Comparison of numerical methods for 1-D hyperbolic-type problems with free boundary

Dissertation Abstract

Comparison of numerical methods for 1-D hyperbolic-type problems with free boundary

1 次元双曲型自由境界問題の数値解法の開発と評価

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1 Introduction

In this thesis, we study hyperbolic-type problems with free boundary. The real case can be considered as a phenomenon of ”peeling tape attached to a surface”. From this phenomenon, we can obtain the energy function. This energy function can be derived into Euler-Lagrange equation. A previous research was conducted solving this equation using numerical method called fixed domain method and showed the high accuracy for its results [2]. However, this method only works when there is only one free boundary point.

On the other hand, this problem has been investigated before by adding smoothing characteristic function. We call this as approximated problem. It can be used to model droplet motion on a plane or bubble touching the water surface in higher dimension using numerical methods called discrete Morse flow. In this study, we want to investigate the accuracy of numerical methods solving the approximated problem. However, the exact solution is not available in all cases. Despite not having exact solution, we use the solution of fixed domain method to get the accuracy. The numerical methods which we investigate are: two types of finite difference method, finite element method, and discrete Morse flow. In addition, we also investigate the treatment in the free boundary to get optimal error and comparing two smoothing characteristic functions. Since the fixed domain method solutions are available in 1-D only, we consider 1-D problem in this study.

2 Physical model of peeling tape problem

In this section, we explain the phenomenon of peeling tape on a plane. Suppose there is a thin film adhered to a plane. We consider this film as our tape. It is peeled off from the plane and starts to expand along the sticked tape. The region where the tape is adhered is considered as a domain Ω. We assume that the tape has the same tension γ at any places. The shape of the tape is represented by a function $u : \Omega \to \mathbf{R}$. The shape of u is obtained by measuring the energy of this phenomenon.

$$J(u) = \int_0^\tau \int_\Omega \left(\frac{\gamma}{2}|\nabla u|^2 - \frac{\rho}{2} u_t^2 \chi_{u>0} + \frac{Q^2}{2}\chi_{u>0} \right) dxdt. \tag{1}$$

We choose $\gamma = \rho = 1$. Let u be the stationary point of (1) and $u \in C^0(\Omega \times (0,\tau) \cap W^{1,2}(\Omega,\tau))$, then u satisfies

$$\Delta u - u_{tt} = 0 \quad \text{in } \{u > 0\}. \tag{2}$$

Moreover, if $u \in C^2(\Omega \times (0,\tau) \cap \{u > 0\})$ and $\partial\{u > 0\}$ is in C^{-1}, then u on the boundary satisfies

$$|\nabla u|^2 - u_t^2 = Q^2 \text{ on } \partial\{u > 0\}. \tag{3}$$

We solve this Euler-Lagrange equation using fixed domain method.

Now we consider an approximation of (1) where the characteristic function is approximated using a smoothing characteristic function. Firstly, we define a smooth function called $\beta_\varepsilon(s)$. $\beta_\varepsilon(s) \in C^2(\mathbf{R})$, $\beta_\varepsilon(s) \geq 0$, and satisfies

$$\beta_\varepsilon(s) \begin{cases} = 0 & s \leq 0, \\ \leq 1/\varepsilon & 0 < s < \varepsilon \text{ and } |\beta_\varepsilon'(s)| \leq \dfrac{C}{\varepsilon^2}, \\ = 0 & \varepsilon \leq s. \end{cases}$$

It is also that $\int_0^\varepsilon \beta_\varepsilon ds = 1$ and we define B_ε

$$B_\varepsilon(u) = \int_0^u \beta_\varepsilon(s)ds \underset{\varepsilon\to 0}{\longrightarrow} \begin{cases} 1 & \text{in } \{u > 0\}, \\ 0 & \text{in } \{u = 0\}, \end{cases}$$

which is the smoothing characteristic function of $\chi_{u>0}$. Then we rewrite the energy function

$$J_\varepsilon(u) = \int_0^\tau \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \chi_{u>0}\frac{1}{2}u_t^2 + \frac{Q^2}{2}B_\varepsilon(u) \right) dxdt. \tag{4}$$

By taking the first variation of (4), our problem becomes

$$\begin{cases} \Delta u - \chi_{u>0}u_{tt} = -\dfrac{Q^2}{2}\beta_\varepsilon(u) & \text{in } \Omega \times (0,\tau), \\ u(x,0) = u_0(\geq 0) \\ u_t = v_0 \\ u(x,t)|_{\partial\Omega} = f(x,t) \text{ with } f(x,0) = u_0 \text{ on } \partial\Omega. \end{cases} \tag{5}$$

We will solve this problem using numerical methods above.

3 Numerical method

3.1 Explicit method 1 (spatial central difference + time forward difference)

We represent equation (5) using explicit method with u_{xx} approximated by central difference. Suppose $u_t = v$ then

$$
\begin{aligned}
\frac{d}{dt}u_i(t) &= v_i(t), \quad i = 1, \ldots, N-1, \\
\chi_{\{u>0\}}(x_i, t)\frac{d}{dt}v_i(t) &= \frac{u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)}{(\Delta x)^2} - \frac{Q^2}{2}(\chi^\varepsilon)'(u_i(t)), \quad i = 1, \ldots, N-1.
\end{aligned} \tag{6}
$$

The initial and boundary conditions are $u_0(t) = f(t)$, $u(0) = g(x)$, $v_0(t) = f'(t)$, and $v(0) = h(x)$. We solve (6) using the 4th order Runge-Kutta.

3.2 Explicit method 2 (spatial and time central difference)

We approximate u_{xx} and u_{tt} from equation (5) using central difference. $[0, \tau]$ is divided into M, $0 = t_0 < t_1 < \ldots < t_M = \tau$.

$$
\begin{aligned}
&\chi_{\{u>0\}}(x_i, t_k)\frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{(\Delta t)^2} = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} - \frac{Q^2}{2}(\chi^\varepsilon)'(u_i^k), \\
&i = 1, \ldots, N-1, \quad \text{and} \quad k = 1, \ldots, M-1,
\end{aligned} \tag{7}
$$

where $u_i^k = u(x_i, t_k)$, $i = 0, \ldots, N$. Then we calculate the solutions with following

$$
\chi_{\{u>0\}}(x_i, t_k)\, u_i^{k+1} = 2u_i^k - u_i^{k-1} + (\Delta t)^2 \left(\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} - \frac{Q^2}{2}(\chi^\varepsilon)'(u_i^k) \right). \tag{8}
$$

The boundary conditions are $u_0(t) = f(t)$ and $u(0) = g(x)$.

4 Finite Element Method

We take the weak form of equation (5) and define a test function $\xi \in C_0^\infty(\Omega)$. Then we obtain

$$
\int_\Omega (\chi_{\{u>0\}} u_{tt} - u_{xx} + \frac{Q^2}{2}(\chi^\varepsilon)'(u))\xi dx = 0, \quad \forall \xi \in C_0^\infty(\Omega). \tag{9}
$$

We approximate solution as $u(x,t) = \sum_{i=0}^{N} a_i(t)\varphi_i(x)$ where φ_i is basis function. The basis function is defined as

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{\Delta x} & x_{i-1} \le x \le x_i, \\ \dfrac{x_{i+1} - x}{\Delta x} & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

$$i = 1, 2. \dots, N-1.$$

Then by integration by parts (9) gives

$$\int_\Omega \left(\chi_{\{u>0\}} \left[\sum_{i=0}^{N} (a_i)_{tt} \varphi_i \right] \xi + \left[\sum_{i=0}^{N} a_i (\varphi_x)_i \right] \xi_x + \frac{Q^2}{2} (\chi^\varepsilon)' (\sum_{i=0}^{N} a_i \varphi_i) \xi \right) dx = 0, \forall \xi \in C_0^\infty(\Omega). \tag{10}$$

We choose basis function φ_j; $j = 1, \dots, N-1$ as our test function and rewrite (10)

$$\sum_{i=0}^{N} \left[(a_i)_{tt} \int_\Omega \chi_{\{u>0\}} \varphi_i \varphi_j dx \right] + \sum_{i=0}^{N} \left[(a_i) \int_\Omega (\varphi_x)_i (\varphi_x)_j dx \right] + \frac{Q^2}{2} \int_\Omega (\chi^\varepsilon)' (\sum_{i=0}^{N} a_i \varphi_i) \varphi_j dx = 0 \tag{11}$$

$$j = 1, 2, \dots, N-1$$

We change into matrix form

$$Ba_{tt} + Aa + \frac{Q^2}{2} C(a) = 0, \tag{12}$$

where $a = a_1, \dots, a_{N-1}$, C is a column matrix whose elements are determined by a and b is determined by boundary values.

We approximate a_{tt} using central difference with $a^k = a(x, t_k)$.

$$B \frac{a^{k+1} - 2a^k + a^{k-1}}{(\Delta t)^2} + Aa^k + \frac{Q^2}{2} C(a^k) = b. \tag{13}$$

The final form is

$$Ba^{k+1} = 2Ba^k - Ba^{k-1} - (\Delta t)^2 \left(Aa^k + \frac{Q^2}{2} C(a^k) + b \right). \tag{14}$$

We solve (14) using steepest decent method.

4.1 Discrete Morse Flow

Minimizing the functional below on function space κ approximates the weak solution of equation (5) [1]

$$J_n(u) = \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2(\Delta t)^2} \chi_{\{u>0\}} dx + \frac{1}{2} \int_\Omega |\nabla u|^2 dx + \frac{Q^2}{2} \int_\Omega \chi^\varepsilon(u) dx. \quad (15)$$

$$\kappa = \{u \in \mathbf{W}^{1,2}(\Omega); u = g \text{ on } \partial\Omega\}$$

Function u_0 and u_1 are the approximate solutions at time level $t = 0$ and $t = \Delta t$ respectively. We define the approximate solution u_n for the next time level $t = n\Delta t$, $n = 2, 3, \ldots, N$ by minimizing (15).

5 Numerical Results

The initial conditions of our experiments are

$$\begin{aligned} l_0 &= \frac{1}{\sqrt{Q^2 + f'(0)^2}} \\ g(x) &= \max(1 - \frac{1}{l_0} x, 0), \\ h(x) &= \begin{cases} f'(0) & 0 < x \leq l_0, \\ 0 & l_0 < x, \end{cases} \end{aligned}$$

and $f(t)$ are

case 1 Peeling speed is constant $f(t) = at + 1$. The exact solution of this case is $u(x, t) = \max(1 + t - \frac{1}{l_0} x, 0)$.

case 2 Peeling speed is increasing $f(t) = (at + 1)^2$.

case 3 Peeling speed is decreasing $f(t) = \sqrt{at + 1}$.

case 4 Peeling speed is stopping at some times $f(t) = 1 + at + \sin\ t$

case 5 Peeling direction is downward (pasting the tape).

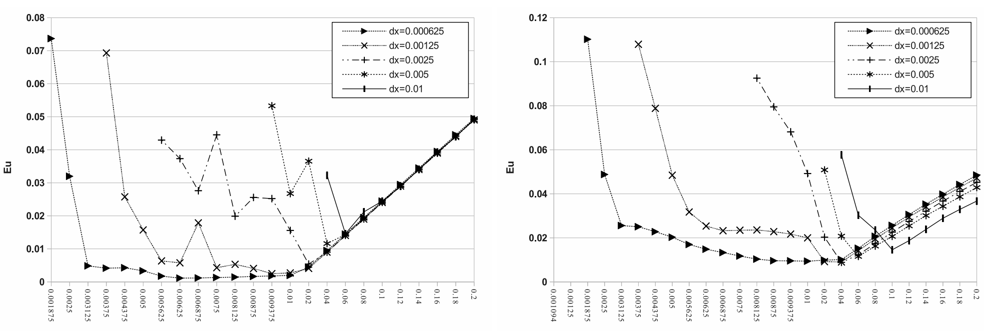
$$
\begin{aligned}
g(x) &= \max(10 - \frac{1}{l_0}x, 0),\\
f(t) &= 10 - at,\\
h(x) &= \begin{cases} f'(0) & 0 < x \le l_0,\\ 0 & l_0 < x, \end{cases}
\end{aligned}
$$

case 6 Peeling directions are upward and downward (oscillating tape) $f(t) = 1 + 0.3 \sin\ t$.

5.1 The error of peeling tape model using smoothing characteristic function

The comparisons are shown in figures 1. From the figures, we can see that the errors of solutions from all cases tend to be small when dx is decreasing in some ε. They show that small and big ε give big error.

In addition, the choice of ε to get minimum error also depends on the gradient of the solution near to free boundary point. To see this, we solve cases 1-4 with different gradients of solutions. We choose only cases 1-4 since they are enough to represent different kinds of solution. By changing parameter a, the gradient of the solution can be set. We set the gradient vary from 2-40. We call this gradient as g_u. The result is the appropriate ε approximately $8 - 360 \times dx$.



(a) E_u case 1 at time level τ=9

(b) E_u case 2 at time level τ=9

Figure 1: errors of numerical solution of cases 1-6

5.2 Comparisons of smoothing characteristic functions

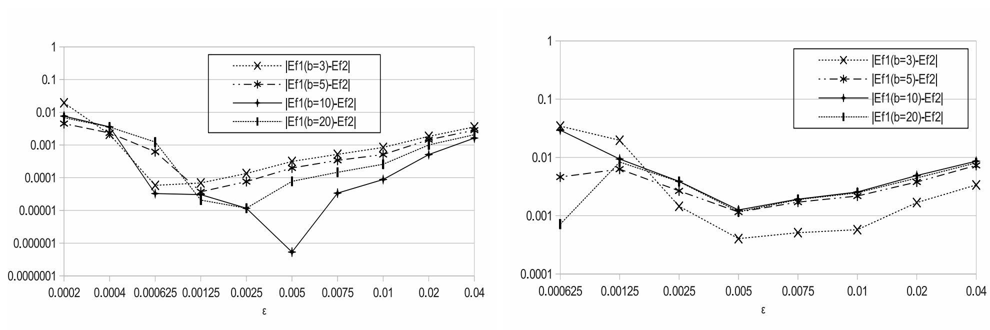
We compare two smoothing characteristic functions (16) and (17). We apply these two functions in equation (5) with some parameters $Q^2 = 1,\ \Omega = [0, 15],\ a = 1,\ dx = 0.005,$ and $dt = 0.9dx$ and solve using explicit method 2.

$$(\chi^{\varepsilon})'(u) = \begin{cases} 1/\varepsilon & 0 < u < \varepsilon, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

or

$$(\chi^{\varepsilon})'(u) = \begin{cases} \dfrac{hu}{a} & 0 < u < a, \\ h & a \le u \le \varepsilon - a, \\ \dfrac{h(\varepsilon - u)}{a} & \varepsilon - a \le u \le \varepsilon, \\ 0 & \text{otherwise,} \end{cases} \tag{17}$$

where $a = \dfrac{\varepsilon}{b}$, b is positive number and $h = \dfrac{1}{\varepsilon - a}$.



(a) $dx = 0.0001$ (b) $dx = 0.000625$

Figure 2: $|\tilde{E}_u^{17} - \tilde{E}_u^{16}|$ case 1 at $\tau = 9$

The comparisons in figure 2 is done by calculating $|\tilde{E}_u^{17} - \tilde{E}_u^{16}|$ where $\tilde{E}_u^{17}$ is $\tilde{E}_u$ using equation (17) (in figure 2 we call $Ef1$) and $\tilde{E}_u^{17}$ is $\tilde{E}_u$ using equation (16) (in figure 2 we call $Ef2$). From the figure, we can see that the error differences between two smoothing characteristic function are relatively similar that are in the order $10^{-3} - 10^{-4}$, if we consider ε which are bigger $4 - 10 \times dx$. Therefore, equation (16) is sufficient to be our smoothing characteristic function.

5.3 Comparisons of numerical methods

We compare four numerical methods by solving cases 1-6 and compare the errors of each methods. We choose the parameters $Q^2 = 1,\ \Omega = [0, 15],\ dx = 0.005,\ \varepsilon = 0.04$ and dt as in table 1. The errors of the methods are shown in table 2. Time complexity of each methods can be seen in table 3.

explicit method 1	explicit method 2	FEM	DMF
$0.9dx$	$0.9dx$	$0.5dx$	$0.1dx$

Table 1: dt for numerical methods

case	explicit method 1	explicit method 2	FEM	DMF
1	0.008	0.011	0.01	0.0097
2	0.02	0.02	0.023	0.012
3	0.0079	0.006	0.0075	0.0074
4	0.013	0.013	0.014	0.0097
5	0.005	0.003	0.005	0.006
6	0.007	0.008	0.007	0.009

Table 2: E_u at time $\tau = 9$ (cases 1-5) and $\tau = 7$ (case 6)

	fixed domain method	explicit method 1	explicit method 2	FEM	DMF
time	2s	3s	3s	10mins	> 15mins

Table 3: Time complexity

The error differences of each numerical methods in table 2 are relatively small (order $10^{-3} - 10^{-4}$). Therefore, we conclude that all methods are good. However, based on the time complexity in table 3, DMF has big time complexity due to its algorithm and small dt. On the other hand, DMF has advantage that it can be added some constraints such as volume constraint to support advanced model like droplet motion. Hence, this method is promising to be used further. We also try several dt for DMF and we find that when $dt = 1/10dx$ the errors of DMF solution approach the errors of other methods. In FEM, we find that $dt \leq 1/2dx$ gives stable solutions.

6 Conclusions

We solve 1D hyperbolic-type problem with free boundary and smoothing characteristic function by explicit method and compare with the exact or fixed domain method solutions. The results tell that the errors of explicit method depend on the selection of smoothing characteristic function parameter ε and dx. In addition, the choice of ε depends on the gradient of the solution. We also compare two kinds of smoothing characteristic functions and find that both have similar errors. Furthermore, we compare four numerical methods solving the peeling tape problem and find that all methods have similar errors. However, based on time complexity, discrete Morse flow is the slowest. We also solve some cases where the free boundary points contains more than one points and they appear or vanish during simulation time. We obtain the solutions using four numerical methods and find that all of them give similar solutions. For the future research, it is interesting to implement this problem in higher dimensions.

References

[1] E. Ginder, K. Svadlenka, *A variational approach to a constrained hyperbolic free boundary problem,* Nonlinear Analysis 71 (2009) e1527-e1537

[2] H. Imai, K. Kikuchi, K. Nakane, S. Omata, T. Tachikawa, *A numerical approach to the Asymptotic Behavior of Solution of a one-dimensional free boundary problem of hyperbolic type,* Japan J. of Ind. and Appl. Math. 18 (2001), 43 - 58.

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学位論文審査報告書（甲）

１．学位論文題目（外国語の場合は和訳を付けること。）

Comparison of numerical methods for 1-D hyperbolic-type problems with free boundary

１次元双曲型自由境界問題の数値解法の開発と評価

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３．審査結果の要旨（600～650字）

Faizal Makhrus 君は、２０１２年１０月に自然科学研究科数物科学専攻に入学し（インドネシア政府奨学金給付生）、それ以降、双曲型偏微分方程式の自由境界問題の数値解法の開発と高速化に取り組んできた。この問題は退化双曲型作用素と外力に未知関数に依存する測度を持つ問題である。この問題は難問で空間１次元の場合の強い解の存在と、同じく空間１次元の場合の近似問題の弱解の存在が知られているだけである。

同君はこの問題に対して、主として空間１次元の場合に様々な数値解法を開発してそれらの詳細な比較検討を行った。まず、自由境界の生成消滅には無力ではあるが、それ以外の場合では強力な固定領域法に基づくソルバーを開発した。これは自由境界に囲まれる領域を固定領域に写像し、そこでガレルキン法を用いて常微分方程式に有限次元化し、４次のルンゲ-クッタ法を適用したものである。これを基準として、変分法に基づく離散勾配流法、測度をなめらかに近似した差分法、有限要素法などと詳細に比較した。また、その近似方程式の進行波解を構成し、より深い議論を行うことを可能にした。これらの結果により、双曲型自由境界問題の数値解法をどのように選択すべきかが状況に応じて明確となった。同君は、この結果を原著論文１本にまとめた。以上により本論文は、博士（理学）を授与するに値すると判断した。

４．審査結果　(1) 判　定（いずれかに○印）　合　格　・　不合格

(2) 授与学位　博　士（　理学　）