* The *state* of the particle is also expressed by a non-zero element of $L^2(\mathbf{R}^d)$. When the state of the particle is $\psi \in L^2(\mathbf{R}^d)$, the probability for the particle to be found in a region $M \subset \mathbf{R}^d$ is given by $(\|\psi \mathbf{1}_M\|_2 / \|\psi\|_2)^2$, where $\mathbf{1}_M$ is the indicator function of the subset M and $\|\cdot\|_2$ is the $L^2(\mathbf{R}^d)$-norm. Furthermore, if $(\cdot,\cdot)$ is the inner product of $L^2(\mathbf{R}^d)$, the measured results of the Hamiltonian H is given by $(H\psi, \psi)$. We note that this value is a real number due to the self-adojointness of H in $L^2(\mathbf{R}^d)$. The time evolution of the state of the particle is described by the *Schrödinger equation*:

$$i\hbar \frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t), \quad x \in \mathbf{R}^d,\ t > 0. \tag{1.1}$$

Here $\hbar = h/2\pi$, the *Planck's constant* h divided by 2π. The solution $\psi(x,t)$ of (1.1) is called *wave function.* In the following, $\hbar$ is taken to be 1 without loss of generality. Equation (1.1) is obtained through *quantization*, a *mysterious* procedure which shifts from classical mechanical system to quantum mechanical system.

Now, let us consider a spinless particle with mass m under the influence of the magnetic potential $A(x)$ and the electric potential $V(x)$. In nonrelativistic case, the corresponding classical Hamiltonian is given by the right-hand side of the equation

$$E = \frac{1}{2m}\left(\xi - \frac{e}{c}A(x)\right)^2 + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d. \tag{1.2}$$

Here E is the total energy, and ξ and x are the momentum and position for the particle. The constants c and e are the light velocity and the electric charge. Then the quantization procedure consists of changing the total energy E on the left to the differential operator $i\frac{\partial}{\partial t}$, and the variables ξ and x on the right to the differential operator $-i\nabla$ and the multiplication operator $x \times \cdot$, respectively, and consequently the kinetic energy $\frac{1}{2m}(\xi - \frac{e}{c}A(x))^2$ and the electric potential $V(x)$ on the right, respectively, to the two operators in $L^2(\mathbf{R}^d)$, the self-adjoint operator $\frac{1}{2m}(-i\nabla - \frac{e}{c}A(x))^2$ and the multiplication operator $V(x)\times$ by function $V(x)$. Then, assuming all these operators to apply to a function $\psi(x,t)$, we obtain the Schrödinger equation (1.1) with

$$H = \frac{1}{2m}\Big(-i\nabla - \frac{e}{c}A(x)\Big)^2 + V(x). \tag{1.3}$$

$H = -\frac{1}{2m}\Delta$ when $A \equiv 0$ and $V \equiv 0$, where Δ is the Laplacian $\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_d^2}$ on $\mathbf{R}^d$.

One can deal with equation (1.1) from probabilistic viewpoint by going from *Minkowski space-time* to *Euclidian space-time.* Namely, we change time t to *imaginary time* $-it$ and put $u(x,t) := \psi(-it, x)$ in (1.1), leading to *imaginary time Schrödinger equation*

$$\frac{\partial}{\partial t}u(x,t) = -Hu(x,t), \quad x \in \mathbf{R}^d,\ t > 0. \tag{1.4}$$

It is well known that, for H in (1.3), the solution of the Cauchy problem for (1.4) with initial data $g \in L^2(\mathbf{R}^d)$ is given by the following *Feynman-Kac-Itô formula* ([21]):

$$u(x,t) = \int_{C_0} e^{-i\frac{e}{c}\int_0^t A(x+B(s))\cdot dB(s) - \frac{i}{2}\frac{e}{c}\int_0^t \mathrm{div}A(x+B(s))ds - \int_0^t V(x+B(s))ds} g(x+B(t))d\mu^m(B). \tag{1.5}$$

Here $\int_0^t A(x+B(s)) \cdot dB(s)$ is called *Itô's stochastic integral* ([21], [13], [1]), and μ^m is a probability measure, called *Wiener measure*, on the space $C_0 := C_0([0,\infty) \to \mathbf{R}^d)$ of continuous paths $B : [0,\infty) \to \mathbf{R}^d$ with $B(0) = 0$ such that

$$\mu^m(B \in C_0; B(t) \in dy) = \Big(\frac{m}{2\pi t}\Big)^{d/2} e^{-\frac{m}{2t}|y|^2}dy, \quad \xi \in \mathbf{R}^d,\ t > 0$$
$$\int_{C_0} e^{i\xi\cdot B(t)}d\mu^m(B) = e^{-t\frac{|\xi|^2}{2m}}, \quad \xi \in \mathbf{R}^d,\ t \geq 0.$$

We note that the function $(\frac{m}{2\pi t})^{d/2}e^{-\frac{m}{2t}|y|^2}$ is the fundamental solution of (1.4) with $H = -\frac{1}{2m}\Delta$, i.e., the *heat equation*

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2m}\Delta u(x,t), \quad x \in \mathbf{R}^d,\ t > 0.$$

and so this function is also called *heat kernel.*

In this thesis, we consider a spinless particle with mass m in relativistic quantum mechanics (for example, pions and electron ignored spin). For this purpose, we treat the imaginary-time Schrödinger equation (1.4) with the quantum Hamiltonian H associated with the classical Hamiltonian symbol on the right-hand side of the equation

$$E = \sqrt{c^2(\xi - eA(x))^2 + m^2c^4} - mc^2 + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d. \tag{1.6}$$

We may also use the expression $\sqrt{c^2(\xi - eA(x))^2 + m^2c^4} + V(x)$ instead, without subtracting *rest energy* mc^2. However, the use of expression (1.6) will enable us to notice equation (1.4) with the *Weyl quantized relativistic Hamiltonian* as H to have a deep connection with *Lévy process* (see Definition 6.3 in subsection 6.1) in probability theory.

The aim of this work is to show the convergence of the solutions of the Cauchy problem for (1.4) as mass-parameter m goes to zero. We call this problem *the zero-mass limit problem*. It is in fact by use of the probabilistic framework with Lévy process just briefly mentioned above that we are going to show it, though it was already shown by using operator theory [6] and pseudo-differential calculus [18]. It may amount to showing a kind of the limit theorems, which Kasahara–Watanabe [14] discussed in the framework of *semimartingales* (see Definition 6.8 and (6.5) in subsection 6.2), for a sequence of point processes and their certain functionals represented by stochastic integrals. In this thesis, we deal with a sequence of slightly more general functionals of special kind of Lévy processes having no Gaussian part, i.e., *pure-jump* Lévy processes. More precisely, these functionals are given by the exponential semimartingales.

Finally, in passing, we note here also that there is another limit problem when the light velocity c goes to infinity (*nonrelativistic limit problem*), for which we refer [4], [6], [19], [17, pp. 257–260].

This thesis is organized as follows: In Section 2, we describe the framework developed by Ichinose mentioned above ([5], [6], [7]) to treat our relativistic problem. In particular, we introduce Weyl quantized relativistic Hamiltonian and give a path integral representation for the solution of the Cauchy problem for the imaginary-time relativistic Schrödinger equation (1.4), which is the very formula for relativistic case corresponding to the Feynman–Kac–Itô formula (1.5) for nonrelativistic case. In Section 3, we state our results concerning the zero-mass limit problem. In Section 4 and Section 5, we prove our results. In Section 6, Appendix, we collect some basic notations from the probability theory, such as Lévy process, semimartingale and their relevant formulas.

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2. Weyl quantized relativistic Hamiltonian and path integral formula.

In expression (1.6) for the relativistic classical Hamiltonian, we may now assume without loss of generality that c and e are also equal to 1. Therefore equation (1.6) turns out

$$E = \sqrt{(\xi - A(x))^2 + m^2} - m + V(x), \quad (\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d. \tag{2.1}$$

For $N = 1, d$, we denote by $C_b^\infty(\mathbf{R}^d; \mathbf{R}^N)$ the space of $\mathbf{R}^N$-valued C^∞-functions on $\mathbf{R}^d$ which are bounded together with all their derivatives. For $A \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^d)$, let us define the operator H_A^m in $L^2(\mathbf{R}^d)$ with domain $C_0^\infty(\mathbf{R}^d)$, which is corresponding to $\sqrt{(\xi - A(x))^2 + m^2}$ in (2.1) as

$$(H_A^m f)(x) := \frac{1}{(2\pi)^d} \text{Os-} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\big(\xi - A(\tfrac{x+y}{2})\big)^2 + m^2} f(y) dy d\xi, \quad f \in C_0^\infty(\mathbf{R}^d). \tag{2.2}$$

Here "Os"means *oscillatory integral* (cf. [16, I, pp.45–53]) and the integral on the right-hand side of (2.2) is defined by the limit

$$\lim_{\varepsilon \downarrow 0} \iint_{\mathbf{R}^d \times \mathbf{R}^d} e^{i(x-y)\cdot\xi} \sqrt{\big(\xi - A(\tfrac{x+y}{2})\big)^2 + m^2} \chi(\varepsilon y, \varepsilon y) f(y) dy d\xi,$$
$$\chi \in \mathcal{S}(\mathbf{R}^d \times \mathbf{R}^d) \text{ with } \chi(0,0) = 1.$$

We note that this limit exists and is independent of the choice of cutoff function χ ([9, Proposition 2.1]). H_A^m is called *Weyl pseudo-differential operator with mid-point prescription.* It can be proved that if

$$A \in C_b^\infty(\mathbf{R}^d; \mathbf{R}^d), \quad V \in C_b^\infty(\mathbf{R}^d; \mathbf{R}), \tag{2.3}$$

then

$$H_{A,V}^m := H_A^m + V \tag{2.4}$$

is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$ ([11, Theorem 2.1 (i)]). Especially, if $A \equiv 0$, then $H_0^m = \sqrt{-\Delta + m^2}$ so that $(H_0^m)^2 = (\sqrt{-\Delta + m^2})^2 = -\Delta + m^2$.

In the following, we see that this operator H_0^m is related to Lévy process. First, we note that the function $\xi \mapsto e^{-[\sqrt{\xi^2+m^2}-m]}$ is positive definite (e.g. [1, Proof of Theorem 1.2.26], [11, Lemma 4.1(i)]), namely

$$\sum_{i,j=1}^{k} e^{-[\sqrt{(\xi_i-\xi_j)^2+m^2}-m]} z_i \overline{z_j} \geq 0, \quad \xi_1, \ldots, \xi_k \in \mathbf{R}^d,\ z_1, \ldots, z_k \in \mathbf{C},\ k \in \mathbf{N}.$$

It is seen by Bochner's theorem ([1, Theorem 1.1.12], [20, Proposition 2.5 (i)]) that $\xi \mapsto e^{-[\sqrt{\xi^2+m^2}-m]}$ is the characteristic function of some probability measure on $\mathbf{R}^d$. It is easy to see that this probability measure is *infinitely divisible* (see Definition 6.4 in subsection 6.1). Let D_0 be the set of the right-continuous paths $X : [0, \infty) \to \mathbf{R}^d$ with left-hand limits and $X(0) = 0$ and put $\mathcal{F} := \sigma(X(s); s \leq t)$. By Theorem 6.1 in subsection 6.1, there exists a probability measure λ^m on $(D_0, \mathcal{F})$ such that $X = \{X(t)\}_{t\geq 0}$ is a *Lévy process* (see Definition 6.3 in subsection 6.1) with respect to λ^m and

$$e^{-t[\sqrt{\xi^2+m^2}-m]} = E^m[e^{i\xi\cdot X(t)}], \quad t \geq 0,\ \xi \in \mathbf{R}^d. \tag{2.5}$$

Here $E^m[\cdots]$ denotes the expectation over D_0 with respect to λ^m. By (2.5) and the *Lévy-Khintchine formula* (see Theorem 6.2 and (6.2) in subsection 6.1), there exists a *Lévy measure* $n^m(dy)$ such that

$$\sqrt{\xi^2+m^2} - m = -\int_{|y|>0} \big(e^{i\xi\cdot y} - 1 - i\xi\cdot y\mathbf{1}_{|y|<1}\big) n^m(dy), \quad \xi \in \mathbf{R}^d. \tag{2.6}$$

Here $n^m(dy)$ has density ([5, (2.2), p.268])

$$n^m(y) = n^m(|y|) = \begin{cases} 2\left(\dfrac{m}{2\pi}\right)^{(d+1)/2} \dfrac{K_{(d+1)/2}(m|y|)}{|y|^{(d+1)/2}}, & m > 0, \\ \dfrac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \dfrac{1}{|y|^{d+1}}, & m = 0. \end{cases} \tag{2.7}$$

Furthermore, (2.6) and the *Lévy-Itô decomposition* (see Theorem 6.3 in subsection 6.1) imply that

$$X(t) = \int_0^t \int_{|y|\geq 1} yN_X(dsdy) + \int_0^t \int_{0<|y|<1} y\widetilde{N_X^m}(dsdy), \quad \lambda^m\text{-a.s. } X \in D_0. \tag{2.8}$$

Here $\int_0^t := \int_{(0,t]}$, $N_X(dsdy)$ is a counting measure on $(0,\infty)\times(\mathbf{R}^d\setminus\{0\})$ defined by

$$N_X(G) := \#\{s>0; (s, X(s)-X(s-))\in G\}, \quad G\in\mathcal{B}(0,\infty)\times\mathcal{B}(\mathbf{R}^d\setminus\{0\}) \tag{2.9}$$

and $\widetilde{N_X^m}(dsdy) := N_X(dsdy) - dsn^m(dy)$. $\{N_X(G)\}_G$ is a stationary Poisson random measure with intensity measure $dsn^m(dy)$ with respect to λ^m (see Definition 6.5 in subsection 6.1).

Next, we consider the probability distribution $\lambda^m(X; X(t)\in dy)$ on $\mathbf{R}^d$. Let $k_0^m(y,t)$ be the fundamental solution of the heat equation of (1.4) with $H_0^m - m$, i.e.,

$$\frac{\partial}{\partial t}u(x,t) = -[H_0^m - m]u(x,t), \quad x\in\mathbf{R}^d,\ t>0.$$

It can be seen that

$$\widehat{k_0^m(\cdot,t)}(\xi) = e^{-t[\sqrt{\xi^2+m^2}-m]}, \quad \xi\in\mathbf{R}^d,\ t>0, \tag{2.10}$$

where for $\varphi\in\mathcal{S}(\mathbf{R}^d)$, we define the Fourier transform of φ by $\widehat{\varphi}(\xi) := \int_{\mathbf{R}^d} e^{-ix\cdot\xi}\varphi(x)dx$. Taking $\xi=0$ in (2.10), we have

$$\int_{\mathbf{R}^d} k_0^m(y,t)dy = 1, \quad t>0. \tag{2.11}$$

The explicit expression of $k_0^m(y,t)$ is given ([5, (2.4), p.269]) by

$$k_0^m(y,t) = \begin{cases} 2\left(\dfrac{m}{2\pi}\right)^{(d+1)/2} \dfrac{te^{mt}K_{(d+1)/2}(m(|y|^2+t^2)^{1/2})}{(|y|^2+t^2)^{(d+1)/4}}, & m>0,\\ \dfrac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}\dfrac{t}{(|y|^2+t^2)^{(d+1)/2}}, & m=0. \end{cases} \tag{2.12}$$

Here K_ν and Γ stand for the modified Bessel function of the third kind of order ν and the Gamma function, respectively ([3]). We note that for each $t>0$, $k_0^0(y,t)$ is the probability density of Cauchy distribution ([20, example 2.12]). By (2.5), (2.10) and $k_0^m(y,t) = k_0^m(-y,t)$, we have

$$\lambda^m(X\in D_0; X(t)\in dy) = k_0^m(y,t)dy. \tag{2.13}$$

(2.13) implies that

$$\lambda^m(X\in D_0; (X(t_1),\ldots,X(t_k))\in dy_1\cdots dy_k) = \prod_{j=1}^k k_0^m(y_j-y_{j-1}, t_j-t_{j-1})dy_1\cdots dy_k, \tag{2.14}$$

where $0 = t_0 < t_1 < \cdots < t_k < \infty$, $k \geq 1$ and $y_0 := 0$.

Now, let us see that $H^m_{A,V}$ can be defined for A and V having singularities. As shown in [5], H^m_A can be written as the singular integral operator given by

$$(H^m_A f)(x) = mf(x) - \lim_{r\downarrow 0} \int_{|y|\geq r} \left[e^{-iy\cdot A(x+\frac{1}{2}y)} f(x+y) - f(x)\right] n^m(dy), \quad f \in C_0^\infty(\mathbf{R}^d). \tag{2.15}$$

The limit on the right of (2.15) exists a.s. x as well as in the $L^2(\mathbf{R}^d)$-norm for $A \in L^{2+\delta}_{\rm loc}(\mathbf{R}^d)$ with $\delta > 0$ ([12, Lemma 4.1]). It can be proved that if

$$A \in L^{2+\delta}_{\rm loc}(\mathbf{R}^d; \mathbf{R}^d) \text{ for some } \delta > 0, \quad 0 \leq V \in L^2_{\rm loc}(\mathbf{R}^d; \mathbf{R}), \tag{2.16}$$

then $H^m_{A,V}$ is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$, and its closure is bounded from below by m ([6, Proposition 2.1], [12, Theorem 4.3]). We have by (2.15)

$$\begin{aligned} h^m_{A,V}[f] :&= (H^m_{A,V} f, f) \\ &= m\|f\|_2^2 + \frac{1}{2} \iint_{|x-y|>0} |e^{-i(x-y)\cdot A(\frac{x+y}{2})} f(x) - f(y)|^2 n^m(x-y) dxdy \\ &\quad + \int_{\mathbf{R}^d} V(x)|f(x)|^2 dx, \quad f \in C_0^\infty(\mathbf{R}^d), \end{aligned} \tag{2.17}$$

Assume

$$A \in L^{1+\delta}_{\rm loc}(\mathbf{R}^d; \mathbf{R}^d) \text{ for some } \delta > 0, \quad 0 \leq V \in L^1_{\rm loc}(\mathbf{R}^d; \mathbf{R}) \tag{2.18}$$

and define $h^m_{A,V}[\cdot]$ as (2.17) with form domain $Q(h^m_{A,V}) := \{f \in L^2(\mathbf{R}^d); h^m_{A,V}[f] < \infty\}$. $C_0^\infty(\mathbf{R}^d)$ is a subspace of $Q(h^m_{A,V})$. The quadratic form $h^m_{A,V}$ is closed with respect to $Q(h^m_{A,V})$-norm $\|\cdot\|_{h^m_{A,V}} := (h^m_{A,V}[\cdot] + \|\cdot\|_2^2)^{1/2}$ and is symmetric, namely,

$$h^m_{A,V}[f,g] := \frac{1}{4}\left(h^m_{A,V}[f+g] - h^m_{A,V}[f-g] + ih^m_{A,V}[f+ig] - ih^m_{A,V}[f-ig]\right)$$

satisfies $h^m_{A,V}[f,g] = \overline{h^m_{A,V}[g,f]}$ ([12, Lemma 3.1]). Therefore, there exists ([15, VI, Theorem 2.1, Theorem 2.6]) a unique selfadjoint operator $H^m_{A,V}$ in $L^2(\mathbf{R}^d)$ with domain $D(H^m_{A,V})$ bounded from below by m such that

$$(H^m_{A,V} f, g) = h^m_{A,V}[f,g], \quad f \in D(H^m_{A,V}),\ g \in Q(h^m_{A,V}).$$

It can be proved that $C_0^\infty(\mathbf{R}^d)$ is dense in $Q(h_{A,V}^m)$ with respect to $\|\cdot\|_{h_{A,V}^m}$, namely $C_0^\infty(\mathbf{R}^d)$ is a form core (cf. [15, p. 317]) of $H_{A,V}^m$ ([12, Theorem 3.4]).

The path integral formula for imaginary time Schrödinger equation (1.4) with Weyl quantized Hamiltonian is as follows:

Theorem 2.1. ([11], [7], [8], [9]) *If (2.18) holds, then*

$$u^m(x,t) := E^m[e^{-S^m(t,x,X)}g(x+X(t))] \tag{2.19}$$

is the solution of (1.4) with $H = H_{A,V}^m - m$ with initial data $u^m(\cdot,0) = g \in L^2(\mathbf{R}^d)$. Here

$$\begin{aligned} S^m(t,x,X) := i\Big(&\int_0^t \int_{|y|\geq 1} A(x+X(s-)+\tfrac{1}{2}y)\cdot y N_X(dsdy) \\ &+ \int_0^t \int_{0<|y|<1} A(x+X(s-)+\tfrac{1}{2}y)\cdot y\widetilde{N_X^m}(dsdy) \\ &+ \int_0^t ds\ p.v. \int_{0<|y|<1} A(x+X(s)+\tfrac{1}{2}y)\cdot y n^m(dy)\Big) \\ &+ \int_0^t V(x+X(s))ds \end{aligned} \tag{2.20}$$

and p.v. means the principal value, namely p.v. $\int_{0<|y|<1} A(x+X(s)+\frac{1}{2}y)\cdot yn^m(dy)$ *is* $L^2(D_0;\lambda^m)$*-limit of*

$$\int_{\varepsilon<|y|<1} A(x+X(s)+\tfrac{1}{2}y)\cdot yn^m(dy).$$

as $\varepsilon \downarrow 0$.

Remark 2.1. (1) If A is locally Hölder-continuous, then the above limit is equal to

$$\int_{0<|y|<1} [A(x+X(s)+\tfrac{1}{2}y) - A(x+X(s))]\cdot yn^m(dy)$$

because $\int_{0<|y|<1} |y|^{1+\delta}dy < \infty$ for $\delta > 0$ by (2.7).
(2) In (2.8) and (2.20) , the integration regions $|y| \geq 1$ and $0 < |y| < 1$ may be replaced by $|y| \geq \delta$ and $0 < |y| < \delta$, respectively, for any $\delta > 0$.

Proof of Theorem 2.1 (sketch). We will prove Theorem 2.1 only under condition (2.3). For $t > 0$, let $T^m(t)$ be a bounded operator of $L^2(\mathbf{R}^d)$ defined by

$$(T^m(t)g)(x) = \int_{\mathbf{R}^d} k_0^m(x-y,t)\exp\left[-iA(\tfrac{x+y}{2})\cdot(y-x) - V(\tfrac{x+y}{2})t\right]g(y)dy, \quad g \in L^2(\mathbf{R}^d).$$

It can be proved that $T^m(\frac{t}{n})^n g$ converges to the solution of (1.4) with $H = H^m_{A,V} - m$ with initial data g as $n \to \infty$ with respect to $L^2(\mathbf{R}^d)$-norm ([11, p.246], [9, Proposition 4.2]). So, if we show $\left(T^m(\frac{t}{n})^n g\right)(x)$ converges to the right-hand side of (2.19) with (2.20) as $n \to \infty$, then we can see the proof of Theorem 2.1 to be accomplished. First, we note that for $n \geq 1$,

$$\left(T^m(\tfrac{t}{n})^n g\right)(x) = \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \prod_{j=1}^n k_0^m(x_j - x_{j-1}, \tfrac{t}{n})\; e^{-S_n(x_1,\ldots,x_n)} g(x_n) dx_1 \cdots dx_n,$$

where $x_0 = x$ and

$$S_n(x_1, \ldots, x_n) := i \sum_{j=1}^n A\left(\frac{x_{j-1} + x_j}{2}\right) \cdot (x_j - x_{j-1}) + \sum_{j=1}^n V\left(\frac{x_{j-1} + x_j}{2}\right) \frac{t}{n}.$$

By (2.14), we have

$$\left(T^m(\tfrac{t}{n})^n g\right)(x) = E^m\left[e^{-S_n(X)} g(x + X(t))\right],$$

where

$$\begin{aligned} S_n(X) &:= S_n\left(x + X(\tfrac{t}{n}), x + X(\tfrac{2t}{n}), \ldots, x + X(t)\right) \\ &= i \sum_{j=1}^n A\left(x + \frac{X(\frac{j-1}{n}t) + X(\frac{j}{n}t)}{2}\right) \cdot \left(X(\tfrac{j}{n}t) - X(\tfrac{j-1}{n}t)\right) \\ &\quad + \sum_{j=1}^n V\left(x + \frac{X(\frac{j-1}{n}t) + X(\frac{j}{n}t)}{2}\right) \frac{t}{n} \\ &=: iS_{n1}(X) + S_{n2}(X). \end{aligned}$$

It is evident that $S_{n2}(X)$ converges to $\int_0^t V(x + X(t))ds$ as $n \to \infty$, which is the forth term on the right-hand side of (2.20).

As for $S_{n1}(X)$, we have by (2.8) and *Itô's formula* (see Theorem 6.6 in subsection 6.2)

$$\begin{aligned} S_{n1}(X) = &\int_0^t \int_{|y|\geq 1} K_n(s, y, X) N_X(dsdy) + \int_0^t \int_{0<|y|<1} K_n(s, y, X) \widetilde{N_X^m}(dsdy) \\ &+ \int_0^t \int_{0<|y|<1} L_n(s, y, X) ds n^m(dy), \end{aligned}$$

where

$$K_n(s, y, X) = \sum_{j=1}^n \mathbf{1}_{(\frac{j-1}{n}t, \frac{j}{n}t]}(s) \Bigg[A\left(x + \frac{X(\frac{j-1}{n}t) + X(s-) + y}{2}\right) \cdot \left(X(s-) - X(\tfrac{j-1}{n}t) + y\right)$$

$$
\begin{aligned}
&\qquad - A\left(x + \frac{X(\frac{j-1}{n}t) + X(s-)}{2}\right) \cdot \left(X(s-) - X(\tfrac{j-1}{n}t)\right)\Bigg],\\
L_n(s,y,X) = \sum_{j=1}^{n} \mathbf{1}_{(\frac{j-1}{n}t, \frac{j}{n}t]}(s) &\Bigg[A\left(x + \frac{X(\frac{j-1}{n}t) + X(s) + y}{2}\right) \cdot \left(X(s) - X(\tfrac{j-1}{n}t) + y\right)\\
&\qquad - A\left(x + \frac{X(\frac{j-1}{n}t) + X(s)}{2}\right) \cdot \left(X(s) - X(\tfrac{j-1}{n}t)\right)\Bigg]\\
&\qquad - \frac{1}{2}\left((y\cdot\nabla) A\right)\left(x + \frac{X(\frac{j-1}{n}t) + X(s)}{2}\right) \cdot \left(X(s) - X(\tfrac{j-1}{n}t)\right)\\
&\qquad - y \cdot A\left(x + \frac{X(\frac{j-1}{n}t) + X(s)}{2}\right)\Bigg].
\end{aligned}
$$

$K_n(s,y,X)$, $L_n(s,y,X)$ converges to $A(x+X(s-)+\frac{1}{2}y)\cdot y$, $\left[A(x+X(s-)+\frac{1}{2}y) - A(x+X(s-))\right]\cdot y$, respectively as $n \to \infty$. Therefore $S_{n1}(X)$ converges to the sum of the first, the second and the third term on the right-hand side of (2.20) as $n \to \infty$. Thus $\left(T^m(\frac{t}{n})^n g\right)(x)$ converges to the right-hand side of (2.19) with (2.20) as $n \to \infty$. □

3. Results.

In this section, we state our results concerning the zero-mass limit problem. First, we see the convergence of the probability measure λ^m.

Theorem 3.1. (cf. [10]) *λ^m converges weakly to λ^0 as $m \downarrow 0$. Namely, for each bounded continuous function Ψ defined on the metric space D_0 with the metric in [2, p.168],*

$$E^m[\Psi(X)] \to E^0[\Psi(X)], \quad \text{as } m \downarrow 0.$$

The following theorems are our main results, where $0 < T < \infty$ can be taken arbitrary.

Theorem 3.2. *If $A \in C_0^1(\mathbf{R}^d;\mathbf{R}^d)$ and $V \in C_0(\mathbf{R}^d;\mathbf{R})$, then $u^m(\cdot,t)$ converges to $u^0(\cdot,t)$ in $C_\infty(\mathbf{R}^d)$ as $m \downarrow 0$, uniformly on $[0,T]$. Here $C_\infty(\mathbf{R}^d)$ is the Banach space of the continuous functions $g : \mathbf{R}^d \to \mathbf{C}$ with $|g(x)| \to 0$ as $|x| \to \infty$ with norm $\|g\|_\infty := \sup_{x\in\mathbf{R}^d}|g(x)|$.*

Theorem 3.3. *If (2.18) holds, then $u^m(\cdot,t)$ converges to $u^0(\cdot,t)$ in $L^2(\mathbf{R}^d)$ as $m \downarrow 0$, uniformly on* $[0,T]$.

Theorem 3.3 implies the strong resolvent convergence of $H^m_{A,V} - m$ to $H^0_{A,V}$ as $m \downarrow 0$ ([15, IX, Theorem 2.16, p.504]). An immediate consequence is the following result for the solution $\psi^m(x,t)$ of the Cauchy problem for the Schrödinger equation (1.1).

Corollary 3.1. *If (2.18) holds, then $\psi^m(\cdot,t)$ converges to $\psi^0(\cdot,t)$ on $L^2(\mathbf{R}^d)$ as $m \downarrow 0$, uniformly on* $[0,T]$.

These results, Theorem 3.2, Theorem 3.3 and Corollary 3.1, are extensions of those proved in Ichinose–Murayama [10] under stronger condition that $A \in C_0^\infty(\mathbf{R}^d;\mathbf{R}^d)$ and $V \in C_0(\mathbf{R}^d;\mathbf{R})$. However, the proof of Theorem 3.2 in this thesis employs the same argument as in [10, Proof of Theorem 3]. The crucial idea of the proofs of Theorem 3.2 and Theorem 3.3 is to do a change of variable "path ". We will prove these theorems by probabilistic method in Section 4 and 5.

In a future work, we will also consider the zero-mass limit problem for the other two different magnetic Schrödinger operators (mentioned in [8], [9]) corresponding to the same classical relativistic Hamiltonian (2.1).

4. Proof of Theorem 3.1 and Theorem 3.2.

In this section, we show Theorem 3.1 and Theorem 3.2.

4.1 Proof of Theorem 3.1.

To prove Theorem 3.1, first, we have to the check the following three facts ([2, Theorem 13.5]):

(i) The finite dimensional distributions with respect to λ^m converge weakly to those with respect to λ^0 as $m \downarrow 0$.

(ii) For each $t > 0$, the probability measure $\lambda^0(X; X(t) - X(t-\varepsilon) \in dy)$ on $\mathbf{R}^d$ converges weakly to Dirac measure concentrated at the point $0 \in \mathbf{R}^d$ as $\varepsilon \downarrow 0$. Namely, for each bounded continuous function φ defined on $\mathbf{R}^d$,

$$E^0\left[\varphi\left(X(t) - X(t-\varepsilon)\right)\right] \to \varphi(0), \quad \text{as } \varepsilon \downarrow 0.$$

(iii) There exist $\alpha > \frac{1}{2}$ and $\beta > 0$, and a nondecreasing continuous function F on $[0, \infty)$ such that

$$E^m\left[|X(s) - X(r)|^\beta |X(t) - X(s)|^\beta\right] \le [F(t) - F(r)]^{2\alpha}, \quad 0 < m < 1,\ 0 \le r < s < t < \infty.$$

First, as for (i), if $0 = t_0 < t_1 < \cdots < t_k < \infty$, $k \in \mathbf{N}$, then by independent and stationary increments property of Lévy process and (2.5), we have

$$\begin{aligned}
E^m\left[e^{i(\xi_1\cdot X(t_1)+\cdots+\xi_k\cdot X(t_k))}\right] &= E^m\left[e^{i\sum_{j=1}^k(\xi_j+\cdots+\xi_k)\cdot(X(t_j)-X(t_{j-1}))}\right] \\
&= \prod_{j=1}^k E^m\left[e^{i(\xi_j+\cdots+\xi_k)\cdot(X(t_j)-X(t_{j-1}))}\right] \\
&= \prod_{j=1}^k E^m\left[e^{i(\xi_j+\cdots+\xi_k)\cdot X(t_j-t_{j-1})}\right] \\
&= \prod_{j=1}^k e^{-(t_j-t_{j-1})[\sqrt{(\xi_j+\cdots+\xi_k)^2+m^2}-m]} \\
&\to \prod_{j=1}^k e^{-(t_j-t_{j-1})(\xi_j+\cdots+\xi_k)^2}, \quad \text{as } m \downarrow 0, \\
&= E^0\left[e^{i(\xi_1\cdot X(t_1)+\cdots+\xi_k\cdot X(t_k))}\right].
\end{aligned}$$

Therefore we have (i).

Next, (ii) follows from the stochastic continuity of $\{X(t)\}_{t\ge0}$ (see Definition 6.3 (iii) in subsection 6.1).

Finally, we confirm (iii). Since $(d/d\tau)\tau^\nu K_\nu(\tau) = -\tau^\nu K_{\nu-1}(\tau)$ $(\tau > 0, \nu > 0)$ ([3, (21), p.79]) and $\nu \mapsto K_\nu(\tau)$ is strictly increasing in $(0, \infty)$ ([3, (21), p.82]), we have $(d/d\tau)(e^\tau \tau^\nu K_\nu(\tau)) = e^\tau \tau^\nu(K_\nu(\tau) - K_{\nu-1}(\tau)) < 0$ if $0 < \nu < \frac{1}{2}$. Therefore $\tau \mapsto e^\tau \tau^\nu K_\nu(\tau)$ is strictly decreasing in $(0, \infty)$ and so [3, (41), (42), (43), p.10]

$$e^\tau \tau^\nu K_\nu(\tau) \le \lim_{\tau\downarrow0} \tau^\nu K_\nu(\tau) = 2^{\nu-1}\Gamma(\nu). \tag{4.1}$$

Then we have for $0 \le r < s < t < \infty$, $\frac{1}{2} < \beta < 1$,

$$\begin{aligned}
E^m\left[|X(s) - X(r)|^\beta |X(t) - X(s)|^\beta\right] &= \int_{\mathbf{R}^d} |y|^\beta k_0^m(y, s-r)dy \int_{\mathbf{R}^d} |y|^\beta k_0^m(y, t-s)dy \\
&= C(d, \beta)^2((s-r)(t-s))^\beta
\end{aligned}$$

$$
\begin{aligned}
&\times e^{m(s-r)}(m(s-r))^{\frac{1-\beta}{2}} K_{\frac{1-\beta}{2}}(m(s-r)) \\
&\times e^{m(t-s)}(m(t-s))^{\frac{1-\beta}{2}} K_{\frac{1-\beta}{2}}(m(t-s)) \\
\le\ & C(d,\beta)^2 2^{-(1+2\beta)} \Gamma(\tfrac{1-\beta}{2})^2 (t-r)^{2\beta},
\end{aligned}
$$

where in the second equality we use [4, Lemma 3.3 (ii)] with a constant $C(d,\beta)$ depending on d and β. Therefore (iii) holds for $\frac{1}{2} < \beta < 1$, $\alpha := \beta$ and $F(p) := C(d,\beta)^{1/\beta} 2^{-(1+2\beta)/2\beta} \Gamma(\frac{1-\beta}{2})^{1/\beta} p$. □

For $t > 0$, $X \in D_0$, let $\phi_t(X)$ be X restricted the domain $[0,\infty)$ to $[0,t]$. (i), (ii) and (iii) imply that $\lambda^m \phi_t^{-1}$ converges weakly to $\lambda^0 \phi_t^{-1}$ as $m \downarrow 0$ for each $t > 0$. Then we have by [2, Lemma 3, p.173] that λ^m converges weakly to λ^0 as $m \downarrow 0$. □

4.2 Proof of Theorem 3.2.

From (2.19), we have to prove that

$$
\begin{aligned}
u^m(x,t) &= E^m[e^{-S^m(t,x,X)} g(x+X(t))] \\
&\to E^0[e^{-S^0(t,x,X)} g(x+X(t))] = u^0(x,t)
\end{aligned}
$$

as $m \downarrow 0$ in $C_\infty(\mathbf{R}^d)$. But its direct proof seems a little troublesome since both the integrand $e^{-S^m(t,x,X)} g(x+X(t))$ and the probability measure λ^m depend on m. So we want to move from $E^m[\cdots]$ to $E^0[\cdots]$ by a *change of variable* (i.e., *change of probability measure*)

$$
\lambda^m = \lambda^0 \Phi_m^{-1} \tag{4.2}
$$

with path space transformation $\Phi_m : D_0 \to D_0$, which enables us to consider the problem on the same probability space (D_0, λ^0).

If there is such a Φ_m, then we can see by (2.5) and (2.6) that the characteristic feature of the path $X(t)$ and the transformed path $\Phi_m(X)(t)$ is expressed in terms of their associated Lévy measures $n^0(dy)$ and $n^m(dy)$. In fact, for $\xi \in \mathbf{R}^d$, $t \ge 0$, we have

$$
\begin{aligned}
E^0[e^{i\xi\cdot X(t)}] &= \exp\left\{ t \left[\int_{|y|>0} \left(e^{i\xi\cdot y} - 1 - i\xi\cdot y \mathbf{1}_{|y|<1}\right) n^0(dy) \right] \right\}, \\
E^0[e^{i\xi\cdot \Phi_m(X)(t)}] &= E^m[e^{i\xi\cdot X(t)}] \\
&= \exp\left\{ t \left[\int_{|y|>0} \left(e^{i\xi\cdot y} - 1 - i\xi\cdot y \mathbf{1}_{|y|<1}\right) n^m(dy) \right] \right\}.
\end{aligned} \tag{4.3}
$$

So it is presumed to hold that

$$n^m(dy) = n^0\phi_m^{-1}(dy) \tag{4.4}$$

for some mapping $\phi_m\colon \mathbf{R}^d \setminus \{0\} \to \mathbf{R}^d \setminus \{0\}$.

We will determine ϕ_m in such a way that (1) $n^m(dy) = n^0\phi_m^{-1}(dy)$, (2) $\phi_m \in C^1(\mathbf{R}^d \setminus \{0\}; \mathbf{R}^d \setminus \{0\})$, (3) ϕ_m is one to one and onto, (4) $\det D\phi_m(z) \neq 0$ for all $z \in \mathbf{R}^d \setminus \{0\}$, where $D\phi_m(z)$ is the Jacobian matrix of ϕ_m at the point z.

Let $U := \{y \in \mathbf{R}^d \setminus \{0\}; |y| \in U'\}$ for $U' \in \mathcal{B}(0,\infty)$. Introducing the spherical coordinates by $z = r\omega$, $r > 0$, $\omega \in S^{d-1}$, we have

$$n^m(U) = \int_U n^m(|y|)dy = C(d)\int_{U'} n^m(r)r^{d-1}dr,$$

where $C(d)$ is the surface area of the d-dimensional unit ball.

Let us assume that $\phi_m^{-1}(z) = l_m(|z|)\frac{z}{|z|}$ for some non-decreasing C^1 function l_m : $(0,\infty) \to (0,\infty)$. Then we have

$$\begin{aligned} n^0\phi_m^{-1}(U) &= \int_U n^0(l_m(|z|))|z|^{-(d-1)}l_m(|z|)^{d-1}l_m'(|z|)dz \\ &= C(d)\int_{U'} n^0(l_m(r))l_m(r)^{d-1}l_m'(r)dr, \end{aligned}$$

where $l_m'(r) = (d/dr)l_m(r)$. Therefore we have

$$n^m(r)r^{d-1} = n^0(l_m(r))l_m(r)^{d-1}l_m'(r), \quad \text{a.s. } r > 0.$$

If $m > 0$, from (2.7), we have

$$-\frac{d}{dr}l_m(r)^{-1} = 2^{-\frac{d-1}{2}}\Gamma(\tfrac{d+1}{2})^{-1}m^{\frac{d+1}{2}}r^{\frac{d-3}{2}}K_{\frac{d+1}{2}}(mr).$$

We solve this differential equation under boundary condition $l_m(\infty) = \infty$ to get

$$l_m(r) = \frac{2^{\frac{d-1}{2}}\Gamma(\frac{d+1}{2})}{m^{\frac{d+1}{2}}\int_r^\infty u^{\frac{d-3}{2}}K_{\frac{d+1}{2}}(mu)du}. \tag{4.5}$$

Here we note that $0 < \int_r^\infty u^{\frac{d-3}{2}}K_{\frac{d+1}{2}}(mu)du < \infty$ by $K_{\frac{d+1}{2}}(\tau) > 0$ for $\tau > 0$, and ([3, (37), (38), p.9])

$$K_{\frac{d+1}{2}}(\tau) = \left(\frac{\pi}{2}\right)^{1/2}\tau^{-1/2}e^{-\tau}(1+o(1)),\ \tau \uparrow \infty.$$

Proposition 4.1. (i) *$l_m(r)$ is a strictly increasing C^∞ function of $r \in (0,\infty)$ and $l_m(+0) = 0$, $l_m(\infty) = \infty$.*
(ii) *For all $r > 0$, $l_m(r)$ converges to r, strictly decreasingly, as $m \downarrow 0$.*

Proof. (4.1) implies $l_m(+0) = 0$. The other claims of (i) follow from (4.5) and the fact that $K_{(d+1)/2}(\tau)$ is a C^∞ function in $(0,\infty)$. The claim (ii) can be proved by the fact that $\tau^\nu K_\nu(\tau)$ is strictly decreasing in $(0,\infty)$ (cf. proof of (iii) in subsection 4.1), (4.1) and the monotone convergence theorem. $\square$

Now, for $m = 0$, let $l_0(r) := r$. Let us put $\phi_0(z) := z$ and for $m > 0$,

$$\phi_m(z) := l_m^{-1}(|z|)\frac{z}{|z|}, \quad z \in \mathbf{R}^d \setminus \{0\}.$$

Then we have (4.4) and

$$\phi_m^{-1}(z) = l_m(|z|)\frac{z}{|z|}, \quad z \in \mathbf{R}^d \setminus \{0\}.$$

We note that

$$\phi_m(z) \to z,\ |\phi_m(z)| = l_m^{-1}(|z|) \uparrow |z| \tag{4.6}$$

as $m \downarrow 0$ by Proposition 4.1 (ii).

By (4.4), the right-hand side of the third equality in (4.3) is equal to

$$\exp\left\{t\left[\int_{|z|>0}\left(e^{i\xi\cdot\phi_m(z)} - 1 - i\xi\cdot\phi_m(z)\mathbf{1}_{|\phi_m(z)|<1}\right)n^0(dy)\right]\right\}.$$

So, let us put $\Phi_0(X) := X$ and for $m > 0$ [1, Theorem 2.3.7 (1) and (1.9), p.109],

$$\Phi_m(X)(t) := \int_0^t\int_{|z|\geq 1}\phi_m(z)N_X(dsdz) + \int_0^t\int_{0<|z|<1}\phi_m(z)\widetilde{N_X^0}(dsdz), \tag{4.7}$$

We note that (4.7) implies (4.3).

Finally, we confirm (4.2). For $0 = t_0 < t_1 < \cdots < t_k < \infty$, $\xi_1, \ldots, \xi_k \in \mathbf{R}^d$, $k \in \mathbf{N}$, by independent and stationary increments property of Lévy process and (4.3), we have

$$\begin{aligned}
E^m\left[e^{i\sum_{j=1}^k \xi_j\cdot X(t_j)}\right] &= \prod_{j=1}^k E^m\left[e^{i(\xi_j+\cdots+\xi_k)\cdot(X(t_j)-X(t_{j-1}))}\right] \\
&= \prod_{j=1}^k E^m\left[e^{i(\xi_j+\cdots+\xi_k)\cdot X(t_j-t_{j-1})}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{k} E^0\big[e^{i(\xi_j+\cdots+\xi_k)\cdot\Phi_m(X)(t_j-t_{j-1}))}\big] \\
&= \prod_{j=1}^{k} E^0\big[e^{i(\xi_j+\cdots\xi_k)\cdot(\Phi_m(X)(t_j)-\Phi_m(X)(t_{j-1}))}\big] \\
&= E^0\big[e^{i\sum_{j=1}^{k}\xi_j\cdot\Phi_m(X)(t_j)}\big].
\end{aligned}
$$

Therefore, we have (4.2). □

In the following, we also write the completion of $(\Omega, \mathcal{F}, \lambda^m)$ by the same $(\Omega, \mathcal{F}, \lambda^m)$ with *augmented natural filtration* with $\{\mathcal{F}(t)\}_{t\geq 0}$ (see subsection 6.2).

Proposition 4.2. *For every sequence $\{m\}$ with $m \downarrow 0$, there exists a subsequence $\{m'\}$ such that*

$$\sup_{t\leq T}|\Phi_{m'}(X)(t) - X(t)| \to 0 \text{ as } m' \downarrow 0, \lambda^0\text{-a.s. } X \in D_0.$$

Proof. From (2.8) and (4.7), we have

$$
\begin{aligned}
\sup_{t\leq T}|\Phi_m(X)(t) - X(t)| \leq &\int_0^T\int_{|z|\geq 1}|\phi_m(z) - z|N_X(dsdz) \\
&+ \sup_{t\leq T}\Big|\int_0^t\int_{0<|z|<1}(\phi_m(z) - z)\widetilde{N_X^0}(dsdz)\Big| \\
=: &\; I_1(m, X) + \sup_{t\leq T}|I_2(t, m, X)|.
\end{aligned}
$$

For $I_1(m, X)$, the integrand of $I_1(m, X)$ converges to zero as $m \downarrow 0$ by (4.6). Hence $I_1(m, X)$ converges to zero as $m \downarrow 0$ because $\int_0^T\int_{|z|\geq 1}|z|N_X(dsdz) < \infty$ (see (6.3) in subsection 6.1).

For $I_2(t, m, X)$, we note that $I_2(t, m, X)$ is the $L^2(D_0; \lambda^0)$-limit of the right-continuous $\{\mathcal{F}(t)\}_{t\geq 0}$-martingale $\{I_2^\varepsilon(t, m, X)\}_{t\geq 0}$ with $I_2^\varepsilon(t, m, X) := \int_0^t\int_{\varepsilon<|z|<1}(\phi_m(z) - z)\widetilde{N_X^0}(dsdz)$ as $\varepsilon \downarrow 0$, with convergence being uniform on $t \leq T$. By taking a subsequence if necessary, $I_2^\varepsilon(t, m, X)$ converges to $I_2(t, m, X)$ as $\varepsilon \downarrow 0$ uniformly on $t \leq T$, λ^0-a.s., and hence $I_2(t, m, X)$ is right-continuous on $t \leq T$, λ^0-a.s. (e.g. [13, p.73, Proof of Theorem 5.1], [20, pp.128–129, Proofs of Lemmas 20.6, 20.7]). Then we use *Doob's martingale inequality* (see Theorem 6.4 in subsection 6.2), we have

$$E^0\left[\sup_{t\leq T}|I_2(t, m, X)|^2\right] \leq 4E^0\left[|I_2(T, m, X)|^2\right]$$

$$\leq 4T \int_{0<|z|<1} |\phi_m(z) - z|^2 n^0(dz),$$

which is also seen to converge to zero as $m \downarrow 0$, by (4.6) since $\int_{0<|z|<1} |z|^2 n^0(dz) < \infty$. $\Box$

By (2.19) and (4.2), we have

$$u^m(x,t) = E^0[e^{-S^m(t,x,\Phi_m(X))} g(x + \Phi_m(X)(t))] \tag{4.8}$$

and then

$$\begin{aligned}\sup_{t\leq T} \|u^m(\cdot,t) - u^0(\cdot,t)\|_\infty \leq \|g\|_\infty \sup_{t\leq T,\ x\in\mathbf{R}^d} E^0 \left[\left|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^0(t,x,X)}\right|\right] \\ + (e^{-T \inf V} \vee 1) E^0 \left[\sup_{t\leq T} \|g(\cdot + \Phi_m(X)(t)) - g(\cdot + X(t))\|_\infty\right].\end{aligned} \tag{4.9}$$

Since $g \in C_\infty(\mathbf{R}^d)$ is uniformly continuous and bounded on $\mathbf{R}^d$, the second term on the right-hand side of (4.9) converges to zero as $m \downarrow 0$.

Next we consider the first term on the right-hand side of (4.9). By $N_{\Phi_m(X)}(dsdy) = N_X(ds\phi_m^{-1}(dy))$, we have

$$\begin{aligned}&S^m(t,x,\Phi_m(X)) \\ &= i\Big(\int_0^t \int_{|z|\geq 1} A(x + \Phi_m(X)(s-) + \tfrac{1}{2}\phi_m(z)) \cdot \phi_m(z) N_X(dsdz) \\ &\quad + \int_0^t \int_{0<|z|<1} A(x + \Phi_m(X)(s-) + \tfrac{1}{2}\phi_m(z)) \cdot \phi_m(z) \widetilde{N_X^0}(dsdz) \\ &\quad + \int_0^t ds \text{ p.v.} \int_{0<|z|<1} A(x + \Phi_m(X)(s) + \tfrac{1}{2}\phi_m(z)) \cdot \phi_m(z) n^0(dz)\Big) \\ &\quad + \int_0^t V(x + \Phi_m(X)(s)) ds \\ &=: i\Big(S_1^m(t,x,X) + S_2^m(t,x,X) + S_3^m(t,x,X)\Big) + S_4^m(t,x,X).\end{aligned} \tag{4.10}$$

By the inequality

$$\begin{aligned}|e^{-(ia+b)} - e^{-(ia'+b')}| &\leq |e^{-ia} - e^{-ia'}| + |b - b'| \\ &\leq |a - a'| + |b - b'|, \quad a, a' \in \mathbf{R}, b, b' \geq 0,\end{aligned} \tag{4.11}$$

we have

$$
\begin{aligned}
&\sup_{t\le T,\ x\in\mathbf{R}^d} E^0\left[\left|e^{-S^m(t,x,\Phi_m(X))}-e^{-S^0(t,x,X)}\right|\right]\\
&=\sup_{t\le T,\ x\in\mathbf{R}^d} e^{-t\inf V}\Big\{E^0\Big[|e^{-i(S_1^m(t,x,X)+S_2^m(t,x,X)+S_3^m(t,x,X))-(S_4^m(t,x,X)-t\inf V)}\\
&\qquad\qquad - e^{-i(S_1^0(t,x,X)+S_2^0(t,x,X)+S_3^0(t,x,X))-(S_4^0(t,x,X)-t\inf V)}|\Big]\Big\}\\
&\le (e^{-T\inf V}\vee 1)\Big\{E^0\Big[\sup_{t\le T}\|e^{-iS_1^m(t,\cdot,X)}-e^{-iS_1^0(t,\cdot,X)}\|_\infty\Big]\\
&\qquad + \sup_{x\in\mathbf{R}^d} E^0\Big[\sup_{t\le T}|S_2^m(t,x,X)-S_2^0(t,x,X)|\Big]\\
&\qquad + E^0\Big[\sup_{t\le T}\|S_3^m(t,\cdot,X)-S_3^0(t,\cdot,X)\|_\infty\Big]\\
&\qquad + E^0\Big[\sup_{t\le T}\|S_4^m(t,\cdot,X)-S_4^0(t,\cdot,X)\|_\infty\Big]\Big\}.
\end{aligned}
\tag{4.12}
$$

Now, let $\{m\}$ be a sequence with $m\downarrow 0$ and $\{m'\}$ any subsequence of $\{m\}$. By Proposition 4.2, there exists a subsequence $\{m''\}$ of $\{m'\}$ such that $\sup_{t\le T}|\Phi_{m''}(X)(t)-X(t)|\to 0$ as $m''\downarrow 0$, λ^0-a.s. $X\in D_0$.

We want to show each of the four terms in the brace $\{\cdots\}$ of the last member of (4.12) converges to zero as $m''\downarrow 0$. First, for the first term, note that

$$
\begin{aligned}
S_1^{m''}(t,x,X)-S_1^0(t,x,X) &= \int_0^t\int_{|z|\ge 1}\big(A(x+\Phi_{m''}(X)(s-)+\tfrac{1}{2}\phi_{m''}(z))\\
&\qquad\qquad - A(x+X(s-)+\tfrac{1}{2}z)\big)\cdot\phi_{m''}(z)N_X(dsdz)\\
&\quad+\int_0^t\int_{|z|\ge 1} A(x+X(s-)+\tfrac{1}{2}z)\cdot(\phi_{m''}(z)-z)N_X(dsdz).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\sup_{t\le T}\|S_1^{m''}(t,\cdot,X)-S_1^0(t,\cdot,X)\|_\infty\\
&\le \int_0^T\int_{|z|\ge 1}\sup_{x\in\mathbf{R}^d}|A(x+\Phi_{m''}(X)(s-)+\tfrac{1}{2}\phi_{m''}(z))-A(x+X(s-)+\tfrac{1}{2}z)||z|N_X(dsdz)\\
&\quad+\sup_{x\in\mathbf{R}^d}|A(x)|\int_0^T\int_{|z|\ge 1}|\phi_{m''}(z)-z|N_X(dsdz),
\end{aligned}
$$

which converges to zero as $m''\downarrow 0$ since $A\in C_0^1(\mathbf{R}^d;\mathbf{R}^d)$ is uniformly continuous on $\mathbf{R}^d$.

Next, for the second term, since $S_2^m(t,x,X)$ is seen to be right-continuous, we have by Schwarz's inequality and Doob's martingale inequality,

$$
\begin{aligned}
&E^0\Big[\sup_{t\le T}|S_2^{m''}(t,x,X)-S_2^0(t,x,X)|\Big]\\
&\le E^0\Big[\sup_{t\le T}|S_2^{m''}(t,x,X)-S_2^0(t,x,X)|^2\Big]^{\frac12}\\
&\le 2E^0\Big[|S_2^{m''}(T,x,X)-S_2^0(T,x,X)|^2\Big]^{\frac12}\\
&\le 2E^0\Big[\int_0^T ds\int_{0<|z|<1}\Big(A(x+\Phi_{m''}(X)(s-)+\tfrac12\phi_{m''}(z))\cdot\phi_{m''}(z)\\
&\qquad\qquad -A(x+X(s-)+\tfrac12 z)\cdot z\Big)^2 n^0(dz)\Big]^{\frac12}\\
&=2E^0\Big[\int_0^T ds\int_{0<|z|<1}\Big(\big[A(x+\Phi_{m''}(X)(s-)+\tfrac12\phi_{m''}(z))-A(x+X(s-)+\tfrac12 z)\big]\cdot\phi_{m''}(z)\\
&\qquad\qquad +A(x+X(s-)+\tfrac12 z)\cdot(\phi_{m''}(z)-z)\Big)^2 n^0(dz)\Big]^{\frac12}
\end{aligned}
$$

By the inequality $(a+b)^2\le 2(a^2+b^2)$ for any $a,b\in\mathbf{R}$, $E^0[\cdots]$ above is less than or equal to

$$
\begin{aligned}
2\Big\{E^0\Big[\int_0^T ds\int_{0<|z|<1}\sup_{x\in\mathbf{R}^d}&|A(x+\Phi_{m''}(X)(s-)+\tfrac12\phi_{m''}(z))\\
&-A(x+X(s-)+\tfrac12 z)|^2|z|^2n^0(dz)\Big]\\
+T\sup_{x\in\mathbf{R}^d}|A(x)|^2\int_{0<|z|<1}&|\phi_{m''}(z)-z|^2n^0(dz)\Big\},
\end{aligned}
$$

which converges to zero as $m''\downarrow 0$.

As for the third term, we note that

$$
S_3^{m''}(t,x,X)=\int_0^t ds\int_{0<|z|<1}\big[A(x+\Phi_{m''}(X)(s)+\tfrac12 z)-A(x+\Phi_{m''}(X)(s))\big]\cdot\phi_{m''}(z)n^0(dz),
$$

because $A\in C_0^1(\mathbf{R}^d;\mathbf{R})$ (cf. Remark 2.1 (1)). Then we have by the mean value theorem,

$$
\begin{aligned}
&S_3^{m''}(t,x,X)-S_3^0(t,x,X)\\
&=\frac12\int_0^t ds\int_{0<|z|<1}n^0(dz)\int_0^1\big[\big(W_{x,X}^{m''}(s,\theta)\phi_{m''}(z)\big)\cdot\phi_{m''}(z)-\big(W_{x,X}^0(s,\theta)z\big)\cdot z\big]d\theta.
\end{aligned}
$$

Here $W^{m''}_{x,X}(s,\theta)$ and $W^0_{x,X}(s,\theta)$ are $d\times d$ matrices defined by

$$\begin{aligned} W^{m''}_{x,X}(s,\theta) &= DA(x+\Phi_{m''}(X)(s)+\tfrac{1}{2}\phi_{m''}(z)\theta),\\ W^0_{x,X}(s,\theta) &= DA(x+X(s)+\tfrac{1}{2}z\theta), \end{aligned}$$

where $DA(\cdot)$ is the Jacobian matrix of A. By the equality

$$\begin{aligned} &\big(W^{m''}_{x,X}(s,\theta)\phi_{m''}(z)\big)\cdot\phi_{m''}(z)-\big(W^0_{x,X}(s,\theta)z\big)\cdot z\\ &=\big(W^{m''}_{x,X}(s,\theta)\phi_{m''}(z)\big)\cdot\big(\phi_{m''}(z)-z\big)+\big((W^{m''}_{x,X}(s,\theta)-W^0_{x,X}(s,\theta))\phi_{m''}(z)\big)\cdot z\\ &\quad+\big(W^0_{x,X}(s,\theta)(\phi_{m''}(z)-z)\big)\cdot z, \end{aligned}$$

the integrand of the third term in the brace $\{\cdots\}$ of the last member of (4.12) is less than or equal to

$$\begin{aligned} &T\sup_{x\in\mathbf{R}^d}\|DA(x)\|\int_{0<|z|<1}|\phi_{m''}(z)-z||z|n^0(dz)\\ &+\frac{1}{2}\int_0^T ds\int_{0<|z|<1}|z|^2n^0(dz)\int_0^1\sup_{x\in\mathbf{R}^d}\|W^{m''}_{x,X}(s,\theta)-W^0_{x,X}(s,\theta)\|d\theta, \end{aligned}$$

where $\|\cdot\|$ is the norm of matrices. This is less than or equal to

$$3T\sup_{x\in\mathbf{R}^d}\|DA(x)\|\int_{0<|z|<1}|z|^2n^0(dz)<\infty$$

and converges to zero as $m''\downarrow 0$ because each component of DA is uniformly continuous on $\mathbf{R}^d$.

Finally, the fourth term in the brace $\{\cdots\}$ of the last member of (4.12) is less than or equal to

$$E^0\left[\int_0^T\|V(\cdot+\Phi_{m''}(X)(s))-V(\cdot+X(s))\|_\infty ds\right],$$

which converges to zero as $m''\downarrow 0$ since $V\in C_0(\mathbf{R}^d;\mathbf{R})$ is uniformly continuous on $\mathbf{R}^d$.

Thus we have $\sup_{t\le T}\|u^{m''}(\cdot,t)-u^0(\cdot,t)\|_\infty\to 0$ as $m''\downarrow 0$, and hence $\sup_{t\le T}\|u^m(\cdot,t)-u^0(\cdot,t)\|_\infty\to 0$ as $m\downarrow 0$. $\square$

5. Proof of Theorem 3.3.

In this section, we prove Theorem 3.3, first in the case that $A(x)$ and $V(x)$ are smooth continuous functions of compact support and then in the general case.

5.1 The case that $A \in C_0^1(\mathbf{R}^d;\mathbf{R}^d)$ and $V \in C_0(\mathbf{R}^d)$.

First we consider the case $g \in C_0^\infty(\mathbf{R}^d)$ and then the general case $g \in L^2(\mathbf{R}^d)$.

Step I: Let $g \in C_0^\infty(\mathbf{R}^d)$. For $R > 0$, we have by Minkowski's inequality

$$\begin{aligned}\|u^m(\cdot,t) - u^0(\cdot,t)\|_2 &\le \|u^m(\cdot,t) - u^0(\cdot,t)\|_{L^2(|x|<R)} + \|u^m(\cdot,t) - u^0(\cdot,t)\|_{L^2(|x|\ge R)}\\ &=: I_1(t,m,R) + I_2(t,m,R).\end{aligned}$$

By Theorem 3.2, we have

$$\sup_{t\le T} I_1(t,m,R) \le \sup_{t\le T} \|u^m(\cdot,t) - u^0(\cdot,t)\|_\infty \sqrt{\mathrm{vol}(B_d(R))},$$

which converges to zero as $m \downarrow 0$. Here $\mathrm{vol}(B_d(R))$ is the volume of the d-dimensional ball with radius R.

Next, we will show $\sup_{t\le T,\, 0\le m\le 1} I_2(t,m,R)$ converges to zero as $R \to \infty$. By Minkowski's inequality and (2.19) with (2.13), we have

$$\begin{aligned}I_2(t,m,R) &\le \|u^m(\cdot,t)\|_{L^2(|x|\ge R)} + \|u^0(\cdot,t)\|_{L^2(|x|\ge R)}\\ &\le e^{-t\inf V}\Big\{\|E^m[|g(\cdot+X(t))|]\|_{L^2(|x|\ge R)} + \|E^0[|g(\cdot+X(t))|]\|_{L^2(|x|\ge R)}\Big\}\\ &= e^{-t\inf V}\Bigg\{\left(\int_{|x|\ge R} dx \int_{\mathbf{R}^d} k_0^m(y,t)|g(x+y)|^2 dy\right)^{\frac12}\\ &\qquad + \left(\int_{|x|\ge R} dx \int_{\mathbf{R}^d} k_0^0(y,t)|g(x+y)|^2 dy\right)^{\frac12}\Bigg\}\\ &= e^{-t\inf V}\Bigg\{\left(\int_{|x|\ge R} (k_0^m(\cdot,t) * h)(x) dx\right)^{\frac12} + \left(\int_{|x|\ge R} (k_0^0(\cdot,t) * h)(x) dx\right)^{\frac12}\Bigg\}\\ &=: e^{-t\inf V}(J(t,m,R) + J(t,0,R)). \qquad (5.1)\end{aligned}$$

Here $h(x) := |g(x)|^2$ and $k_0^m(\cdot,t) * h$ is the convolution of $k_0^m(\cdot,t)$ and h. Let χ be a nonnegative C_0^∞ function with $0 \le \chi(x) \le 1$ in $\mathbf{R}^d$ such that $\chi(x) = 1$ if $|x| \le \frac12$ and $= 0$ if $|x| \ge 1$. By the inequality $\mathbf{1}_{|x|<R} \ge \chi(\frac{x}{R})$, (2.11), Parseval's equality and (2.10), we have for $m \ge 0$

$$\begin{aligned}J(t,m,R)^2 &\le \int_{\mathbf{R}^d} \left(1 - \chi(\tfrac{x}{R})\right)(k_0^m(\cdot,t) * h)(x) dx\\ &= \int_{\mathbf{R}^d} h(y) dy - \int_{\mathbf{R}^d} (k_0^m(\cdot,t) * h)(x)\chi(\tfrac{x}{R}) dx\end{aligned}$$

$$
\begin{aligned}
&= \widehat{h}(0) - \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \widehat{k_0^m(\cdot,t)}(\xi)\widehat{h}(\xi)\overline{\widehat{\chi(\tfrac{\cdot}{R})}(\xi)}d\xi \\
&= \widehat{h}(0) - \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-t[\sqrt{\xi^2+m^2}-m]}\widehat{h}(\xi)R^d\overline{\widehat{\chi}(R\xi)}d\xi \\
&= \widehat{h}(0) - \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} e^{-t\left[\sqrt{\frac{\eta^2}{R^2}+m^2}-m\right]}\widehat{h}(\tfrac{\eta}{R})\overline{\widehat{\chi}(\eta)}d\eta \\
&= \frac{1}{(2\pi)^d}\Big\{\widehat{h}(0)\int_{\mathbf{R}^d}\Big(1-e^{-t\left[\sqrt{\frac{\eta^2}{R^2}+m^2}-m\right]}\Big)\overline{\widehat{\chi}(\eta)}d\eta \\
&\qquad + \int_{\mathbf{R}^d}\Big(\widehat{h}(0)) - \widehat{h}(\tfrac{\eta}{R})\Big)e^{-t\left[\sqrt{\frac{\eta^2}{R^2}+m^2}-m\right]}\overline{\widehat{\chi}(\eta)}d\eta\Big\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sup_{t\le T,\,0\le m\le 1} J(t,m,R)^2 &\le \frac{1}{(2\pi)^d}\Big\{\widehat{h}(0)\int_{\mathbf{R}^d}\Big(1-e^{-T\frac{\eta}{R}}\Big)\overline{\widehat{\chi}(\eta)}d\eta \\
&\qquad + \int_{\mathbf{R}^d}\Big(\widehat{h}(0)) - \widehat{h}(\tfrac{\eta}{R})\Big)\overline{\widehat{\chi}(\eta)}d\eta\Big\} \\
&\to 0, \quad \text{as } R\to\infty, \qquad (5.2)
\end{aligned}
$$

because $\chi \in C_0^\infty(\mathbf{R}^d)$ and so $\widehat{\chi} \in \mathcal{S}(\mathbf{R}^d) \subset L^1(\mathbf{R}^d)$. Therefore we have by (5.1), (5.2)

$$
\begin{aligned}
\sup_{t\le T, 0\le m\le 1} I_2(t,m,R) &\le 2(e^{-T\inf V}\vee 1)\sup_{t\le T,\,0\le m\le 1} J(t,m,R) \\
&\to 0 \quad \text{as } R\to\infty.
\end{aligned}
$$

and conclude this step.

Step II: Let $g \in L^2(\mathbf{R}^d)$. There is a sequence $\{g_n\} \subset C_0^\infty(\mathbf{R}^d)$ such that $g_n \to g$ in $L^2(\mathbf{R}^d)$ as $n\to\infty$. Put $u_n^m(x,t) := E^m[e^{-S^m(t,x,X)}g_n(x+X(t))]$. Then we have

$$
\begin{aligned}
\|u^m(\cdot,t) - u^0(\cdot,t)\|_2 &\le \|u^m(\cdot,t) - u_n^m(\cdot,t)\|_2 + \|u_n^m(\cdot,t) - u_n^0(\cdot,t)\|_2 \\
&\quad + \|u_n^0(\cdot,t) - u^0(\cdot,t)\|_2 \\
&\le 2e^{-t\inf V}\|g_n - g\|_2 + \|u_n^m(\cdot,t) - u_n^0(\cdot,t)\|_2.
\end{aligned}
$$

By Step I, we have

$$
\limsup_{m\downarrow 0}\sup_{t\le T}\|u^m(\cdot,t) - u^0(\cdot,t)\|_2 \le 2(e^{-T\inf V}\vee 1)\|g_n - g\|_2,
$$

which converges to zero as $n\to\infty$. $\square$

5.2 The general case that $A \in L^{1+\delta}_{\mathbf{loc}}(\mathbf{R}^d;\mathbf{R}^d)$ and $0 \le V \in L^1_{\mathbf{loc}}(\mathbf{R}^d;\mathbf{R})$.

Choose a sequence $\{A_j\} \subset C_0^\infty(\mathbf{R}^d;\mathbf{R}^d)$ with $|A_j(x)| \le |A(x)|$ a.s. x and $A_j \to A$ in $L^{1+\delta}_{\rm loc}(\mathbf{R}^d;\mathbf{R}^d)$, and a sequence $\{V_j\} \subset C_0^\infty(\mathbf{R}^d)$ with $0 \le V_j(x) \le V(x)$ a.s. x, $V_j \to V$ in $L^1_{\rm loc}(\mathbf{R}^d;\mathbf{R})$.

Lemma 5.1. *Let $\{m\}$ be a sequence with $\sup_{t\le T} |\Phi_m(X)(t) - X(t)| \to 0$ as $m \downarrow 0$ λ^0-a.s. $X \in D_0$. Then we have for $R > 0$*

$$\limsup_{m\downarrow 0} \sup_{t\le T} \int_{|x|<R} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big]dx \to 0, \quad \textit{as } j \to \infty,$$

$$\sup_{t\le T} \int_{|x|<R} E^0\Big[\big|e^{-S^0(t,x,X)} - e^{-S^{0,j}(t,x,X)}\big|^2\Big]dx \to 0, \quad \textit{as } j \to \infty,$$

where $S^{m,j}(t,x,\Phi_m(X))$ is $S^m(t,x,\Phi_m(X))$ in (4.10) with A and V replaced by A_j and V_j.

Proof. We assume $0 < \delta < 1$ without loss of generality because $L^q_{\rm loc} \subset L^p_{\rm loc}$ for $0 < p < q$. Fix $R > 0$. For $k \in \mathbf{N}$, $X \in D_0$, let

$$\sigma_k(X) := \begin{cases} \inf\{s > 0; |X(s-)| > k\}, & \text{if } \{\cdots\} \ne \emptyset, \\ 0, & \text{if } \{\cdots\} = \emptyset, \end{cases}$$

be the *first hitting time* for $\{y; |y| > k\}$. Then for each $X \in D_0$,

$$\sigma_k(X) \uparrow \infty \text{ as } k \to \infty, \text{ and } |X(s-)| \le k \text{ if } 0 < s \le \sigma_k(X).$$

For any $k \in \mathbf{N}$, we have

$$\begin{aligned} D(t,m,j) &:= \int_{|x|<R} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big]dx \\ &= \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big]dx \\ &\quad + \int_{|x|<R} E^0\Big[\mathbf{1}_{\sigma_k(\Phi_m(X))<t}\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big]dx \\ &=: D_1(t,m,j,k) + D_2(t,m,j,k). \end{aligned} \tag{5.3}$$

We will show that

$$\limsup_{m\downarrow 0} \sup_{t\le T} D_1(t,m,j,k) \to 0 \quad \text{as } j \to \infty \text{ for each fixed } k \in \mathbf{N},$$

$$\limsup_{j\to\infty}\limsup_{m\downarrow 0}\sup_{t\le T} D_2(t,m,j,k)\to 0 \qquad \text{as } k\to\infty.$$

If we show these convergence, then by (5.3), we can see the proof of Lemma 5.1 to be accomplished. First, for $D_2(t,m,j,k)$, we have

$$D_2(t,m,j,k)\le 4\,(\mathrm{vol}B_d(R))\,E^0\Big[\mathbf{1}_{\sigma_k(\Phi_m(X))<t}\Big].$$

By the inequality

$$\limsup_{m\downarrow 0}\mathbf{1}_{\sigma_k(\Phi_m(X))<t}\le \mathbf{1}_{\sigma_{k-1}(X)<t},\quad \lambda^0\text{-a.s. } X\in D_0,\ k\ge 2,\ t>0,$$

we have

$$\begin{aligned}\limsup_{j\to\infty}\limsup_{m\downarrow 0}\sup_{t\le T} D_2(t,m,j,k) &\le 4\,(\mathrm{vol}B_d(R))\limsup_{m\downarrow 0}\sup_{t\le T}E^0\Big[\mathbf{1}_{\sigma_k(\Phi_m(X))<t}\Big]\\ &\le 4\,(\mathrm{vol}B_d(R))\,E^0\Big[\limsup_{m\downarrow 0}\mathbf{1}_{\sigma_k(\Phi_m(X))<T}\Big]\\ &\le 4\,(\mathrm{vol}B_d(R))\,E^0\Big[\mathbf{1}_{\sigma_{k-1}(X)<T}\Big],\end{aligned}$$

which converges to zero as $k\to\infty$.

Next, for $D_1(t,m,j,k)$, we have by (4.11)

$$\begin{aligned}D_1(t,m,j,k) &\le 2\int_{|x|<R}E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|e^{-S^m(t,x,\Phi_m(X))}-e^{-S^{m,j}(t,x,\Phi_m(X))}\big|\Big]dx\\ &\le 2\Big\{\int_{|x|<R}E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|e^{-iS_1^m(t,x,X)}-e^{-iS_1^{m,j}(t,x,X)}\big|\Big]dx\\ &\quad+\int_{|x|<R}E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|S_2^m(t,x,X)-S_2^{m,j}(t,x,X)\big|\Big]dx\\ &\quad+\int_{|x|<R}E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|S_3^m(t,x,X)-S_3^{m,j}(t,x,X)\big|\Big]dx\\ &\quad+\int_{|x|<R}E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\big|S_4^m(t,x,X)-S_4^{m,j}(t,x,X)\big|\Big]dx\Big\}\\ &=: 2\sum_{l=1}^{4}I_l(t,m,j,k),\end{aligned}\tag{5.4}$$

where $S_l^{m,j}(t,x,X)$ is $S_l^m(t,x,X)$ in (4.10) with A and V replaced by A_j and V_j.

We have to show that each $\limsup_{m\downarrow 0}\sup_{t\le T}I_l(t,m,j,k)$, $l=1,2,3,4$, converges to zero as $j\to\infty$, for any $k\in\mathbf{N}$.

In the following, fix $k \in \mathbf{N}$. First, for $I_1(t,m,j,k)$,

$$\begin{aligned}
&\limsup_{m\downarrow 0}\sup_{t\le T} I_1(t,m,j,k)\\
&\le E^0\Big[\limsup_{m\downarrow 0}\sup_{t\le T}\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\int_{|x|<R}\big|e^{-iS_1^m(t,x,X)}-e^{-iS_1^{m,j}(t,x,X)}\big|dx\Big].
\end{aligned}$$

The integrand above is less than or equal to 2. On the other hand, we have by (4.6) we have

$$\begin{aligned}
&\int_{|x|<R}\big|e^{-iS_1^m(t,x,X)}-e^{-iS_1^{m,j}(t,x,X)}\big|dx\\
&\le \int_{|x|<R}\big|S_1^m(t,x,X)-S_1^{m,j}(t,x,X)\big|dx\\
&\le \int_{|x|<R}\int_0^t\int_{|z|\ge 1}|A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad -A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))||z|N_X(dsdz)\\
&=\sum_{s\le t}\mathbf{1}_{|X(s)-X(s-)|\ge 1}|X(s)-X(s-)|\\
&\quad\times\int_{|x|<R}|A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(X(s)-X(s-)))\\
&\qquad\qquad -A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(X(s)-X(s-)))|dx\\
&\le \sum_{s\le t}\mathbf{1}_{|X(s)-X(s-)|\ge 1}|X(s)-X(s-)|\\
&\quad\times\int_{|z|<R+k+\frac{1}{2}\sup_{s\le t}|X(s)-X(s-)|}|A(z)-A_j(z)|dz
\end{aligned}$$

on $t\le\sigma_k(\Phi_m(X))$. It follows that

$$\begin{aligned}
&\limsup_{m\downarrow 0}\sup_{t\le T}\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\int_{|x|<R}\big|e^{-iS_1^m(t,x,X)}-e^{-iS_1^{m,j}(t,x,X)}\big|dx\\
&\le \sum_{s\le T}\mathbf{1}_{|X(s)-X(s-)|\ge 1}|X(s)-X(s-)|\\
&\quad\times\int_{|z|<R+k+\frac{1}{2}\sup_{s\le T}|X(s)-X(s-)|}|A(z)-A_j(z)|dz.
\end{aligned}$$

This is the finite sum (see (6.3) in subsection 6.1) and converges to zero as $j\to\infty$ because $A_j\to A$ in $L^{1+\delta}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d)$ and so in $L^1_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d)$. Therefore, we have $\limsup_{m\downarrow 0}\sup_{t\le T} I_1(t,m,j,k)$ converges to zero as $j\to\infty$.

Next, for $I_2(t,m,j,k)$, we have

$$
\begin{aligned}
&I_2(t,m,j,k)\\
&= \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\Big| \int_0^t \int_{0<|z|<1} \big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\widetilde{N_X^0}(dsdz)\Big|\Big]dx\\
&\le \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\Big| \iint_{G^{m,j}(t,x,X)} \big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\widetilde{N_X^0}(dsdz)\Big|\Big]dx\\
&\quad + \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\Big| \iint_{G^{m,j}(t,x,X)^{\complement}} \big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\widetilde{N_X^0}(dsdz)\Big|\Big]dx\\
&=: \int_{|x|<R} J_1^{t,m,j,k}(x)dx + \int_{|x|<R} J_2^{t,m,j,k}(x)dx.
\end{aligned}
$$

Here $G^{m,j}(t,x,X)$ is a subset of $(0,t]\times\{z; 0<|z|<1\}$ defined by

$$
\begin{aligned}
G^{m,j}(t,x,X) := \Big\{(s,z); \Big|\big[&A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&- A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\Big| > 1\Big\}.
\end{aligned}
$$

By the inequality $|\widetilde{N_X^m}(dsdz)| \le N_X(dsdz) + dsn^0(dz)$ and $E^0[\widetilde{N_X^m}(dsdz)] = dsn^0(dz)$, we have

$$
\begin{aligned}
&\int_{|x|<R} J_1^{t,m,j,k}(x)dx\\
&\le \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \iint_{G^{m,j}(t,x,X)} \Big|\big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\Big|(N_X(dsdz)+dsn^0(dz))\Big]dx\\
&\le 2\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \iint_{G^{m,j}(t,x,X)} \Big|\big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\Big|^{1+\delta} dsn^0(dz)\Big]dx\\
&\le 2\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \int_0^t \int_{0<|z|<1} \Big|A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\Big|^{1+\delta} |z|^{1+\delta} dsn^0(dz)\Big]dx
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^t ds \int_{0<|z|<1} |z|^{1+\delta} n^0(z)dz E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \\
&\quad\times \int_{|x|<R} \Big|A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z)) - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\Big|^{1+\delta} dx\Big] \\
&\le T\int_{0<|z|<1} |z|^{1+\delta} n^0(z)dz \int_{|u|<R+k+\frac{1}{2}} |A(u)-A_j(u)|^{1+\delta} du.
\end{aligned}
$$

It follows that $\limsup_{m\downarrow 0}\sup_{t\le T}\int_{|x|<R} J_1^{t,m,j,k}(x)dx$ converges to zero as $j\to\infty$ because $\int_{0<|z|<1}|z|^{1+\delta}n^0(dz)<\infty$ by (2.7). Next, we have by Schwarz's inequality

$$
\begin{aligned}
&\int_{|x|<R} J_2^{t,m,j,k}(x)dx \\
&\le \sqrt{\mathrm{vol}(B_d(R))} \\
&\times \Bigg(\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\Big| \iint_{G^{m,j}(t,x,X)^{\complement}} \big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z)) \\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\widetilde{N_X^0}(dsdz)\Big|\Big]^2 dx\Bigg)^{1/2} \\
&\le \sqrt{\mathrm{vol}(B_d(R))} \\
&\quad\times \Bigg(\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))}\Big| \iint_{G^{m,j}(t,x,X)^{\complement}} \big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z)) \\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\widetilde{N_X^0}(dsdz)\Big|^2\Big] dx\Bigg)^{1/2} \\
&\le \sqrt{\mathrm{vol}(B_d(R))} \\
&\quad\times \Bigg(\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \iint_{G^{m,j}(t,x,X)^{\complement}} \Big|\big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z)) \\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\Big|^2 ds n^0(dz)\Big] dx\Bigg)^{1/2} \\
&\le \sqrt{\mathrm{vol}(B_d(R))} \\
&\quad\times \Bigg(\int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \iint_{G^{m,j}(t,x,X)^{\complement}} \Big|\big[A(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z)) \\
&\qquad\qquad - A_j(x+\Phi_m(X)(s-)+\tfrac{1}{2}\phi_m(z))\big]\cdot\phi_m(z)\Big|^{1+\delta} ds n^0(dz)\Big] dx\Bigg)^{1/2}
\end{aligned}
$$

$$\leq \sqrt{\mathrm{vol}(B_d(R))}\Big(T\int_{0<|z|<1}|z|^{1+\delta}n^0(z)dz\int_{|u|<R+k+\frac12}|A(u)-A_j(u)|^{1+\delta}du\Big)^{1/2}.$$

It follows that $\limsup_{m\downarrow 0}\sup_{t\leq T}\int_{|x|<R}J_2^{m,j}(t,x)dx$ converges to zero as $j\to\infty$. Therefore $\limsup_{m\downarrow 0}\sup_{t\leq T}I_2(t,m,j,k)$ converges to zero as $j\to\infty$.

For $I_3(t,m,j,k)$, by Fatou's lemma, the change of variable $\phi_m(z)=y$ and the fact that $n^m(y)$ converges to $n^0(y)$, strictly increasing, as $m\downarrow 0$ for any fixed y ([6, Lemma 3.1 (ii)]), we have

$$\begin{aligned}
&I_3(t,m,j,k)\\
&=\int_{|x|<R}E^0\Big[\mathbf{1}_{t\leq\sigma_k(\Phi_m(X))}\Big|\int_0^t ds\lim_{\varepsilon\downarrow 0}\int_{\varepsilon<|z|<1}\big[A(x+\Phi_m(X)(s)+\tfrac12\phi_m(z))\\
&\qquad\qquad -A_j(x+\Phi_m(X)(s)+\tfrac12\phi_m(z))\big]\cdot\phi_m(z)n^0(dz)\Big|\Big]dx\\
&\leq\liminf_{\varepsilon\downarrow 0}\int_{|x|<R}E^0\Big[\mathbf{1}_{t\leq\sigma_k(\Phi_m(X))}\int_0^t ds\Big|\int_{\varepsilon<|z|<1}\big[A(x+\Phi_m(X)(s)+\tfrac12\phi_m(z))\\
&\qquad\qquad -A_j(x+\Phi_m(X)(s)+\tfrac12\phi_m(z))\big]\cdot\phi_m(z)n^0(dz)\Big|\Big]dx\\
&=\liminf_{\varepsilon\downarrow 0}\int_0^t dsE^0\Big[\mathbf{1}_{t\leq\sigma_k(\Phi_m(X))}\int_{|x|<R}\Big|\int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)}\big[A(x+\Phi_m(X)(s)+\tfrac12 y)\\
&\qquad\qquad -A_j(x+\Phi_m(X)(s)+\tfrac12 y)\big]\cdot yn^m(dy)\Big|dx\Big]\\
&\leq T\liminf_{\varepsilon\downarrow 0}\int_{|w|<R+k}\Big|\int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)}\big[A(w+\tfrac12 y)-A_j(w+\tfrac12 y)\big]\cdot yn^m(y)dy\Big|dw\\
&=T\liminf_{\varepsilon\downarrow 0}\int_{|w|<R+k}\Big|\int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)}\big[A(w+\tfrac12 y)-A_j(w+\tfrac12 y)\big]\cdot y\\
&\qquad\qquad\times\big((n^m(y)-n^0(y))+n^0(y)\big)dy\Big|dw\\
&\leq T\liminf_{\varepsilon\downarrow 0}\Big\{\int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)}|y|(n^0(y)-n^m(y))dy\int_{|w|<R+k}\big|A(w+\tfrac12 y)-A_j(w+\tfrac12 y)\big|dw\\
&\qquad +\int_{|w|<R+k}\Big|\int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)}\big[A(w+\tfrac12 y)-A_j(w+\tfrac12 y)\big]\cdot yn^0(y)dy\Big|dw\Big\}\\
&=:T\liminf_{\varepsilon\downarrow 0}(K_1^{m,j,k,\varepsilon}+K_2^{m,j,k,\varepsilon}).
\end{aligned}$$

By (4.6) and the equality $\int_{|y|>0}(n^0(y)-n^m(y))dy=m$ ([6, Lemma 3.1 (iii)]), we have

$$K_1^{m,j,k,\varepsilon}\leq m\int_{|u|<R+k+\frac12}|A(u)-A_j(u)|du.$$

Next, since $y_i n^0(y)$ $(y = (y_1, \dots, y_d))$ is the Calderon–Zygmund kernel ([5, p.275]), by Hölder's inequality and the Calderon–Zygmund theorem ([22, Theorem 2, p35]) with a constant C_δ depending only on δ, we have

$$
\begin{aligned}
K_2^{m,j,k,\varepsilon} &\le (\mathrm{vol}(B_d(R+k)))^{\frac{\delta}{1+\delta}} \\
&\quad \times \left(\int_{|w|<R+k} \left| \int_{l_m^{-1}(\varepsilon)<|y|<l_m^{-1}(1)} \left[A(w+\tfrac{1}{2}y) - A_j(w+\tfrac{1}{2}y)\right] \cdot y n^0(y) dy \right|^{1+\delta} dw \right)^{\frac{1}{1+\delta}} \\
&\le (\mathrm{vol}(B_d(R+k)))^{\frac{\delta}{1+\delta}} C_\delta \left(\int_{|w|<R+k} \left| A(w) - A_j(w) \right|^{1+\delta} dw \right)^{\frac{1}{1+\delta}}.
\end{aligned}
$$

Therefore $\limsup_{m\downarrow 0} \sup_{t\le T} I_3(t,n,m,j)$ converges to zero as $j \to \infty$.

Finally, for $I_4(t,m,j,k)$, we have

$$
\begin{aligned}
I_4(t,m,j,k) &= \int_{|x|<R} E^0\Big[\mathbf{1}_{t\le\sigma_k(\Phi_m(X))} \int_0^t (V(x+\Phi_m(s)) - V_j(x+\Phi_m(s))ds\Big] dx \\
&\le T \int_{|u|<R+k} (V(u) - V_j(u)) du.
\end{aligned}
$$

It follows that $\limsup_{m\downarrow 0} \sup_{t\le T} I_4(t,m,j,k)$ converges to zero as $j \to \infty$ because $V_j \to V$ in $L^1_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R})$.

This shows $\limsup_{m\downarrow 0} \sup_{t\le T} D_1(t,m,j,k)$ converges to zero as $j \to \infty$ for all k and completes the proof of the first convergence in Lemma 5.1. The proof of the second convergence in Lemma 5.1 is the same argument as above for $m = 0$. $\square$

We are now in a position to prove Theorem 3.3. We will follow the arguments used in subsection 5.1 under condition (2.16) when g is a C_0^∞ function.

Let us put

$$
u^{m,j}(x,t) := E^0[e^{-S^{m,j}(t,x,\Phi_m(X))} g(x+\Phi_m(X)(t))].
$$

Then we have

$$
\begin{aligned}
\sup_{t\le T} \|u^m(\cdot,t) - u^0(\cdot,t)\|_2 &\le \sup_{t\le T} \|u^m(\cdot,t) - u^{m,j}(\cdot,t)\|_2 + \sup_{t\le T} \|u^{m,j}(\cdot,t) - u^{0,j}(\cdot,t)\|_2 \\
&\quad + \sup_{t\le T} \|u^{0,j}(\cdot,t) - u^0(\cdot,t)\|_2.
\end{aligned}
$$

Now, as the proof of Theorem 3.2, let $\{m\}$ be a sequence with $m \downarrow 0$ and $\{m'\}$ any subsequence of $\{m\}$. By Proposition 4.2, there exists a subsequence $\{m''\}$ of $\{m'\}$ such that $\sup_{t \le T} |\Phi_{m''}(X)(t) - X(t)| \to 0$ as $m'' \downarrow 0$, λ^0-a.s. $X \in D_0$.

Since $A_j \in C_0^\infty(\mathbf{R}^d;\mathbf{R}^d)$ and $V_j \in C_0^\infty(\mathbf{R}^d;\mathbf{R})$, we have by Theorem 3.3 in subsection 5.1

$$\begin{aligned}\limsup_{m''\downarrow 0} \sup_{t\le T} \|u^{m''}(\cdot,t) - u^0(\cdot,t)\|_2 \le{}& \limsup_{m''\downarrow 0} \sup_{t\le T} \|u^{m''}(\cdot,t) - u^{m'',j}(\cdot,t)\|_2 \\ &+ \sup_{t\le T} \|u^{0,j}(\cdot,t) - u^0(\cdot,t)\|_2. \end{aligned} \tag{5.5}$$

For $m \ge 0$ and $R > 0$, we have by Schwarz's inequality and (2.13)

$$\begin{aligned}
&\|u^m(\cdot,t) - u^{m,j}(\cdot,t)\|_2 \\
&= \Big(\int_{\mathbf{R}^d} \Big| E^0\Big[\big(e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big) g(x+\Phi_m(X)(t))\Big]\Big|^2 dx\Big)^{1/2} \\
&\le \Big(\int_{\mathbf{R}^d} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2 |g(x+\Phi_m(X)(t))|^2\Big] dx\Big)^{1/2} \\
&= \Big(\int_{|x|\ge R} + \int_{|x|<R} \Big)^{1/2} \\
&\le \Big(4\int_{|x|\ge R} E^0\Big[|g(x+\Phi_m(X)(t))|^2\Big] dx \\
&\quad + \|g\|_\infty^2 \int_{|x|<R} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big] dx\Big)^{1/2} \\
&= \Big(4\int_{|x|\ge R} dx \int_{\mathbf{R}^d} k_0^m(y,t) |g(x+y)|^2 dy \\
&\quad + \|g\|_\infty^2 \int_{|x|<R} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big] dx\Big)^{1/2} \\
&= \Big(4J(t,m,R)^2 + \|g\|_\infty^2 \int_{|x|<R} E^0\Big[\big|e^{-S^m(t,x,\Phi_m(X))} - e^{-S^{m,j}(t,x,\Phi_m(X))}\big|^2\Big] dx\Big)^{1/2}.
\end{aligned}$$

Hence we have by Lemma 5.1 and (5.2)

$$\begin{aligned}
&\limsup_{m''\downarrow 0} \sup_{t\le T} \|u^{m''}(\cdot,t) - u^{m'',j}(\cdot,t)\|_2 \vee \sup_{t\le T} \|u^0(\cdot,t) - u^{0,j}(\cdot,t)\|_2 \\
&\le \Bigg[4 \sup_{t\le T,\ 0\le m\le 1} J(t,m,R)^2 \\
&\quad + \|g\|_\infty^2 \Big\{ \limsup_{m''\downarrow 0} \sup_{t\le T} \int_{|x|<R} E^0\Big[\big|e^{-S^{m''}(t,x,\Phi_{m''}(X))} - e^{-S^{m'',j}(t,x,\Phi_{m''}(X))}\big|^2 dx
\end{aligned}$$

$$\vee \sup_{t\le T}\int_{|x|<R} E^0\Big[\big|e^{-S^0(t,x,X)}-e^{-S^{0,j}(t,x,X)}\big|^2\Big]dx\Big\}\Big]^{1/2}$$

$$\to 2 \sup_{t\le T,\ 0\le m\le 1} J(t,m,R) \quad \text{as } j\to\infty,$$

$$\to 0 \quad \text{as } R\to\infty. \tag{5.6}$$

By (5.5) and (5.6), we have $\sup_{t\le T}\|u^{m''}(\cdot,t)-u^0(\cdot,t)\|_2\to 0$ as $m''\downarrow 0$ and hence we conclude the proof of Theorem 3.3. □

6. Appendix.

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space.

6.1 Lévy process.

Definition 6.1. Let ν be a probability measure on $\mathbf{R}^d$. The $\mathbf{R}^d$-valued stochastic process $B=\{B(t)\}_{t\ge 0}$ is called *d-dimensional Brownian motion (or Wiener process) with the initial distribution* ν if the following conditions are satisfied:
(i) $\mathbf{P}(B(0)\in dy)=\nu(dy)$.
(ii) For $0\le s<t$, $B(t)-B(s)$ is independent of $\sigma(B(r);r\le s)$ and its distribution is d-dimensional Gaussian distribution with mean vector 0, covariance matrix $(t-s)E_d$, where E_d is the d-dimensional unit matrix;

$$E^{\mathbf{P}}[e^{i\xi\cdot(B(t)-B(s))}]=e^{-\frac{t-s}{2}|\xi|^2},\quad \xi\in\mathbf{R}^d.$$

Here $E^{\mathbf{P}}$ denotes the expectation over Ω with respect to $\mathbf{P}$.
(iii) The sample path $t\mapsto B(t)$ is continuous a.s.
Especially, when $\nu(dy)=\delta_a(dy)$ (Dirac measure concentrated at a point $a\in\mathbf{R}^d$), i.e., $B(0)=a$ a.s., B is called *d-dimensional Brownian motion (or Wiener process) starting at* a.

Brownian motion is the model for the irregular motion of pollen suspended in the water.

Definition 6.2. The real stochastic process $N=\{N(t)\}_{t\ge 0}$ is called *Poisson process with parameter* $c>0$ if $N(t)$ has Poisson distribution with mean ct for any $t\ge 0$, i.e.,

$$\mathbf{P}(N(t)=k)=e^{-ct}\frac{(ct)^k}{k!},\quad k=0,1,2,\dots.$$

$N(t)$ is the number of times which some event causes from time 0 to time t. For example, the number of the traffic accident in a day, the number of the radiant rays radiated from a radioactive substance and the number of the visitors to the window, etc. We note that the characteristic function of $N(t)$ is

$$E^{\mathbf{P}}[e^{i\xi N(t)}] = \exp\left\{ct(e^{i\xi} - 1)\right\}, \quad \xi \in \mathbf{R}.$$

Brownian motion and Poisson process belong to a class called *Lévy process*, which is one of the basic classes in the stochastic process theory.

Definition 6.3. The $\mathbf{R}^d$-valued stochastic process $X = \{X(t)\}_{t\geq 0}$ is called *Lévy process* if the following conditions are satisfied:
(i) (*independent increments*) For $0 \leq t_0 < t_1 < \cdots < t_k < \infty$, $k \in \mathbf{N}$, $X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})$ are independent.
(ii) (*stationary increments*) For $s, t \geq 0$, the probability distribution of $X(t+s) - X(s)$ does not depend on s, i.e., $\mathbf{P}(X(t+s) - X(s) \in dy) = \mathbf{P}(X(t) \in dy)$.
(iii) (*stochastically continuous*) For $t \geq 0$ and $\varepsilon > 0$,

$$\lim_{s\to t} \mathbf{P}(|X(t) - X(s)| > \varepsilon) = 0.$$

(iv) The sample path $t \mapsto X(t)$ is right-continuous, has left-hand limits and $X(0) = 0$ a.s.

If X is a Lévy process, then we have by Definition 6.3 (i), (ii)

$$\begin{aligned} E^{\mathbf{P}}[e^{i\xi\cdot X(t)}] &= E^{\mathbf{P}}[e^{i\sum_{l=1}^{k}\xi\cdot(X(\frac{l}{k}t) - X(\frac{l-1}{k}t))}] \\ &= \prod_{l=1}^{k} E^{\mathbf{P}}[e^{i\xi\cdot(X(\frac{l}{k}t) - X(\frac{l-1}{k}t))}] \\ &= \left(E^{\mathbf{P}}[e^{i\xi\cdot X(\frac{t}{k})}]\right)^k, \quad \xi \in \mathbf{R}^d, k \in \mathbf{N}. \end{aligned}$$

Therefore, the distribution of $X(t)$ is *infinitely divisible* for any $t \geq 0$. Here the definition of an infinitely divisible distribution is given as follows:

Definition 6.4. A probability measure P on $\mathbf{R}^d$ is called a *infinitely divisible distribution* if, for each $k \in \mathbf{N}$, there exists a probability measure P_k on $\mathbf{R}^d$ such that P is equal to k-fold convolution of P_k, i.e.,

$$E^{P}[e^{i\xi\cdot y}] = \left(E^{P_k}[e^{i\xi\cdot y}]\right)^k, \quad \xi \in \mathbf{R}^d.$$

Here E^P (resp. E^{P_k}) denotes the expectation over $\mathbf{R}^d$ with respect to P (resp. P_k).

The following theorem states that Lévy process corresponds to infinitely divisible distribution:

Theorem 6.1. ([20, Theorem 7.10]) (i) *If* $\{X(t)\}_{t\geq 0}$ *is a Lévy process, then for any* $t \geq 0$, $\mathbf{P}(X(t) \in dx)$ *is infinitely divisible and it holds that*

$$E^{\mathbf{P}}[e^{i\xi\cdot X(t)}] = \left(E^{\mathbf{P}}[e^{i\xi\cdot X(1)}]\right)^t, \quad \xi \in \mathbf{R}^d.$$

(ii) *Conversely, if* P *is an infinitely divisible distribution, then there exists a Lévy process* $\{X(t)\}_{t\geq 0}$ *such that* $\mathbf{P}(X(1) \in dx) = P(dx)$.

The following theorem gives a representation of the characteristic function of the infinitely divisible distribution:

Theorem 6.2. (Lévy-Khintchine formula) ([20, Theorem 8.1], [1, Theorem 1.2.14]) (i) *If* P *is infinitely divisible, then there exist a vector* $\gamma \in \mathbf{R}^d$, *a nonnegative definite symmetric* $d \times d$ *matrix* A *and Lévy meausre* $n(dy)$, *that is a* σ*-finite measure on* $\mathbf{R}^d \setminus \{0\}$ *satisfying* $\int_{|y|>0}(1 \wedge |y|^2)n(dy) < \infty$, *such that*

$$E^P[e^{i\xi\cdot y}] = \exp\left\{i\gamma\cdot\xi - \frac{1}{2}\xi\cdot A\xi + \int_{|y|>0}\left(e^{i\xi\cdot y} - 1 - i\xi\cdot y\mathbf{1}_{|y|<1}\right)n(dy)\right\}, \quad \xi \in \mathbf{R}^d. \tag{6.1}$$

γ, A *and* $n(dy)$ *are uniquely determined by* P.

(ii) *Conversely, if* $\gamma \in \mathbf{R}^d$, A *is a nonnegative definite symmetric* $d\times d$ *matrix and* $n(dy)$ *is a Lévy measure, then there exists an infinitely divisible distribution whose characteristic function is given by the right-hand side of (6.1).*

We note by Theorem 6.1 and Theorem 6.2 that Lévy process corresponds to triplet $(\gamma, A, n(dy))$. Namely, if X is a Lévy process, then

$$E^{\mathbf{P}}[e^{i\xi X(t)}] = \exp\left[t\left\{i\gamma\cdot\xi - \frac{1}{2}\xi\cdot A\xi + \int_{|y|>0}\left(e^{i\xi\cdot y} - 1 - i\xi\cdot y\mathbf{1}_{|y|<1}\right)n(dy)\right\}\right], \quad \xi \in \mathbf{R}^d, t \geq 0, \tag{6.2}$$

for some γ, A and $n(dy)$. Conversely, if γ, A and $n(dy)$ are given, then there exists a Lévy process X such that (6.2) holds. We call $(\gamma, A, n(dy))$ the *generating triplet* (or *generator*) of the Lévy process X ([20, p.65]). For example, d-dimensional Brownian motion and Poisson process with parameter $c > 0$ correspond to $(0, E_d, 0)$ and $(0, 0, c\delta_1(dy))$, respectively.

Next, we discuss the sample path $t \mapsto X(t)$ of Lévy process X with the generating triplet $(\gamma, A, n(dy))$.

Definition 6.5. For $G \in \mathcal{B}(0,\infty) \times \mathcal{B}(\mathbf{R}^d \setminus \{0\})$, let us define

$$N(G) := \#\{s > 0; (s, X(s) - X(s-)) \in G\}.$$

We note that for $t > 0$, $\varepsilon > 0$ ([2, p122]),

$$N\big((0,t] \times \{y; |y| \geq \varepsilon\}\big) = \#\{0 < s \leq t; |X(s) - X(s-)| \geq \varepsilon\} < \infty. \tag{6.3}$$

$N(dsdy)$ is a counting measure on $(0,\infty) \times (\mathbf{R}^d \setminus \{0\})$. More precisely, $\{N(G)\}_G$ is a *Possion random measure* on $(0,\infty) \times (\mathbf{R}^d \setminus \{0\})$ with *intensity measure* $dsn(dy)$. Namely
(i) for each G, $N(G)$ has Poisson distribution with mean $dsn(dy)(G)$,
(ii) if $G_1, \ldots, G_k$ are disjoint, then $N(G_1), \ldots, N(G_k)$ are independent.
In (i), we interpret that $N(G) = \infty$ a.s. if $dsn(dy)(G) = \infty$. (ii) states that the different times or the different size jumps are independent.

Put

$$\widetilde{N}(G) := \begin{cases} N(G) - dsn(dy)(G), & \text{if } dsn(dy)(G) < \infty, \\ 0, & \text{if } dsn(dy)(G) = \infty. \end{cases}$$

$\{\widetilde{N}(G)\}_G$ is called *compensated Possion random measure.* It is trivial that $E^{\mathbf{P}}[\widetilde{N}(G)] = 0$.

Theorem 6.3. (Lévy-Itô decomposition) ([20, Theorem 19.2], [1, Theorem 2.4.16]) (i) *For each* $t \geq 0$,

$$X(t) = t\gamma + \sigma B(t) + \int_0^t \int_{|y|\geq 1} yN(dsdy) + \int_0^t \int_{0<|y|<1} y\widetilde{N}(dsdy), \tag{6.4}$$

where B *is* d'*-dimensional Brownian motion starting at* 0, σ *is a* $d \times d'$ ($d' \leq d$) *real valued matrix such that* ${}^t\sigma\sigma = A$ *and* $\int_0^t := \int_{(0,t]}$.
(ii) *Three stochastic processes* $\{t\gamma + \sigma B(t)\}_{t\geq 0}$, $\left\{\int_0^t \int_{|y|\geq 1} yN(dsdy)\right\}_{t\geq 0}$ *and* $\left\{\int_0^t \int_{0<|y|<1} y\widetilde{N}(dsdy)\right\}_{t\geq 0}$ *are independent.*

Remark 6.1. (1) By Definition 6.5, $\int_0^t \int_{|y|\geq 1} yN(dsdy)$ is equal to the finite sum (cf. (6.3))

$$\sum_{s\leq t} \mathbf{1}_{|X(s)-X(s-)|\geq 1}(X(s) - X(s-)).$$

(2) $\int_0^t \int_{0<|y|<1} y\widetilde{N}(dsdy)$ is the $L^2(\Omega; \mathbf{P})$-limit of

$$\int_0^t \int_{\varepsilon<|y|<1} y\widetilde{N}(dsdy) = \int_0^t \int_{\varepsilon<|y|<1} yN(dsdy) - t\int_{\varepsilon<|y|<1} yn(dy)$$

as $\varepsilon \downarrow 0$.

(3) The characteristic functions of $t\gamma + \sigma B(t)$, $\int_0^t \int_{|y|\geq 1} yN(dsdy)$ and $\int_0^t \int_{0<|y|<1} y\widetilde{N}(dsdy)$ are

$$\begin{aligned}
&\exp\left\{t(i\gamma\cdot\xi - \tfrac{1}{2}\xi\cdot A\xi)\right\}, \\
&\exp\left\{t\int_{|y|\geq 1} \left(e^{i\xi\cdot y} - 1\right)n(dy)\right\} \text{ and} \\
&\exp\left\{t\int_{0<|y|<1} \left(e^{i\xi\cdot y} - 1 - i\xi\cdot y\right)n(dy)\right\},
\end{aligned}$$

respectively (cf. Theorem 6.2).

6.2 Martingale and semimartingale.

In this subsection, let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\{\mathcal{F}(t)\}_{t\geq 0}$ is a filtration of $\mathcal{F}$, i.e., $\mathcal{F}(t)$ is a sub σ-algebra of $\mathcal{F}$ and $\mathcal{F}(s) \subset \mathcal{F}(t)$ for $0 \leq s \leq t$. Furthermore, we make the *usual hypothesis* ([1], [13]):
(i) (completeness) $\mathcal{N} := \{A \in \mathcal{F}; \mathbf{P}(A) = 0\} \subset \mathcal{F}(0)$
(ii) (right continuity) $\bigcap_{\varepsilon>0} \mathcal{F}(t+\varepsilon) = \mathcal{F}(t)$ $(t \geq 0)$.
The completeness of the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and condition (i) are made from the mathematical convenience. Condition (ii) is needed to consider the stopping time defined after.

Let $X = \{X(t)\}_{t\geq 0} = \{X(t,\omega)\}_{t\geq 0,\, \omega\in\Omega}$ be a real stochastic process.

Definition 6.6. X is called a $\{\mathcal{F}(t)\}_{t\geq 0}$-*martingale* (*supermartingale, submartingale*) if the following conditions are satisfied:
(i) $E^{\mathbf{P}}[|X(t)|] < \infty$ for each $t \geq 0$.

(ii) X is $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted, i.e., $X(t)$ is $\mathcal{F}(t)$-measurable for each $t \geq 0$.
(iii) For $0 \leq s \leq t$, $E^{\mathbf{P}}[X(t)|\mathcal{F}(s)] = X(s)$ (resp. $\leq$, $\geq$) a.s., i.e., for $A \in \mathcal{F}(s)$, $E^{\mathbf{P}}[X(t)\mathbf{1}_A] = E^{\mathbf{P}}[X(s)\mathbf{1}_A]$ (resp. $\leq, \geq$).
Here $E^{\mathbf{P}}[X(t)|\mathcal{F}(s)]$ is the *conditional expectation* of $X(t)$ with respect to $\mathcal{F}(s)$.

The martingale (supermartingale, submartingale) is a mathematical model for the fair (resp. disadvantage, profitable) gambling. If 1-dimensional Brownian motion B and Poisson process N with parameter $c > 0$ are $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted, then B and $\{N(t)-ct\}_{t\geq 0}$ are $\{\mathcal{F}(t)\}_{t\geq 0}$-martingales.

The following theorem is useful when we estimate a martingale:

Theorem 6.4. (Doob's martingale inequaity) ([13, I, Theorem 6.10], [1, Theorem 2.1.5]) *If $X = \{X(t)\}_{t\geq 0}$ is a martingale such that $E^{\mathbf{P}}[|X(t)|^p] < \infty$ for some $p > 1$ and any $t \geq 0$, then for $T > 0$, the following inequalities hold:*

$$\mathbf{P}\left(\sup_{t\leq T}|X(t)| \geq \varepsilon\right) \leq \frac{1}{\varepsilon^p}E^{\mathbf{P}}[|X(T)|^p], \quad \varepsilon > 0,$$
$$E^{\mathbf{P}}\left[\sup_{t\leq T}|X(t)|^p\right] \leq \left(\frac{p}{p-1}\right)^p E^{\mathbf{P}}[|X(T)|^p].$$

Next, we define some concepts to introduce semimartingale.

Definition 6.7. (1) A mapping $f : [0,\infty) \to \mathbf{R}^d$ is called *càdlàg* if f is right-continuous and has left-hand limit. Càdlàg is short for *continue à droite et limité à goche* in Frecnch.
(2) A mapping $\sigma : \Omega \to [0,\infty]$ is called a $\{\mathcal{F}(t)\}_{t\geq 0}$-*stopping time* (or *Markov time*) if $\{\omega \in \Omega; \sigma(\omega) \leq t\} \in \mathcal{F}(t)$ for each $t \geq 0$.
(3) X is called a *local* $\{\mathcal{F}(t)\}_{t\geq 0}$-*martingale* if it is $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted and there is a sequence of $\{\mathcal{F}(t)\}_{t\geq 0}$-stopping time σ_k such that $\sigma_k < \infty$ ($k \in \mathbf{N}$), $\sigma_k \uparrow \infty$ a.s. and $\{X(t \wedge \sigma_k)\}_{t\geq 0}$ is a $\{\mathcal{F}(t)\}_{t\geq 0}$-martingale for any $k \in \mathbf{N}$.
(4) X is called a *increasing process* if the following conditions are satisfied:
(i) X is $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted.
(ii) $X(0) = 0$, the sample path $t \mapsto X(t)$ is right-continuous and increasing a.s. (hence $X(t) \geq 0$ a.s. for any $t \geq 0$.)
(5) X is called *predictable* if the mapping $(t,\omega) \mapsto X(t,\omega)$ is measurable with respect to the smallest σ-algebra on $(0,\infty) \times \Omega$ generated by all $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted left-continuous real stochastic processes defined on $[0,\infty) \times \Omega$.

Theorem 6.5. (Doob-Meyer decomposition) ([1, Theorem 2.2.6]) *Any càdlàg submartingale X has a unique decomposition $X(t) = X(0)+M(t)+C(t)$, where M is local martingale with $M(0) = 0$ a.s. and C is an increasing, predictable process.*

Definition 6.8. X is called a $\{\mathcal{F}(t)\}_{t\geq 0}$-*semimartingale* if it is $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted and for any $t \geq 0$,

$$X(t) = X(0) + M(t) + C(t) \quad \text{a.s. } t \geq 0,$$

where M is a local $\{\mathcal{F}(t)\}_{t\geq 0}$-martingale with $M(0) = 0$ a.s. and C is a $\{\mathcal{F}(t)\}_{t\geq 0}$-adapted process such that $t \mapsto C(t)$ is of bounded variation on any finite interval a.s..

Clearly, any martingale is a semimartingale. Any 1-dimensional Lévy process is a semimartingale ([1, Proposition 2.7.1]) with *augmented natural filtration* $\{\mathcal{F}(t)\}_{t\geq 0}$ (cf. [1, Theorem 2.1.10]), where $\mathcal{F}(t)$ is the smallest σ-algebra which contains $\bigcap_{\varepsilon>0}\sigma(X(s); s \leq t+\varepsilon)$ and $\mathcal{N}$. In fact, If the Lévy process X has a decomposition in Theorem 6.3, then we may put $M(t) := \sigma B(t) + \int_0^t \int_{0<|y|<1} y\widetilde{N}(dsdy)$, $C(t) := t\gamma + \int_0^t \int_{|y|\geq 1} yN(dsdy)$.

Stochastic calculus, namely, the infinitesimal calculus for sample functions of stochastic processes, was established to assign meaning to ordinary differential equations involving continuous stochastic processes. Stochastic calculus is developed for semimartingales, in particular, *Itô process*:

$$\begin{aligned} Y(t) = Y(0) &+ \int_0^t f_1(s)ds + \int_0^t f_2(s)dB(s) \\ &+ \int_0^t \int_{|y|>0} g_1(s,y)N(dsdy) + \int_0^t \int_{|y|>0} g_2(s,y)\widetilde{N}(dsdy). \end{aligned} \tag{6.5}$$

As for the condition of f_1, f_2, g_1, g_2, we refer to [1], [13]

Finally, we introduce *Itô's formula* for Lévy process. This formula is corresponding to the chain rule in the infinitesimal calculus:

Theorem 6.6. (Itô's formula) ([13, II, Theorem 5.1], [1, Theorem 4.4.7]) *If $X = \{X(t)\}_{t\geq 0}$ is a d-dimensional Lévy process represented by (6.4), then for each $F \in C^2(\mathbf{R}^d)$, $\{F(X(t))\}_{t\geq 0}$ is a Itô process given by*

$$F(X(t)) = F(0) + \sum_{i=1}^{d}\sum_{j=1}^{d'} \sigma_{ij} \int_0^t \frac{\partial F}{\partial x_i}(X(s))dB_j(s) + \int_0^t (\gamma \cdot \nabla F)(X(s))ds$$

$$
\begin{aligned}
&+ \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{d'}\sigma_{ik}\sigma_{jk}\int_0^t \frac{\partial F}{\partial x_i \partial x_j}(X(s))ds \\
&+ \int_0^t\int_{|y|\geq 1}\{F(X(s-)+y) - F(X(s-))\}\, N(dsdy) \\
&+ \int_0^t\int_{0<|y|<1}\{F(X(s-)+y) - F(X(s-))\}\, \widetilde{N}(dsdy) \\
&+ \int_0^t\int_{0<|y|<1}\{F(X(s)+y) - F(X(s)) - (y\cdot\nabla F)\,(X(s))\}\, dsn(dy). \qquad (6.6)
\end{aligned}
$$

Here we put $\sigma := (\sigma_{ij})_{1\leq i\leq d, 1\leq j\leq d'}$ *and* $B(s) := (B_1(s), \ldots, B_{d'}(s))$.

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