Numerical simulation of a hyperbolic free boundary problem with volume conservation constraint

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1 Model Equation

In this paper, the hyperbolic free-boundary problem under the volume conservation condition is treated. The typical phenomenon corresponding to this problem is the motion of a bubble on water surface. The aim is to realize the motion in numerical simulation. The bubble moves on water surface with changing its shape and support. The inertia force of the bubble film or a nonuniform distribution of surface-active agent can be the driving force of its motion.

We use the graph of scalar function u to describe the shape of the bubble. The zero level set of u coincides with the water surface. The set where the bubble touches the water surface is called *free-boundary.* The bubble is assumed to keep its volume, i.e., the volume of air surrounded by the film is preserved ($\int_\Omega u\,dx = V = \text{const.}$). The film of bubble is assumed not to go under the water surface ($u \geq 0$). Therefore, the problem becomes a free-boundary problem of degenerate hyperbolic type with volume constraint. The following equation describes the phenomena well:

$$\chi_{u>0} u_{tt} = \Delta u - R^2 \chi'_\varepsilon(u) + \lambda_\varepsilon \chi_{u>0} \quad (x \in \Omega,\ 0 < t < \tau), \tag{1.1}$$

where Ω is a bounded domain of $\mathbf{R}^m$ ($m \in \mathbf{N} \setminus \{0\}$).

Here, the individual terms of equation (1.1) has the following meaning: The term u_{tt} shows the acceleration of the vertical movement of the film and Δu represents the force originating in the elastic energy of surface tension of the film. The term $\chi_{u>0}$ is the characteristic function of the set $\{u > 0\}$ and $\chi_\varepsilon \in C^2(\mathbf{R})$ is a smoothing function of χ satisfying

$$\chi_\varepsilon(s) = \begin{cases} 0 & (s \leq 0), \\ 1 & (\varepsilon \leq s), \end{cases}$$

with interpolating in $0 < s < \varepsilon$ in such a way that $|\chi'_\varepsilon(s)| \leq C/\varepsilon$. The term $R^2\chi'_\varepsilon(u)$ in (1.1) describes the adhesive force. It is due to this restriction force that the generation of the new surface or movement of free-boundary could be disturbed. It is assumed that the coefficient R does not depend on time and is bounded from below and above ($0 < R_m \leq R \leq R_M$). The above-mentioned three terms represent kinetic effects. The first feature of this equation lies in the coefficient $\chi_{u>0}$ on the left-hand side. Because of this coefficient, non-negativity of the solution is guaranteed.

Function λ_ε, which appears in the last term of (1.1) is a Lagrange multiplier coming from the volume-preservation condition. The explicit form of the Lagrange multiplier λ_ε is obtained as follows:

$$\lambda_\varepsilon (t) = \frac{1}{V} \int_\Omega \left[u u_{tt} \chi_{u>0} + |\nabla u|^2 + R^2 u \chi'_\varepsilon (u) \right] dx \tag{1.2}$$

by formal calculation with volume constraint condition under the assumption that λ_ε does not depend on space variables. It is the second feature of this equation that such non-local term appears. The integral representation above makes the problem more difficult. However, an approximate solution to (1.1) can be calculated by use of a time-semidiscretized functional which is called *discrete Morse flow (DMF) of hyperbolic type* without explicitly considering the volume constraint.

2 Discrete Morse Flow and Approximate Solution

We introduce an approximate problem to (1.1). Here, we give the volume constraint in the admissible space for finding a minimizer of a time-semidiscretized functional corresponding to the Lagrangian.

Problem 2.1. *Let Ω be a bounded domain in $\mathbf{R}^m$. For $n = 2, 3, \ldots$, find minimizer u_n of the following functional:*

$$J_n (u) := \int_\Omega \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \chi_{u>0} \, dx + \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega R^2 \chi_\varepsilon (u) \, dx, \tag{2.1}$$

in the function set

$$\mathcal{K}_V := \left\{ u \in W^{1,2} (\Omega, \mathbf{R}) ; u = 0 \text{ on } \partial\Omega, \int_\Omega u \chi_{u>0} \, dx = V \right\}.$$

Functions $u_0, u_1 \in \mathcal{K}_V$ with $u_1 = u_0 + h v_0$ are given and the sequence $\{u_n\}$ is to be determined inductively. Moreover, by use of these minimizers, construct a weak solution to (1.1).

According to the minimality property, we can show the non-negativity and existence of the minimizer. Furthermore we get the following theorem about the regularity of minimizer.

Theorem 2.1. *For all $\widetilde{\Omega} \subset\subset \Omega$, there exists a positive constant α independent of n with $0 < \alpha < 1$, such that minimizers u_n belong to $C^\alpha(\widetilde{\Omega})$.*

This is indispensable for the application of first variation formula to J_n.

In addition, we carry out interpolation in time of minimizers $\{u_n\}$ and construct the approximate weak solution to (1.1).

We define $\overline{u}^h$ and u^h on $\Omega \times (0, \infty)$ by

$$\overline{u}^h (x, t) = u_n (x), \tag{2.2}$$

$$u^h (x, t) = \frac{t - (n-1) h}{h} u_n (x) + \frac{nh - t}{h} u_{n-1} (x), \tag{2.3}$$

$$\overline{\lambda}^h (t) = \lambda_n, \tag{2.4}$$

for $(x,t) \in \Omega \times ((n-1)h, nh]$, $n \in \mathbf{N}$. Here,

$$\lambda_n = \frac{1}{V}\int_\Omega \left[\frac{u - 2u_{n-1} + u_{n-2}}{h^2} u\chi_{u>0} + |\nabla u|^2 + R^2 u \chi'_\varepsilon(u)\right] dx$$

is a Lagrange multiplier.

We can construct the approximate weak solution to the bubble problem in terms of $\overline{u}^h$ and u^h.

Definition 2.1 (Approximate solution). *Let $\{u_n\} \subset \mathcal{K}_V$ and let $\overline{u}^h$ and u^h be defined as above. If the following conditions*

$$\int_h^T \int_\Omega \left(\frac{u_t^h(t) - u_t^h(t-h)}{h}\phi + \nabla \overline{u}^h \nabla \phi + R^2 \left(\chi'_\varepsilon\left(u^h\right)\right)\phi\right) dx\, dt$$

$$= \int_h^T \int_\Omega \overline{\lambda}^h \phi dx \qquad {}^\forall \phi \in C_0^\infty\left(\Omega \times [0,T) \cap \left\{u^h > 0\right\}\right), \tag{2.5}$$

$$u^h \equiv 0 \qquad \text{in } \Omega \times (h,T) \setminus \left\{u^h > 0\right\}, \tag{2.6}$$

and the initial conditions $u^h(0) = u_0$, $u^h(h) = u_0 + hv_0$ are satisfied, then we call $\overline{u}^h$ and u^h **approximate solutions** *to the bubble problem.*

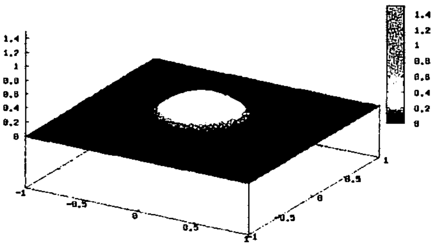
3 Numerical Result

Here we present the numerical method and experimental results. We apply a finite element method with minimizing algorithm and find minimizer of the approximate functional $J_n(u)$ defined above via steepest descent method for a fixed time step n. The time step h and diameter of each finite element are chosen small enough related to the approximation parameter ε.

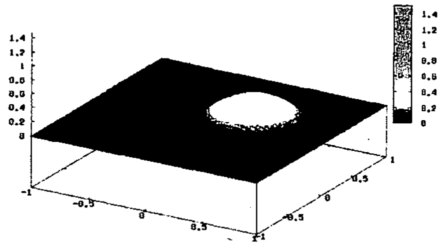
We show the numerical result under the Neumann boundary condition (see Figure 1). We choose the parameters as follows: $h = 5 \times 10^{-3}$, $\varepsilon = 0.1$ and $R^2 = 0.35$. The bubble which has such a initial velocity that it moves in the diagonal direction to the boundary moves toward the boundary and touches it. The more the bubble leans against the boundary, the smaller the area of the surface of the bubble becomes. The bubble moves along the boundary and finally finds the corner of the domain and settles there.

4 Conclusion

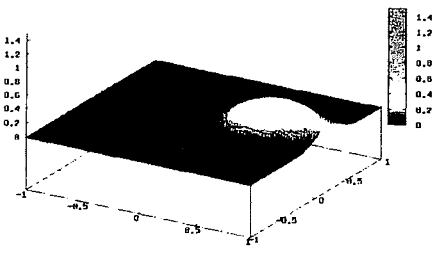
A numerical method for solving the hyperbolic free-boundary problem under the volume conservation constraint was developed. This problem corresponds typically to the motion of bubble restrained on water surface. The model equation is a degenerate hyperbolic including a non-local term (Lagrange multiplier) coming from the volume constraint. We have introduced a variational method called discrete Morse flow to solve this problem and it gave good numerical results. This model can also be applied to the motion of oil on the bottom of water or to problems related to the phenomenon of a water-drop dripping from ceiling. This work has many applications and is significant for the future studying of hyperbolic free-boundary problems.



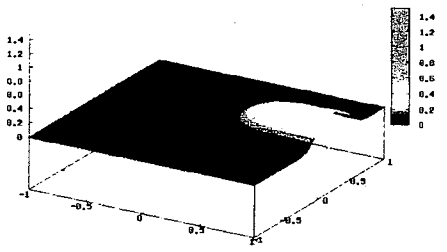
(a) $t = 0.0$



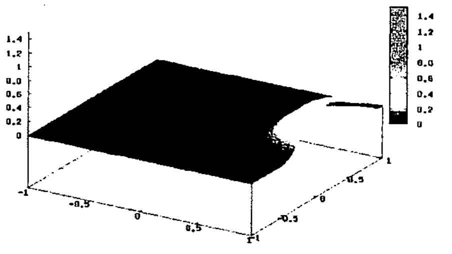
(b) $t = 1.0$



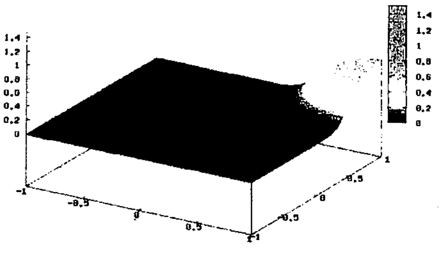
(a) $t = 2.0$



(b) $t = 2.2$



(c) $t = 3.1$



(d) $t = 8.0$

Figure 1: Under the Neumann boundary condition, after touching the boundary, the bubble moves along the boundary. The bubble stops and keeps the smallest surface when reaching the corner of Ω.

学位論文審査結果の要旨

平成１８年８月３日に口頭発表、その後に審査委員会を開催し、慎重審議の結果以下の通り判定した。なお、口頭発表における質疑を最終試験に代えるものとした。

本論文は、双曲型自由境界問題の数値解法に関して、具体的な問題として水面に浮かぶ泡のダイナミクスをモデルに、波動方程式に基づいて運動する泡の表面の挙動を、泡が囲い込む空気が体積保存するという条件の下で、数値解法の開発・近似弱解の構成を行ったものである。さらにここで開発された方法によって実際にシミュレーションを行い、従来解析が困難であった現象を的確に再現することに成功している。本論文の中心的方法は、時間差分空間微分型変分汎関数の最小値を求める離散勾配流法と呼ばれるものであるが、この方法論を今まで適用が困難であった双曲型自由境界問題を取り扱えるように改良し、さらに体積保存条件も無理なく取り扱う工夫を行った。また、近似弱解の構成では、変分汎関数の最小性を使った精密な評価から得られる離散解のヘルダー連続性を示し、近似解としての意味づけがうまくいくことを示した。

以上の研究成果は、数学的には双曲型自由境界問題、体積保存の時間依存問題に新しい知見をもたらし、数値的には接触角で運動する液滴のダイナミクスなど時間に依存する結果がほとんど知られていない分野を切り開くものであり、本論文は博士（理学）授与に値するものと判定した。