Life span of solution for the cauchy problem to nonlinear reaction-diffusion system

氏　　　　名	熊　　暁　華(XIONG　XIAOHUA)
生　年　月　日	
本　　　　籍	中国
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論文審査委員(主査)	小俣　正朗(理学部・教授)
論文審査委員(副査)	伊藤　秀一(理学部・教授)　後藤　俊一(理学部・助教授) 一瀬　　孝(理学部・教授)　田村　博志(理学部・助教授)

学 位 論 文 要 旨

Abstract. In this paper we discuss life span of solution of Cauchy problem to nonlinear reaction-diffusion systems. By using Fourier transformation, fixed point theorem, energy and pointwise estimate methods, we obtain the existence of the solution. Estimates of the global solution and the life span of the local solution are also obtained.

In this paper we discuss the Cauchy system of equations

$$\begin{cases} u_t - m_1\Delta u - m_2\Delta v = f_1(u, v, D_x u, D_x v, D_x^2 u, D_x^2 v) \\ v_t - m_3\Delta u - m_4\Delta v = f_2(u, v, D_x u, D_x v, D_x^2 u, D_x^2 v) \\ \qquad u(0,x) = \varphi(x), v(0,x) = \psi(x), \end{cases} \tag{1}$$

where $(t,x) \in \mathbf{R}_+ \times \mathbf{R}^n$, $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $D_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n})$; $D_x^2 = (\frac{\partial^2}{\partial x_i \partial x_j}; i,j = 1,2,\cdots,n)$ and $m_i (i = 1,2,3,4)$ are constants.
Let $A_0 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_3 \end{pmatrix}$. Assumptions to f_i $(i = 1,2)$ and A_0 are the following:
$A1)$ $f_i(\xi)$ $(\xi \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^{n^2} \times \mathbf{R}^{n^2})$ is smooth and $f_i(\xi) = O(|\xi|^{1+\alpha})$ $(|\xi| \to 0)$, where α is a positive integer and $i = 1,2$.

$A2)$ Eigenvalues of A_0 are positive and A_0 has the form $\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$ or

$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ with the condition that $X^t A_0 X \geq 0$ for any $X \in \mathbf{R}^2$, here $a, b, c \in \mathbf{R}$.

Before giving main theorems we construct a closed set as:

$$X_{s,E,T} := \left\{ V(t,x) = \begin{pmatrix} v_1(t,x) \\ v_2(t,x) \end{pmatrix}; \quad D_{s,T}(V) \leq E \right\} \tag{2}$$

with integer $s \geq n+5$ where

$$D_{s,T}(V) = \sup_{0 \leq t < T} (1+t)^{\frac{n}{2}} \parallel V \parallel_{W^{s-n-3,\infty}(\mathbf{R}^n)} + \sup_{0 \leq t < T} \parallel V \parallel_{W^{s,1}(\mathbf{R}^n)}$$

$$+ \left(\int_0^T \sum_{|k|=2} \parallel D_x^k V(\tau, \cdot) \parallel_{H^s}^2 d\tau \right)^{\frac{1}{2}}. \tag{3}$$

Main results in the paper are the following two theorems.

Theorem 1 *Suppose that* $n > \dfrac{2}{\alpha}$ *and conditions A1), A2) hold. The initial data* Φ *is assumed to be in* $H^{s+1}(\mathbf{R}^n) \cap W^{s,1}(\mathbf{R}^n)$ *with* $s \geq n+5$ *and satisfies*

$$\|\Phi\|_{W^{s,1}(\mathbf{R}^n)} + \|\Phi\|_{H^{s+1}(\mathbf{R}^n)} \leq \delta E. \tag{4}$$

If δ *and* E *are small enough, then there exists a unique weak global solution* U *of (1) in* $X_{s,E,\infty}$.

Theorem 2 *Suppose that* $n \leq \dfrac{2}{\alpha}$ *and conditions A1), A2) hold. The initial data* Φ *is assumed to be in* $H^{s+1}(\mathbf{R}^n) \cap W^{s,1}(\mathbf{R}^n)$ *with* $s \geq n+5$ *and satisfies (4). If* δ *and* E *are small enough, then there exist* $T(\delta) > 0$ *and a unique weak local solution* $U = U(t,x)$ $(0 \leq t < T(\delta))$ *of (1) in* $X_{s,E,T(\delta)}$. *Moreover, the life span* $T(\delta)$ *of the solution to (1) satisfies*

$$T(\delta) = O(\delta^{-\frac{2\alpha}{n}}) \quad \text{when} \quad n = \frac{2}{\alpha},$$

and

$$T(\delta) = O(\delta^{-\frac{2\alpha}{2+n-n\alpha}}) \quad \text{when} \quad n < \frac{2}{\alpha}.$$

Fourier transformation is used in getting the solution of the following homogeneous system

$$\begin{cases} U_t - A_0 \Delta U = 0 \\ U(0,x) = \Phi(x), \end{cases} \tag{5}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$ and $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$.

Denote the solution of (5) as

$$U = S(t)\Phi. \tag{6}$$

We get

$$\|S(t)\Phi\|_{W^{N,\infty}(\mathbf{R}^n)} \leq C(1+t)^{-\frac{n}{2}}\|\Phi\|_{W^{N+n+1,1}(\mathbf{R}^n)}, \quad t \geq 0, \tag{7}$$

$$\|S(t)\Phi\|_{W^{N,1}(\mathbf{R}^n)} \leq C\|\Phi\|_{W^{N,1}(\mathbf{R}^n)}, \quad t \geq 0, \tag{8}$$

where C is independent of t.

The closed set $X_{s,E,T}$ was constructed by using estimates of the solution of (5) in $W^{s-n-3,\infty}(\mathbf{R}^n)$ and $W^{s,1}(\mathbf{R}^n)$.

Next we consider the following linear nonhomogeneous system

$$\begin{cases} U_t - A_0\Delta U = F(t,x) \\ \quad t = 0: \quad U = \Phi. \end{cases} \tag{9}$$

The estimates of the solution to (9) will play an essential role to treat nonlinear problems. We give the next lemma for existence and energy estimates.

Lemma 1 *For any $T > 0$ and integer $s \geq 0$, if we assume $\Phi \in H^{s+1}(\mathbf{R}^n)$ and $F \in L^2(0,T;H^s(\mathbf{R}^n))$, then there exists a unique solution $U = U(t,x)$ to system (9) satisfying*

$$U \in L^2(0,T;H^{s+2}(\mathbf{R}^n)) \tag{10}$$

and

$$U_t \in L^2(0,T;H^s(\mathbf{R}^n)). \tag{11}$$

Moreover, the following estimates also hold:

$$\|U(t)\|^2_{H^s(\mathbf{R}^n)} \leq C(T)\{\|\Phi\|^2_{H^s(\mathbf{R}^n)} + \int_0^t \|F(\tau)\|^2_{H^s(\mathbf{R}^n)}d\tau\}, \tag{12}$$

$$\int_0^t \|\frac{d}{d\tau}U(\tau)\|^2_{H^s(\mathbf{R}^n)}d\tau \leq C\{\|\Phi\|^2_{H^{s+1}(\mathbf{R}^n)} + \int_0^t \|F(\tau)\|^2_{H^s(\mathbf{R}^n)}d\tau\} \tag{13}$$

and

$$\int_0^t \sum_{|k|=2}(\|D_x^k U(\tau,\cdot)\|^2_{H^s(\mathbf{R}^n)}d\tau$$

$$\leq C(\|\Phi\|^2_{H^{s+1}(\mathbf{R}^n)} + \int_0^t \|F(\tau,\cdot)\|^2_{H^s(\mathbf{R}^n)}d\tau). \tag{14}$$

Using the existence of the solution of the linear nonhomogeneous system we define an operator P on $X_{s,E,T}$ as

$$P : V \to U = P(V),$$

here U is the solution of the following system

$$\begin{cases} U_t - A_0\Delta U = F(\Lambda V) \\ \quad t = 0: \quad U = \Phi, \end{cases} \tag{15}$$

where $\Lambda V = (v_1, v_2, D_x v_1, D_x v_2, D_x^2 v_1, D_x^2 v_2)$ and $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. By using Duhamel principle the solution of (15) can be written as the following formula

$$U = S(t)\Phi + \int_0^t S(t-\tau)F(\Lambda V(\tau,\cdot))d\tau. \tag{16}$$

To prove main theorems we prove the operator P maps $X_{s,E,T}$ into itself and is a contraction map. By using fixed point theorem we know there exists a vector function $U \in X_{s,E,T}$ such that U is the solution of (1). To do this we need the estimates of $F(\Lambda V)$ in $W^{s,1}(\mathbf{R}^n)$ and $W^{s-2,1}(\mathbf{R}^n)$ together with the energy estimates of (14). For the estimates of $F(Z)$ we have following two lemmas.

Lemma 2 *Suppose* $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ *satisfies the condition* A1) *and let* s (≥ 0) *be any integer. If* $Z(x)$*, a vector function on* $\mathbf{R}^n$*, belongs to* $W^{s,q}(\mathbf{R}^n) \cap L^p(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$*, then we have*

$$\|F(Z)\|_{W^{s,r}(\mathbf{R}^n)} \leq C\|Z\|_{W^{s,q}(\mathbf{R}^n)}\|Z\|_{L^p(\mathbf{R}^n)}\|Z\|^{\alpha-1}_{L^\infty(\mathbf{R}^n)}, \tag{17}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ $(1 \leq p, q, r \leq \infty)$ *and* C *depends only on* s.

Lemma 3 *Suppose* $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ *satisfies the condition* A1) *and let* s (≥ 0) *be any integer. If* $\overline{Z}$ *and* $\overline{\overline{Z}}$*, vector functions on* $\mathbf{R}^n$*, belong to*

$W^{s,q}(\mathbf{R}^n) \cap W^{s,p}(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$, *then*

$$\begin{aligned}
&\|F(\overline{Z}) - F(\overline{\overline{Z}})\|_{W^{s,r}(\mathbf{R}^n)} \leq C\{\|Z^*\|_{L^p(\mathbf{R}^n)}(\|\overline{Z}\|_{W^{s,q}(\mathbf{R}^n)} \\
&+\|\overline{\overline{Z}}\|_{W^{s,q}(\mathbf{R}^n)}) + \|Z^*\|_{W^{s,q}(\mathbf{R}^n)}(\|\overline{Z}\|_{L^p(\mathbf{R}^n)} + \|\overline{\overline{Z}}\|_{L^p(\mathbf{R}^n)})\} \\
&\cdot(\|\overline{Z}\|_{L^\infty(\mathbf{R}^n)} + \|\overline{\overline{Z}}\|_{L^\infty(\mathbf{R}^n)})^{\alpha-1},
\end{aligned} \tag{18}$$

where $Z^* = \overline{Z} - \overline{\overline{Z}}$, $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q}$ $(1 \leq p, q, r \leq \infty)$ *and* C *depends only on* s.

The important part of this paper is to get energy inequalities in estimating the solution $U(t,x)$ in $H^s(\mathbf{R}^n)$. In order to get the estimate of $U(t,x)$ in $W^{s,1}(\mathbf{R}^n)$, we need estimates of $V(t,x)$ in $H^{s+2}(\mathbf{R}^n)$ by using the estimate of $\|F(\Lambda V(\tau,\cdot)\|_{W^{s,1}(\mathbf{R}^n)}$. Energy inequalities play an important role in proving main theorems.

To get estimates of the life span of the local solution, we first suppose the solution of (15) exists in $(0,T)$, by choosing the initial data small enough we prove P maps $X_{s,E,T}$ into itself and is a contraction map. Estimates of $T = T(\delta)$ corresponding to initial value δE are also obtained.

学位論文審査結果の要旨

提出された学位論文ならびに学位審査書類に基づき，2 回にわたり審査会を開催し，面接審査，口頭発表を行ったうえ，以下の通り判定した。

本論文は反応拡散方程式系の初期値問題の解のライフスパン（寿命）を下から評価したものである。この種の問題は，H. Fujita（1966）が，$u_t = \Box u + u^{1+\alpha}$の初期値問題の爆発解を取り扱ったものにさかのぼる。

S. Klainerman, Li Ta-tsien, Chen Yun-mei 等の研究により，$u_t = \Box u + f(x, u, D_x u, D_x^2 u)$（ただし $f = f(\xi)$ とすると $f(\xi) = O(|\xi|^{1+a})$，$|\xi| \to 0$，ここでαは自然数）の場合は最終的に次のような結果となった。「$n > \dfrac{2}{\alpha}$ のときは，十分小さい初期値を選ぶと大域解が存在し，$n \cdot \dfrac{2}{\alpha}$ の場合は，ライフスパンの下からの評価が有限になる。」

熊君は，これをシステムの場合に拡張し，スカラーの場合と同様な結果を出している。システムの場合の難しさは，(1)ラプラシアンの前の係数行列によってはスカラーの場合と違う作用素となり，その評価をやり直さなければならない，(2)アプリオリ評価に工夫が必要になる，などの点にあり，学術的な価値が認められる。

さらに，熊君は新たに，「初期値の大きさ」と「解の存在保証時間」との関係も導出している。技術的には，様々な不等式，線形化方程式のフーリエ変換による取り扱いと作用素の評価，それに基づく不動点定理など，広範囲の技術を使いこなしている。また，同題の副論文も英文の国際的な学術雑誌に単著で掲載予定となっている。以上より，博士（理学）の学位に相当すると判断される。